Nonlinear Mathematical Physics 1997, V.4, N 1–2, 49–61.

On Integration of the Nonlinear d'Alembert-Eikonal System and Conditional Symmetry of Nonlinear Wave Equations

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Abstract

We study integrability of a system of nonlinear partial differential equations consisting of the nonlinear d'Alembert equation $\Box u = F(u)$ and nonlinear eikonal equation $u_{x_\mu} u_{x^\mu} = G(u)$ in the complex Minkowski space $R(1,3)$. A method suggested makes it possible to establish necessary and sufficient compatibility conditions and construct a general solution of the d'Alembert-eikonal system for all cases when it is compatible. The results obtained can be applied, in particular, to construct principally new (non-Lie, non-similarity) solutions of the non-linear d'Alembert, Dirac, and Yang-Mills equations. Solutions found in this way are shown to correspond to conditional symmetry of the equations enumerated above. Using the said approach, we study in detail conditional symmetry of the nonlinear wave equation $\Box w = F_0(w)$ in the four-dimensional Minkowski space. A number of new (non-Lie) reductions of the above equation are obtained giving rise to its new exact solutions which contain arbitrary functions.

1 Introduction

In the present paper we report some results on the study of the nonlinear d'Alembert-eikonal system, which are shown to be intimately related to the problem of investigation of conditional symmetry of the multidimensional nonlinear wave equation, obtained in collaboration with I.V. Revenko and W.I. Fushchych (Institute of Mathematics, Kyiv). Saying about the d'Alembert-eikonal system, we mean the system of two nonlinear partial differential equations (PDEs) which consists of the nonlinear d'Alembert and eikonal equations taken together

$$\Box u \equiv \partial_\mu \partial^\mu u = F(u), \quad (\partial_\mu u)(\partial^\mu u) = G(u). \tag{1}$$

Here $u = u(x) \in C^2(\mathbf{R}^4, \mathbf{R}^1)$; $\partial_\mu = \partial/\partial x_\mu$, $\mu = \overline{0,3}$; $F(u)$, $G(u)$ are some smooth functions. Hereafter, summation over repeated indices in the Minkowski space with the metric tensor $g_{\mu\nu} = \delta_{\mu\nu} \times (1, -1, -1, -1)$ is understood.

The above system plays an exceptional role when studying reductions of Poincaré invariant PDEs for scalar, spinor, and vector fields. Let us briefly consider a simplest example, reduction of the nonlinear wave equation

$$\Box w = F_0(w), \qquad F_0 \in C^1(\mathbf{R}^1, \mathbf{R}^1). \tag{2}$$

It is well-known that the maximal symmetry group of Eq.(2) with an arbitrary F_0 is the Poincaré group $P(1,3)$. Furthermore, the most general Poincaré-invariant Ansatz reducing it to an ordinary differential equation (ODE) reads

$$w(x) = \varphi\Big(u(x)\Big), \tag{3}$$

where $u(x)$ is an absolute invariant of a three-parameter subgroup of the Poincaré group.

Now, let us ask a simple, at the first sight, question: Do invariant solutions exhaust the set of all possible Ansätze of the form (3) reducing Eq.(2) to ODE? Negative answers to these kinds of questions lead to creation of a concept of *conditional symmetry* of partial differential equations ([1, 2]).

Inserting the Ansatz (3) into Eq.(2), we have

$$(\partial_\mu u)(\partial^\mu u)\varphi''(u) + (\Box u)\varphi'(u) = F_0\Big(\varphi(u)\Big). \tag{4}$$

As we require Eq.(4) be ODE with respect to u, coefficients of φ'', φ' should be functions of u only, i.e. the function u has to satisfy the d'Alembert-eikonal system (1)!

It is straightforward to check that each invariant of a three-parameter subgroup of the Poincaré group satisfies system (1) with properly chosen $F(u)$, $G(u)$. But the inverse assertion is not true. The set of invariants of the group $P(1,3)$ is a small part of the whole set of solutions of the d'Alembert-Hamilton system.

As shown in [3]–[5], the Ansatz

$$\psi(x) = \Big\{if\Big(u(x)\Big)\gamma_\mu\partial_\mu u(x) + g\Big(u(x)\Big)\Big\}\chi, \tag{5}$$

where γ_μ, $\mu = \overline{0,3}$ are 4×4 Dirac matrices, χ is a four-component constant column and $u(x)$ is a solution of the d'Alembert-eikonal system, reduces the nonlinear Dirac equation to a system of two ODEs for the functions $f(u)$, $g(u)$. In [6], an Ansatz for the Yang-Mills field has been suggested, which maps a subset of solutions of Eq.(1) into a subset of solutions of the $SU(2)$ Yang-Mills equations.

A possibility to construct such physically different fields as the Yukawa (scalar), Dirac (spinor) and Yang-Mills (vector) fields via solutions of the system (1) evidences a fundamental nature of the d'Alembert-eikonal system. That is why, it attracted much attention, a number of very interesting results having been obtained.

To the best of our knowledge, the first paper on exact solutions of Eqs.(1) was published by Jacobi (see the list of references in [7]). He studied a three-dimensional elliptic analog of system (1) with $F = G = 0$

$$u_{x_1x_1} + u_{x_2x_2} + u_{x_3x_3} = 0, \quad u_{x_1}^2 + u_{x_2}^2 + u_{x_3}^2 = 0 \tag{6}$$

with a complex-valued function $u(\vec{x})$ and constructed the following class of its exact solutions:

$$C_0(u) + C_1(u)x_1 + C_2(u)x_2 + C_3(u)x_3 = 0, \tag{7}$$

where $C_0(u), \ldots, C_3(u)$ are arbitrary smooth functions satisfying the equality

$$C_1^2(u) + C_2^2(u) + C_3^2(u) = 0. \tag{8}$$

Later on, Smirnov and Sobolev [8]–[10] obtained independently formulae (7), (8) and proved that they give a general solution of system (6).

Some important results on exact solutions of system of PDEs (1) were obtained by Bateman [11], Cartan [12] and Erugin [13].

Recently, Collins [14] using differential geometry methods has obtained a general solution of the three-dimensional d'Alembert-eikonal system. But his approach can not be applied to the four-dimensional system of PDEs (1).

We have developed a technique making it possible to study compatibility of the overdetermined system of PDEs (1) and to construct its general solution. Here we present only principal results omitting the proofs (they can be found in [15]–[17]).

2 List of principal results

In the following, we study the generalized d'Alembert-eikonal system supposing that $u = u(x)$ is a complex-valued function of four complex variables $x_0,\ x_1,\ x_2,\ x_3$.

Evidently, provided $G(u) \neq 0$, system (1) is reduced to the form

$$\Box \tilde{u} = \tilde{F}(\tilde{u}), \quad (\partial_\mu \tilde{u})(\partial^\mu \tilde{u}) = 1$$

by means of the change of the dependent variable

$$u \to \tilde{u} = \int^{u} \Big(G(\tau)\Big)^{-1/2} d\tau.$$

Consequently, instead of (1) we can study a system of PDEs of the form

$$\Box u = F(u), \quad (\partial_\mu u)(\partial^\mu u) = \lambda, \tag{9}$$

where $\lambda = 0,\ 1$.

Theorem 1 *The overdetermined system of PDEs (1) is compatible if and only if the function $F(u)$ has the form*

$$F(u) = \frac{\lambda N}{u + C}\,, \tag{10}$$

where C is an arbitrary complex constant, $N = 0, 1, 2, 3$.

Thus, compatible system of PDEs (1) is equivalent to the following one:

$$\Box u = \frac{\lambda N}{u}, \quad (\partial_\mu u)(\partial^\mu u) = \lambda, \tag{11}$$

where $\lambda = 0$ or $\lambda = 1$.

A general solution of system of nonlinear PDEs (11) is given by the following assertions.

Theorem 2 *General solution of the system of nonlinear PDEs (11) with $N = 0,\ \lambda = 0$ reads*

$$A_\mu(u,\,\tau)x^\mu + A(u,\,\tau) = 0,$$

where $\tau = \tau(x,u)$ is determined in implicit way

$$B_\mu(u,\tau)x^\mu + B(u,\,\tau) = 0$$

and $A_\mu(u,\,\tau),\ B_\mu(u,\,\tau),\ A(u,\,\tau),\ B(u,\,\tau)$ are arbitrary complex-valued functions satisfying the conditions

$$A_\mu A^\mu = A_\mu B^\mu = B_\mu B^\mu = 0, \quad B_\mu \frac{\partial A^\mu}{\partial \tau} = 0.$$

Theorem 3 *General solution of system of nonlinear PDEs (11) with $N = 3,\ \lambda = 1$ reads*

$$u^2 = \Big(x_\mu + A_\mu(\tau)\Big)\Big(x^\mu + A^\mu(\tau)\Big),$$

where the function $\tau = \tau(x)$ is determined in implicit way

$$\Big(x_\mu + A_\mu(\tau)\Big)B^\mu(\tau) = 0$$

and functions $A_\mu(\tau),\ B_\mu(\tau)$ satisfy the relations

$$A'_\mu B^\mu = 0, \quad B_\mu B^\mu = 0.$$

Theorem 4 *General solution of system of nonlinear PDEs (11) with $N = 2,\ \lambda = 1$ reads*

$$1.\ \ u^2 = \Big(x_\mu + R_\mu(\tau)\Big)\Big(x^\mu + R^\mu(\tau)\Big) + \Big[B_\mu(\tau)\Big(x^\mu + R^\mu(\tau)\Big)\Big]^2,$$

where the function $\tau = \tau(x)$ is determined in implicit way

$$\Big(x_\mu + R_\mu(\tau)\Big)B'^\mu(\tau) = 0$$

and $R_\mu,\ B_\mu$ are arbitrary smooth functions satisfying the relations

$$R'_\mu = T(\tau)B_\mu,\ B_\mu B^\mu = -1,\ B'_\mu B'^\mu = 0$$

with an arbitrary $T(\tau)$;

$$2.\ \ u^2 = \Big(x_\mu + R_\mu(\tau)\Big)\Big(x^\mu + R^\mu(\tau)\Big) + \Big[d_\mu\Big(x^\mu + R^\mu(\tau)\Big)\Big]^2,$$

where the function $\tau = \tau(x)$ is determined in implicit way

$$\Big(x_\mu + R_\mu(\tau)\Big)R'^\mu(\tau) + \Big(x_\mu + R_\mu(\tau)\Big)d^\mu d_\nu R'^\nu(\tau) = 0,$$

$d_\mu = const,\ d_\mu d^\mu = -1;$ *R_μ are arbitrary functions satisfying the relation*

$$R'_\mu R'^\mu + (d_\mu R'^\mu)^2 = 0.$$

Theorem 5 *General solution of system of nonlinear PDEs (11) with $N = 1,\ \lambda = 1$ reads*

$$u^2 = \Big(a_\mu x_\mu + h_1(\theta_\mu x^\mu)\Big)^2 + \Big(b_\mu x^\mu + h_2(\theta_\mu x^\mu)\Big)^2,$$

where $h_1,\ h_2$ are arbitrary smooth functions; $a_\mu,\ b_\mu,\ \theta_\mu$ are arbitrary complex constants satisfying the relations

$$a_\mu a^\mu = 1,\ b_\mu b^\mu = -1,\ a_\mu b^\mu = a_\mu \theta^\mu = b_\mu \theta^\mu = \theta_\mu \theta^\mu = 0.$$

Theorem 6 *General solution of system of nonlinear PDEs (11) with $N = 0,\ \lambda = 1$ reads*

$$u = A_\mu(\tau) x^\mu + R_1(\tau),$$

where the function $\tau = \tau(x)$ is determined in implicit way

$$B_\mu(\tau) x^\mu + R_2(\tau) = 0$$

and $A_\mu(\tau),\ B_\mu(\tau),\ R_1(\tau),\ R_2(\tau)$ are arbitrary smooth functions satisfying the relations

$$A_\mu A^\mu = 1, \quad A'_\mu B^\mu = 0, \quad A_\mu B^\mu = 0, \quad B_\mu B^\mu = 0.$$

In the above formulae, prime denotes differentiation with respect to τ.

The principal idea of the proof of the above theorems is a linearization of the eikonal equation $u_{x_0}^2 - u_{x_1}^2 - u_{x_2}^2 - u_{x_3}^2 = 1$ by means of a suitable contact transformation

$$y'_\mu = f_\mu(y,\, v,\, \underset{1}{v}),\ v' = g(y,\, v,\, \underset{1}{v}),\ v'_{y_\mu} = f_\mu(y,\, v,\, \underset{1}{v}), \tag{12}$$

where $\underset{1}{v} = \{\partial v/\partial y_\mu,\ \mu = \overline{0,3}\}$.

Solving a linear equation and substituting the result into the nonlinear d'Alembert equation transformed according to (12), we arrive at an overdetermined system of PDEs with three independent variables. It is reduced to an integrable form by a sequence of transformations (12) with $\mu = 0, 1, 2$.

The theorems 2–6 give a description of the general solution of the nonlinear d'Alembert-eikonal system in a parametric form. But for some special choices of arbitrary functions, it is possible to obtain particular solutions in explicit form. Below, we present without derivation real solutions of the system of PDEs (9).

1) $N = 0, \quad \lambda = 1$

$$u(x) = x_0; \tag{13}$$

2) $N = 1, \quad \lambda = 1$

$$u(x) = \pm(x_0^2 - x_3^2)^{1/2}; \tag{14}$$

3) $N = 2, \quad \lambda = 1$

$$u(x) = \pm(x_0^2 - x_1^2 - x_3^2)^{1/2}; \tag{15}$$

4) $N = 3, \quad \lambda = 1$

$$u(x) = \pm(x_0^2 - x_1^2 - x_2^2 - x_3^2)^{1/2}; \tag{16}$$

5) $N=0,\quad \lambda=-1$

$$
\begin{aligned}
&u(x)=x_1\cos W_1(x_0+x_3)+x_2\sin W_1(x_0+x_3)+W_2(x_0+x_3),\\
&x_0+x_1\sin W_1\Big(u(x)+x_3\Big)+x_2\cos W_1\Big(u(x)+x_3\Big)+\\
&\qquad W_2\Big(u(x)+x_3\Big)=0;
\end{aligned}
\tag{17}
$$

6) $N=1,\quad \lambda=-1$

$$
u(x)=\pm\Big\{\Big(x_1+W_1(x_0+x_3)\Big)^2+\Big(x_2+W_2(x_0+x_3)\Big)^2\Big\}^{1/2};
\tag{18}
$$

7) $N=2,\quad \lambda=-1$

$$
\begin{aligned}
&\pm u(x)+C=x_0\sinh(\tau/C)-x_1\cosh(\tau/C),\\
&\quad \tau=-x_2\pm\Big\{x_0^2-x_1^2+\Big(C\pm u(x)\Big)^2\Big\}^{1/2};\\
&\pm u(x)-C=x_1\sin(\tau/C)+x_2\cos(\tau/C),\\
&\quad \tau=-x_0\pm\Big\{x_1^2+x_2^2-\Big(-C\pm u(x)\Big)^2\Big\}^{1/2};\\
&x_0\sinh\tau-x_3\cosh\tau=2^{-1/2}\{\pm(-u^2(x)-x_\mu x^\mu)^{1/2}\pm u(x)\},\\
&\quad \tau=\arcsin\Big\{\Big(\sqrt{2}(x_1^2+x_2^2)^{1/2}\Big)^{-1}\Big(\pm u(x)\mp(-u^2(x)-x_\mu x^\mu)\Big)^{1/2}\Big\}-\\
&\qquad \arcsin\Big\{x_2(x_1^2+x_2^2)^{-1/2}\Big\},\\
&u(x)=\pm(x_1^2+x_2^2+x_3^2)^{1/2};
\end{aligned}
\tag{19}
$$

8) $N=3,\quad \lambda=-1$

$$
\begin{aligned}
&\pm\Big(u^2(x)-x_3^2\Big)^{1/2}+C=x_0\sinh(\tau/C)-x_1\cosh(\tau/C),\\
&\quad \tau=-x_2\pm\Big\{x_0^2-x_1^2+\Big(C\pm[u^2(x)-x_3^2]^{1/2}\Big)^2\Big\}^{1/2};\\
&\pm\Big(u^2(x)-x_3^2\Big)^{1/2}-C=x_1\sin(\tau/C)+x_2\cos(\tau/C),\\
&\quad \tau=-x_0\pm\Big\{x_1^2+x_2^2-\Big(C\mp[u^2(x)-x_3^2]^{1/2}\Big)^2\Big\}^{1/2}.
\end{aligned}
\tag{20}
$$

Here $W_1, W_2\subset C^2(\mathbf{R}^1,\mathbf{R}^1)$ are arbitrary functions, C is a real non-null constant.

3 Conditional symmetry of the multi-dimensional nonlinear wave equation

As pointed down in the Introduction, substitution of the Ansatz (3), where $f(u),\ g(u)$ are solutions of Eqs.(1), into the nonlinear wave equation (2) reduces it to ODE

$$
g(u)\varphi''(u)+f(u)\varphi'(u)=F_0(\varphi).
\tag{21}
$$

Let us show that the class of Ansätze obtained in this way is substantially wider than the one constructed by means of the symmetry reduction procedure. Within the framework of the said procedure, to reduce Eq.(2) to ODE, one has to construct Ansätze invariant under the three-parameter subgroups of its symmetry group (see, e.g. [18]–[20]). It is well-known that, provided F_0 is an arbitrary function, the maximal symmetry group admitted by PDE (2) is the ten-parameter Poincaré group $P(1,3)$ having the generators

$$P_\mu = g_{\mu\nu}\partial^\nu, \quad J_{\mu\nu} = x_\mu P_\nu - x_\nu P_\mu. \tag{22}$$

Furthermore, the general form of the said Ansätze is given by formula (3), where $u(x)$ is an invariant of some three-parameter subgroup of the group $P(1,3)$. An exhaustive description of invariants of the Poincaré group having generators (22) is given in [21]. In particular, it is established that any invariant of a three-parameter subgroup of the group $P(1,3)$ can be reduced by an appropriate transformation from the Poincaré group either to the form (13)–(16) or to the form

$$x_0 + x_3, \quad x_1 + \theta \ln(x_0 + x_3), \quad x_1 + \theta(x_0 + x_3)^2, \quad x_1^2 + x_2^2, \quad x_1^2 + x_2^2 + x_3^2,$$

where θ is a constant.

But the invariants listed above are very special cases of formulae (17)–(19) which in their turn determine only particular solutions of the d'Alembert-Hamilton system.

Such a substantial extension of the class of Ansätze reducing the nonlinear wave equation is achieved at the expense of its conditional symmetry.

Consider, as an illustration, the Ansatz

$$w(x) = \varphi\Big(x_1 + \rho(x_0 + x_3)\Big), \tag{23}$$

where ρ is an arbitrary smooth function, obtained by substitution of the first formula from (17) with $W_1 = 0$, $W_2 = \rho$ into (3).

Despite of the fact that the Ansatz (23) is not Poincaré-invariant, it reduces PDE (2) to the ODE $\varphi'' = -F_0(\varphi)$. This phenomena can not be in principle understood in the framework of the classical Lie approach, because existence of such Ansätze is a consequence of conditional invariance of the nonlinear wave equation.

Really, the manifold (23) is invariant under the three-parameter Abelian Lie group with the generators

$$Q_1 = \partial_0 - \partial_3, \quad Q_2 = \partial_0 + \partial_3 - 2\rho'\partial_1, \quad Q_3 = \partial_2$$

(this fact is established by direct computation). Obviously, the operator Q_2 can not be represented as a linear combination of the operators P_μ, $J_{\mu\nu}$ with constant coefficients which means that equation (2) is not invariant under the Lie algebra $A = \langle Q_1, Q_2, Q_2 \rangle$.

We will prove that PDE (2) is conditionally-invariant under the algebra A. Acting by the second prolongations of the operators Q_a on (2), we have

$$\underset{2}{Q_1} L = 0, \quad \underset{2}{Q_2} L = 4\rho''\partial_1 Q_1 w, \quad \underset{2}{Q_3} L = 0,$$

where $L = \Box w - F_0(w)$.

Hence, it follows that the system of PDEs

$$\Box w = F_0(w), \quad Q_a w = 0, \quad a = 1, 2, 3$$

is invariant under the Lie algebra A, the same as what was to be proved.

All Ansätze obtained by substitution of the formulae for $u(x)$ listed in (17)–(20) (with the only exception of the last formula from (19)) into (3) correspond to conditional invariance of the nonlinear wave equation and give rise to the new (non-Lie) reductions of PDE (2). Hence it follows, in particular, that the nonlinear d'Alembert equation admits an *infinite* conditional symmetry.

4 Some generalizations

An expression

$$w(x) = \varphi(\omega_1, \omega_2), \tag{24}$$

where $\omega_i = \omega_i(x) \in C^2(\mathbf{R}^4, \mathbf{R}^1)$ are supposed to be functionally-independent, is a natural generalization of the Ansatz (3). The functions $\omega_1(x),\ \omega_2(x)$ are determined by the requirement that the substitution of (24) into (2) yields a two-dimensional PDE for a function $\varphi = \varphi(\omega_1, \omega_2)$. As a result, we obtain the overdetermined system of PDEs [22]

$$\begin{aligned} &\Box\omega_1 = f_1(\omega_1, \omega_2), \quad \Box\omega_2 = f_2(\omega_1, \omega_2), \quad \omega_{1x_\mu}\omega_{1x^\mu} = g_1(\omega_1, \omega_2), \\ &\omega_{2x_\mu}\omega_{2x^\mu} = g_2(\omega_1, \omega_2), \quad \omega_{1x_\mu}\omega_{2x^\mu} = g_3(\omega_1, \omega_2), \\ &\operatorname{rank} \left\| \frac{\partial \omega_i}{\partial x_\mu} \right\|_{i=1}^{2}\,{}_{\mu=0}^{3} = 2 \end{aligned} \tag{25}$$

and besides, the function $\varphi(\omega_1, \omega_2)$ satisfies a two-dimensional PDE

$$g_1\varphi_{\omega_1\omega_1} + g_2\varphi_{\omega_2\omega_2} + 2g_3\varphi_{\omega_1\omega_2} + f_1\varphi_{\omega_1} + f_2\varphi_{\omega_2} = F(\varphi). \tag{26}$$

Consider the following problem: to describe all smooth real functions $\omega_1(x),\ \omega_2(x)$ such that the Ansatz (24) reduces Eq.(2) to ODE with respect to the variable ω_1. It means that one has to put coefficients $g_2,\ g_3,\ f_2$ in (26) equal to zero. In other words, it is necessary to construct a general solution of the system of nonlinear PDEs

$$\begin{aligned} &\Box\omega_1 = f_1(\omega_1, \omega_2), \quad \omega_{1x_\mu}\omega_{1x^\mu} = g_1(\omega_1, \omega_2), \\ &\omega_{1x_\mu}\omega_{2x^\mu} = 0, \quad \omega_{2x^\mu}\omega_{2x_\mu} = 0, \quad \Box\omega_2 = 0. \end{aligned} \tag{27}$$

With an appropriate choice of a function $G(\omega_1, \omega_2)$, the change of variables

$$v = G(\omega_1, \omega_2), \quad u = \omega_2$$

reduces system (27) to the form

$$\Box v = f(u, v), \quad v_{x_\mu}v_{x^\mu} = \lambda, \tag{28}$$

$$u_{x_\mu}v_{x^\mu} = 0, \quad u_{x_\mu}u_{x^\mu} = 0, \quad \Box u = 0, \tag{29}$$

$$\operatorname{rank} \left\| \begin{matrix} v_{x_0} v_{x_1} v_{x_2} v_{x_3} \\ u_{x_0} u_{x_1} u_{x_2} u_{x_3} \end{matrix} \right\| = 2, \tag{30}$$

where λ is a real parameter taking the values $-1, 0, 1$.

Theorem 7 *Eqs.(28)–(30) are compatible if and only if*

$$\lambda = -1, \quad f = -N\Big(v + h(u)\Big)^{-1}, \tag{31}$$

where $h \in C^1(\mathbf{R}^1, \mathbf{R}^1)$ *is an arbitrary function,* $N = 0, 1, 2, 3$.

Theorem 8 *The general solution of the system of Eqs.(28)–(30) being determined within a transformation from the group P(1,3) is given by the following formulae:*

a) under $f = -3\Big(v + h(u)\Big)^{-1}, \quad \lambda = -1$

$$\begin{aligned} &\Big(v + h(u)\Big)^2 = (-A'_\nu A'^\nu)^{-1}(A'_\mu x^\mu + B')^2 + (-A'_\nu A'^\nu)^{-3} \times \\ &\qquad\qquad (\varepsilon^{\mu\nu\alpha\beta} A_\mu A'_\nu A''_\alpha x_\beta + C)^2, \\ &A_\mu x^\mu + B = 0; \end{aligned} \tag{32}$$

b) under $f = -2\Big(v + h(u)\Big)^{-1}, \quad \lambda = -1$

$$\begin{aligned} &\Big(v + h(u)\Big)^2 = (-A'_\nu A'^\nu)^{-1}(A'_\mu x^\mu + B')^2, \\ &A_\mu x^\mu + B = 0, \end{aligned} \tag{33}$$

where $A_\mu = A_\mu(u),\ B = B(u),\ C = C(u)$ *are arbitrary smooth functions satisfying the relations*

$$A_\mu A^\mu = 0, \quad A'_\mu A'^\mu \neq 0, \tag{34}$$

c) under $f = -\Big(v + h(u)\Big)^{-1}, \quad \lambda = -1$

$$\begin{aligned} &\Big(v + h(x_0 - x_3)\Big)^2 = \Big(x_1 + C_1(x_0 - x_3)\Big)^2 + \Big(x_2 + C_2(x_0 - x_3)\Big)^2, \\ &u = C_0(x_0 - x_3), \end{aligned} \tag{35}$$

where $C_0,\ C_1,\ C_2$ *are arbitrary smooth functions;*

d) under $f = 0,\ \lambda = -1$

$$\begin{aligned} 1)\quad &v = (-A'_\nu A'^\nu)^{-3/2} \varepsilon^{\mu\nu\alpha\beta} A_\mu A'_\nu A''_\alpha x_\beta + C, \\ &A_\mu x^\mu + B = 0, \end{aligned} \tag{36}$$

where $A_\mu = A_\mu(u),\ B = B(u),\ C = C(u)$ *are arbitrary smooth functions satisfying the relations (34);*

$$\begin{aligned} 2)\quad &v = x_1 \cos\Big(C_1(x_0 - x_3)\Big) + x_2 \sin\Big(C_1(x_0 - x_3)\Big) + C_2(x_0 - x_3), \\ &u = C_0(x_0 - x_3), \end{aligned} \tag{37}$$

where $C_0,\ C_1,\ C_2$ *are arbitrary smooth functions.*

In the above formulae (32), (36), we denote by $\varepsilon_{\mu\nu\alpha\beta}$ *the completely anti-symmetric fourth-order tensor (the Levi-Civita tensor), i.e.,*

$$\varepsilon_{\mu\nu\alpha\beta} = \begin{cases} \ \ 1, & (\mu,\nu,\alpha,\beta) = \mathrm{cycle}\,(0,1,2,3), \\ -1, & (\mu,\nu,\alpha,\beta) = \mathrm{cycle}\,(1,0,2,3), \\ \ \ 0, & \textit{in the remaining cases,} \end{cases}$$

prime denotes differentiation with respect to u.

Substitution of the results obtained above into formula (24) yields the following collection of Ansätze for the nonlinear d'Alembert equation (2):

$$\begin{aligned}
&1)\quad w(x) = \varphi\Big(\Big[\Big(-A'_\nu(u)A'^\nu(u)\Big)^{-1}\Big(A'_\mu(u)x^\mu + B'(u)\Big)^2 + \\
&\qquad \Big(-A'_\nu(u)A'^\nu(u)\Big)^{-3}\Big(\varepsilon^{\mu\nu\alpha\beta}A_\mu(u)A'_\nu(u)A''_\alpha(u)x_\beta + C(u)\Big)^2\Big]^{1/2},\ u\Big); \\
&2)\quad w(x) = \varphi\Big(\Big(-A'_\nu(u)A'^\nu(u)\Big)^{1/2}\Big(A'_\mu(u)x^\mu + B'(u)\Big),\ u\Big); \\
&3)\quad w(x) = \varphi\Big(\Big[\Big(x_1 + C_1(x_0 - x_3)\Big)^2 + \Big(x_2 + C_2(x_0 - x_3)\Big)^2\Big]^{1/2}, x_0 - x_3\Big); \\
&4)\quad w(x) = \varphi\Big(\Big(-A'_\nu(u)A'^\nu(u)\Big)^{-3/2}\Big(\varepsilon^{\mu\nu\alpha\beta}A_\mu(u)A'_\nu(u)A''_\alpha(u)x_\beta + C(u)\Big),\ u\Big); \\
&5)\quad w(x) = \varphi\Big(x_1\cos C_1(x_0 - x_3) + x_2\sin C_1(x_0 - x_3) + C_2(x_0 - x_3), x_0 - x_3\Big).
\end{aligned} \tag{38}$$

Here $B,\ C,\ C_1,\ C_2$ are arbitrary smooth functions of the corresponding arguments, $A_\mu(u)$ are arbitrary smooth functions satisfying the condition $A_\mu A^\mu = 0$ and the function $u = u(x)$ is determined by the second formula from (32).

Substitution of expressions (38) into (24) gives the following equations for $\varphi = \varphi(u, v)$:

$$\begin{aligned}
&1)\quad \varphi_{vv} + \frac{3}{v}\varphi_v = -F(\varphi), \\
&2)\quad \varphi_{vv} + \frac{2}{v}\varphi_v = -F(\varphi), \\
&3)\quad \varphi_{vv} + \frac{1}{v}\varphi_v = -F(\varphi), \\
&4)\quad \varphi_{vv} = -F(\varphi), \\
&5)\quad \varphi_{vv} = -F(\varphi).
\end{aligned}$$

All Ansätze listed in (38) correspond to conditional symmetry of the nonlinear wave equation (2). It means that for each Ansatz from (38), there exist two differential operators $Q_a = \xi_{a\mu}(x)\partial_\mu$, $a = 1, 2$ such that

$$Q_a w(x) \equiv Q_a\varphi(\omega_1, \omega_2) = 0, \quad a = 1, 2$$

and besides, the system of PDEs

$$\begin{cases} \Box w - F(w) = 0, \\ Q_a w = 0, \quad a = 1, 2 \end{cases}$$

is invariant in Lie's sense under the one-parameter groups with the generators $Q_1,\ Q_2$. For example, the fourth Ansatz from (17) is invariant under the operators: $Q_1 = A_\mu(u)\partial_\mu$, $Q_2 = A'_\mu(u)\partial_\mu$. A direct computation shows that the following relations hold:

$$\underset{2}{Q_i}(\Box\omega) = -(A'^{\alpha}x_\alpha + B')^{-1}A_\mu\partial_\mu Q_i w, \quad i = 1, 2,$$
$$[Q_1,\, Q_2] = 0,$$

where $\underset{2}{Q_i}$ stands for the second prolongation of the operator Q_i. Hence it follows that the nonlinear wave equation (2) is conditionally-invariant under the two-dimensional commutative Lie algebra having the basis elements $Q_1,\ Q_2$.

Below we give new exact solutions of the nonlinear wave equation (2) obtained with the use of the technique described above. We adduce only those ones that can be written down explicitly [22]

1. $F(w) = \lambda w^3$

1) $w(x) = \dfrac{1}{a\sqrt{2}}(x_1^2 + x_2^2 + x_3^2 - x_0^2)^{-1/2}\tan\Big\{-\dfrac{\sqrt{2}}{4}\ln\Big(C(u)\times$
$(x_1^2 + x_2^2 + x_3^2 - x_0^2)\Big)\Big\}$,
where $\lambda = -2a^2 < 0$,

2) $w(x) = \dfrac{2\sqrt{2}}{a}C(u)\Big(1 \pm C^2(u)(x_1^2 + x_2^2 + x_3^2 - x_0^2)\Big)^{-1}$,
where $\lambda = \pm a^2$;

2. $F(w) = \lambda w^5$

1) $w(x) = a^{-1}(x_1^2 + x_2^2 - x_0^2)^{-1/4}\Big\{\sin\ln\Big(C(u)(x_1^2 + x_2^2 - x_0^2)^{-1/2}\Big) + 1\Big\}^{1/2} \times$
$\Big\{2\sin\ln\Big(C(u)(x_1^2 + x_2^2 - x_0^2)^{-1/2}\Big) - 4\Big\}^{-1/2}$,
where $\lambda = a^4 > 0$,

2) $w(x) = \dfrac{3^{1/4}}{\sqrt{a}}C(u)\Big(1 \pm C^4(u)(x_1^2 + x_2^2 - x_0^2)\Big)^{-1/2}$,
where $\lambda = \pm a^2$.

In the above formulae, $C(u)$ is an arbitrary, twice continuously differentiable function on

$$u(x) = \frac{x_0x_1 \pm x_2\sqrt{x_1^2 + x_2^2 - x_0^2}}{x_1^2 + x_2^2},$$

$a \neq 0$ is an arbitrary real parameter.

5 Conclusion

The present paper demonstrates once more that possibilities to construct in explicit form new exact solutions of the nonlinear wave equation (2) (as compared with those obtainable by the standard symmetry reduction technique [20, 23]) are far from being exhausted. A source of new (non-Lie) reductions is the conditional symmetry of Eq.(2).

Roughly speaking, a principal idea of the method of conditional symmetries is the following: to be able to reduce a given PDE, it is enough to require an invariance of a *subset* of its solutions with respect to some Lie transformation group. And what is more, this subset is not obliged to coincide with the whole set. This specific subsets can be chosen in different ways: one can fix in some way an Ansatz for a solution to be found (the method of Ansätze [4, 24] or the direct reduction method [25]) or one can impose an additional differential constraint (the method of Q-conditional [1, 2, 26] or non-classical symmetries [27, 28]). But all the above approaches have a common feature: to find a new (non-Lie) reduction of a given PDE, one has to solve some nonlinear overdetermined system of differential equations. For example, to describe Ansätze of the form (3), (24) reducing Eq.(2) to ODEs, one has to integrate systems of nonlinear PDEs (1), (28)–(30). This is a "price" to be paid for new possibilities to reduce a given nonlinear PDE to equations with a less number of independent variables and to construct its explicit solutions.

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