

Functional Algebras and Dimensional Reduction in the LPDEs Integration Problem

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Abstract

The paper is devoted to application of the noncommutative integration method for linear partial differential equations. This method is based on the noncommutative integration theory for finite-dimensional Hamiltonian systems and is generalized for so-called functional algebras.

Introduction

In Ref. [1] a noncommutative integration method (NIM) has been developed for linear partial differential equations (LPDEs). The method is based on the noncommutative integration theory for finite-dimensional Hamiltonian systems (see Refs. [2, 3] and literature cited there for more details) and notion of a λ -representation of Lie algebras introduced in Ref. [1].

Integration of a given equation,

$$H(x, \partial_x)\psi(x) = 0, \quad x \in R^m, \quad (1)$$

is handled using the noncommutative set L of operators satisfied special conditions.

We use basic notions and notations of Ref. [1]. Remember that by integrability of Eq. (1) we mean the construction of a parametric family of solutions $\psi(x, \lambda)$, $\lambda \in \Lambda \subset C^{m-1}$, where Λ is a domain in the parameters space C^{m-1} , $x \in Q^m \subset R^m$, Q^m is a domain in the space R^m , $\psi \in C^\infty(Q^m \times \Lambda)$. All considered functions are taken to be smooth due to locality of our constructions, in particular, $\psi \in C^\infty(Q \times \Lambda)$. The local completeness of the solutions family $\psi(x, \lambda)$ is equivalent to availability of $(m-1)$ essential parameters λ in the solution $\psi(x, \lambda)$, by definition. The case, when L is a Lie algebra of symmetry operators of the equation under consideration constrained by the condition

$$\dim L + \text{ind } L = 2(m-1), \quad (2)$$

is considered in Ref. [1]. Here $\dim L$, $\text{ind } L$ are a dimension and index of the algebra L , respectively, m is the number of independent variables in (1). If L is a Lie algebra of a general type, Eq. (1) can be reduced to the equation

$$\tilde{H}(u, \partial_u, \lambda)\tilde{\psi}(u, \lambda) = 0 \quad (3)$$

with a lower number of independent variables $u_\alpha, \alpha = 1, \dots, m'$,

$$m' = m - \frac{1}{2}(\dim L + \text{ind } L).$$

A principal possibility of the basis construction in the solutions space has been shown in Ref. [1] for the Klein-Gordon equation in a curved space when this equation is not integrated by the method of separation of variables.

The aim of this paper is to apply NIM for generalized algebraic constructions, so-called functional algebras (or, in shorthand form, F -algebras), for which commutators of generators are, in general, nonlinear functions of generators (linear functions correspond to a Lie algebra). It happens to be possible because notions, introduced for Lie algebras and required for integrability, are inherited by F -algebras. The special case of F -algebras are quadratic algebras [4, 5]. They have found a wide application in quantum field theory and used for integration of ordinary differential equations [6–8]. In the present paper quadratic algebras are applied for integration of LPDEs.

Symmetry F -algebras in the noncommutative integration method

Consider a space F of linear differential operators acting in the functional space $C^\infty(Q^m)$. Functionally independent operators Y_i , $i, j, l, r = 1, \dots, k = \dim F$ form a basis of the space F and satisfy the following commutation relations:

$$[Y_i, Y_j] = c_{ij}(Y), \quad (4)$$

where c_{ij} are symmetrical functions of Y_i . By definition, functions c_{ij} satisfy the Jacobi identity:

$$[Y_l, c_{ij}(Y)] + [Y_i, c_{jl}(Y)] + [Y_j, c_{li}(Y)] = 0. \quad (5)$$

We name such a space F with the operation (4) the F -algebra. This notion is similarly to Poisson algebras which are used in the theory [9] of finite-dimensional Hamiltonian systems. Linear functions $c_{ij}(Y) = c_{ij}^l Y_l$, $c_{ij}^l = \text{const}$ correspond to a Lie algebra, if $c_{ij}(Y)$ are quadratic in Y_i then F -algebra is referred to as quadratic algebra [4–8]. Noncommutative integration of Eq.(1) using F -algebras presents additional difficulties in comparison with the Lie algebras case. It is connected with solution of some auxiliary equations of high orders. This problem becomes easier for quadratic algebras. By this reason these last are of special interest in NIM.

Let $U(F)$ be a linear space with the basis

$$X_{\alpha_1 \dots \alpha_r} = \frac{1}{r!} \sum_{\sigma \in S_r} Y_{\alpha_{\sigma(1)}} \dots Y_{\alpha_{\sigma(r)}}, \quad X_{\alpha_1 \dots \alpha_r}|_{r=0} = 1.$$

Here $r = 0, 1, \dots$; $\alpha_r = 1, \dots, k$. Summation goes with respect to all permutations σ of the permutation group S_r of the set $(1, \dots, r)$. Clearly, the $U(F)$ is an associative algebra relative to standard multiplication of linear differential operators.

Definition 1. $U(F)$ is called the envelope algebra of F -algebra.

Let us also introduce a dual space $U^*(F)$. Its elements are linear functionals $\xi : U(F) \rightarrow C^1$,

$$\xi(X_{\alpha_1 \dots \alpha_r}) = \xi\left(\frac{1}{r!} \sum_{\sigma} Y_{\alpha_{\sigma(1)}} \dots Y_{\alpha_{\sigma(r)}}\right) = \frac{1}{r!} \sum_{\sigma} \xi(Y_{\alpha_{\sigma(1)}}) \dots \xi(Y_{\alpha_{\sigma(r)}}) = \xi_{\alpha_1} \dots \xi_{\alpha_r},$$

where $\xi_{\alpha_l} = \xi(Y_{\alpha_l})$.

It is easy to redefine basic notions of the general theory of Lie algebras for F -algebras.

The mapping $ad : F \rightarrow F$ is:

$$adX(Y) = [X, Y], \quad X, Y \in F.$$

Coadjoint representation ad^* is given by the equation:

$$ad_X^* \xi(Y) = \xi([X, Y]), \quad \forall \xi \in F^*, \quad X, Y \in F.$$

An annulet of the covector ξ is defined as follows:

$$\text{Ann } \xi = \{X \in F \mid ad_X^* \xi = 0\}.$$

By definition, the covector ξ is a generic position one, if the dimension of its annulet is minimal. Index of F -algebra ($\text{ind } F$) is an annulet dimension of a generic-position covector:

$$\text{ind } F = \inf_{\xi \in F^*} (\dim \text{Ann } \xi).$$

Since $\dim \text{Ann } \xi = \dim F - \text{rank}(\xi(c_{ij}(Y)))$, then

$$\text{ind } F = \dim F - \sup_{\xi \in F^*} \text{rank}(\xi(c_{ij}(Y))).$$

To build up an irreducible representation of F -algebra we need a set of differential operators $l(\lambda, \partial_\lambda, J)$ acting in the space $C^\infty(Q^m \times \Lambda)$ such that $[l_i, l_j] = -c_{ij}(l)$ with the same functions $c_{ij}(l)$ that are in (4). Here $\lambda \in C^s$, $s = \frac{1}{2}(\dim F - \text{ind } F)$, $J \in C^r$, $r = \text{ind } F$.

All Casimir operators, i.e. elements of center $Z(U(F))$ of the algebra $U(F)$, are functions of the variables J_1, \dots, J_r in this representation.

Definition 2. Representation of F -algebra by operators l_i is said to be λ -representation of this algebra.

The basis of a representation space V_J of F -algebra is determined by the following system:

$$Y_i(x, \partial_x) \psi_J(x, \lambda) = l_i(\lambda, \partial_\lambda; J) \psi_J(x, \lambda). \quad (6)$$

The parameters J enumerate the representation, and the parameters λ do basis vectors of the representation space V_J . The compatibility of the system (6) and also irreducibility and exactness of the F -algebra representation are due to the construction λ -representation. In particular, the function $\psi_J(x, \lambda)$ is an eigenfunction of the Casimir operators:

$$K_p(Y) \psi_J(x, \lambda) \equiv \omega_p(J) \psi_J(x, \lambda), \quad p = 1, \dots, r.$$

Here $\omega_p(J)$ are mutually independent functions of the parameters J_1, \dots, J_r .

To build up the λ -representation more effectively, it is necessary to classify F -algebras into semi-simple, solvable, nilpotent, etc. For this purpose we define the ideal N of F -algebra such that

$$[N, F] \subset U(N) \leftrightarrow [U(N), F] \subset U(N).$$

The sequence of ideals $F^{(n)}$ is introduced by analogy with Lie algebras: $F^{(0)} = F$, $F^{(n)}$ is a minimal subspace of the space $F^{(n-1)}$ such that $[F^{(n-1)}, F^{(n-1)}] \subset U(F^{(n)})$. If $F^{(n)} = 0$

for some n , then we call such F -algebra solvable. Obviously, every solvable F -algebra contains a commutative ideal (if $F^{(n)} = 0$, then $F^{(n-1)}$ is the commutative ideal). An F -algebra having no commutative ideals is called semi-simple.

Similar to the Lie-algebras theory, an analogue of the Levy-Maltsev theorem can be formulated as follows: F -algebra is expanded into the semi-direct sum of solvable (R) and semi-simple (S) algebras,

$$F = R \ltimes S; \quad [S, S] \subset U(S), \quad [R, R] \subset U(R), \quad [R, S] \subset U(R).$$

Let us introduce one more sequence of ideals $F_{(n)}$: $F_{(0)} = F$, $F_{(n+1)}$ is the minimal subspace $F_{(n)}$ such that $[F, F_{(n)}] \subset U(F_{(n+1)})$. If $F_{(n)} = 0$ for some n , then we call such an F -algebra nilpotent. Since $F^{(n)} \subset F_{(n)}$, every nilpotent F -algebra is solvable. The opposite is not true. From the definition, it follows that every nilpotent F -algebra has a nontrivial center (if $F_{(n)} = 0$, then $F_{(n-1)} = Z(F)$ is the center).

The Casimir operators of semi-simple F -algebras are polynomials in operators X_α . For solvable F -algebras, the Casimir operators usually have a finite order, and to search for them would be a separate problem.

Let us turn to the integration problem of a scalar equation. An analogue of the noncommutative integration theorem is formulated in the following way: the Eq. (1) allowing a symmetry F -algebra such that

$$\dim F + \text{ind } F = 2(m - 1), \quad (7)$$

is integrable in the noncommutative sense.

An integration algorithm for the Eq.(1) using symmetry F -algebra of operators Y_i is similar to one considered in Ref. [1], when operators Y_i form a Lie algebra.

The F -algebras formed by three operators Y_i of the first order and one operator X of the second order are considered in this paper. Commutative relations for such algebras become

$$\begin{aligned} [Y_p, Y_q] &= c_{ij}^l Y_l, \\ [Y_p, X] &= A_p X + c_p^{ij} Y_i Y_j + c_p^i Y_i + c_p. \end{aligned}$$

Here $c_{pq}^i, A_p, c_p^{ij} = c_p^{ji}, c_p^i, c_p$ are structure constants of a square F -algebra and they satisfy Jacobi identity. All indices run from 1 to 3. All the structure constants are tensors with respect to basis transformation in the Lie algebra generated by operators $\{Y_1, Y_2, Y_3\}$. Also allowable is the transformation

$$X \rightarrow aX + b^{pq} Y_p Y_q + c^p Y_p + c, \quad a \neq 0.$$

Classification of four- and five-dimensional square-algebras of the above structure is presented in the Appendix. They admit to integrate differential equations with four independent variables. Due to eq.(7) these F -algebras are subject to the condition:

$$\dim F + \text{ind } F = 6. \quad (8)$$

Moreover, F -algebras including the commutative three of operators not higher than the second order are not presented in Appendix because equations possessing such threes of operators are integrated by the separation of variables method.

Consider the case 2.e) of the four-dimensional square-algebra from the Appendix as a nontrivial example. This square-algebra is solvable, $\{X, Y_2\}$ is the commutative ideal. Assume that Y_i are differential operators of the first order and X is of the second order one with four independent variables. Let us find the metric tensor of a non-Stackel type such that

$$[\Delta, X] = [\Delta, Y_i] = 0. \quad (9)$$

Here $\Delta = \sqrt{g}\partial_{x^\alpha}(g^{\alpha\beta}/\sqrt{g})\partial_{x^\beta}$, $g \equiv \det(g^{\alpha\beta})$. The conditions (8), (9) allow us to integrate the Klein-Gordon equation in a Riemannian space with the metric tensor $g^{\alpha\beta}$:

$$H\varphi(x) = (\Delta - \varepsilon)\varphi(x) = 0. \quad (10)$$

But, the non-Stackel-type metrics exists not for any square-algebra. In particular, in our case (2.e) of the Appendix) we must put $b = 0$.

Operators Y_i, X and non-zero components of the metrics $g^{\alpha\beta}$ are of the form

$$Y_1 = \partial_2, \quad Y_2 = \partial_3, \quad Y_3 = -\partial_1 + x^2\partial_2;$$

$$\begin{aligned} X = \exp(-x^1) \Big\{ & (4/(x^4)^2)\partial_{11} + 2(\alpha + \beta/(x^4)^2)\partial_{13} + (4/x^4)\partial_{14} + \\ & \left[\frac{1}{4}(2\alpha\beta + \beta^2/(x^4)^2) + x^2 \exp(x^1) \right] \partial_{33} + (\alpha x^4 + \beta/x^4)\partial_{34} + \\ & \partial_{44} - 4/(x^4)^2\partial_1 - \beta/(x^4)^2\partial_3 - 1/x^4\partial_4 \Big\}; \end{aligned}$$

$$g^{11} = -4/(x^4)^2, \quad g^{13} = \alpha - \beta/(x^4)^2, \quad g^{23} = -\alpha^3(x^4)^2 \exp(-x^1)/2,$$

$$g^{33} = \gamma - \alpha^2(x^4)^2/2 - \beta^2/(4x^4)^2, \quad g^{44} = 1.$$

Here, α, β, γ are arbitrary constants; $\partial_i \equiv \partial/\partial x^i$, $\partial_{ij} \equiv \partial^2/\partial x^i \partial x^j$.

Since operators Y_1, Y_2 form a commutative ideal, the functions of one variable λ only and two parameters J_1, J_2 correspond to them in the λ -representation. The remaining operators are found easily from commutation relations. Finally, we obtain:

$$l_1 = e^\lambda(\partial_\lambda + J), \quad l_2 = \omega, \quad l_3 = \partial_\lambda, \quad L = \omega^2 e^{-\lambda}.$$

Here, the redesignations: $\lambda_1 = \lambda, J_1 = J, J_2 = \omega$ are made for convenience. Operator L corresponds to operator X . The first three equations of system (6) give:

$$\varphi_{J\omega}(x, \lambda) = \exp[J(x^1 - \lambda) + \omega x^3] \psi_{J\omega}(u, x^4),$$

where $u \equiv e^{x^1}(x^2 - e^{-\lambda})$. Substituting the function $\varphi_{J\omega}(x, \lambda)$ into the remaining equation of system (6) and into (10) we obtain two equations of the second order with two independent variables for the function $\psi_{J\omega}(u, x^4)$:

$$\tilde{X}(u, x^4, \partial_u, \partial_{x^4})\psi_{J\omega}(u, x^4) = 0, \quad (11)$$

$$\tilde{\Delta}(u, x^4, \partial_u, \partial_{x^4})\psi_{J\omega}(u, x^4) = \varepsilon\psi_{J\omega}(u, x^4). \quad (12)$$

The Eqs. (11), (12) are compatible since operator $\tilde{\Delta}$ is a symmetry operator for the operator \tilde{X} : $[\tilde{\Delta}, \tilde{X}] = R\tilde{X}$; here R is some operator.

Eqs. (11), (12) cannot be solved by the separation of variables method because the conditions of the appropriate theorem (see Ref. [10]) are not fulfilled. In particular, there is no function f such that $[\tilde{\Delta}, f\tilde{X}] = 0$. In our example the system (6) possesses an invariant, i.e., the function ξ that $[Y_i - l_i, \xi] = [X - L, \xi] = 0$. Since $\xi = x^4/u$, then $[\tilde{X}, \xi] = 0$. Consider the first-order operator $Y = [\tilde{\Delta}, \xi]$. From the Jacobi identity we have: $[Y, \tilde{X}] = [R, \xi]\tilde{X}$, i.e. Y is the first-order symmetry operator for Eq. (11). By transformation of variables and the function we bring the operator Y into the diagonal form $Y = \partial/\partial\eta$. As this takes place, the operator X will depend on one variable. Solving Eq. (11) and after that Eq. (12), we finally obtain the Klein-Gordon equation basis of solutions in the following form:

$$\varphi_{J\omega}(x, \lambda) = \exp[J(x^1 - \lambda) + \omega x^3 + S] \cdot \{h_+(\xi) \exp(\sigma_+(\xi)h) + h_-(\xi) \exp(\sigma_-(\xi)h)\},$$

$$\xi = x^4(e^{x^1}(x^2 - e^{-\lambda}))^{-\frac{1}{2}}, \quad \eta = (x^4)^2/\xi, \quad \sigma_{\pm}(\xi) = \omega[-\alpha^3\xi^3 - 4\alpha\xi \mp (\alpha^2\xi^2 - 4)^{\frac{1}{2}}]/8,$$

$$h_{\pm}(\xi) = h_{\pm}^0 \exp \left[\int \frac{C(\xi) \pm D(\xi)}{B(\xi) \pm A(\xi)} d\xi \right], \quad h_{\pm}^0 = \text{const};$$

$$A(\xi) = \alpha\xi(\alpha^2\xi^2 - 4)^{\frac{5}{2}}, \quad B(\xi) = 2(\alpha^2\xi^2 - 4)^2,$$

$$S(\xi, \eta) = -(J + \omega\beta/4) \ln \eta + \omega\alpha\eta\xi(1 + \alpha^2\xi^2/2)/4,$$

$$C(\xi) = (\alpha^2\xi^2 - 4)^{\frac{1}{2}}[\alpha^5\beta\xi^4/4 + \alpha^5\xi^4J - \alpha^3\beta\omega\xi^2 + 2\alpha^3\xi^2 +$$

$$2\alpha^2\gamma\omega\xi^2 - 16\alpha J - 8J - 8\gamma\omega - 2\varepsilon(\alpha^2\xi^2 - 4)/\omega],$$

$$D(\xi) = \alpha^4\beta\omega\xi^3/2 + 2\alpha^4J\xi^3 - 4\alpha^2\beta\omega\xi - 16\alpha^2J\xi - 8\alpha^2\xi + 8\beta\omega/\xi + 32(J + 1)/\xi.$$

In conclusion we note that the obtained solution $\varphi_{J\omega}(x, \lambda)$ is the eigenfunction of the four commutative operators:

$$\begin{aligned} Y_2\varphi &= \omega\varphi, & (Y_3Y_2^2 - Y_1X)\varphi &= -J\omega^2\varphi, \\ X\varphi &= \omega^2e^{-\lambda}\varphi, & \Delta\varphi &= \varepsilon\varphi. \end{aligned}$$

But, since the Casimir operator of the square-algebra under consideration, $Y_3Y_2^2 - Y_1X$, has the third order, the separation of variables is impossible in this system.

Appendix

Four-dimensional square-algebras

$$1. [Y_2, Y_3] = Y_1, \quad (\{Y_p, Y_q\} \equiv \frac{1}{2}(Y_pY_q + Y_qY_p)),$$

- a) $[Y_2, X] = aY_3^2 + Y_2 + b, \quad [Y_3, X] = cY_2^2 - Y_3 + d, \quad a, c \neq 0;$
- b) $[Y_2, X] = 2\{Y_2, Y_3\} + aY_3^2 + b, \quad [Y_3, X] = cY_2^2 - Y_3^2 + dY_2 + e;$
- c) $[Y_2, X] = aY_3^2 + bY_3, \quad [Y_3, X] = cY_2^2 + dY_2.$

2. $[Y_1, Y_3] = Y_1 \quad [Y_3, X] = X,$

- a) $[Y_1, X] = Y_2^2 + aY_3^2 + b, \quad a \neq 0;$ b) $[Y_1, X] = \{Y_2, Y_3\} + b;$
c) $[Y_1, X] = Y_2^2 + aY_3, \quad a \neq 0;$ d) $[Y_1, X] = Y_3^2 + aY_2;$
e) $[Y_1, X] = Y_3^2 + b;$ f) $[Y_1, X] = Y_2^2 + b;$
g) $[Y_1, X] = Y_2;$ h) $[Y_1, X] = Y_3;$
i) $[Y_1, X] = 1.$

Five-dimensional square-algebras

1. $[Y_2, Y_3] = Y_1, [Y_1, Y_4] = \alpha Y_1, [Y_2, Y_4] = Y_2,$
 $[Y_3, Y_4] = (\alpha - 1)Y_3, \quad \alpha \neq 0, 1.$

- a) $[Y_2, X] = aY_3^2 + Y_2, [Y_3, X] = bY_2^2 - Y_3, [Y_4, X] = c$
if $\alpha = \frac{3}{2}$, then $b = 0$, if $\alpha = 3$, then $a = 0;$
b) $[Y_4, X] = -X, [Y_2, X] = aY_2^2 + b\{Y_2, Y_3\} + cY_3^2 + dY_1,$
 $[Y_3, X] = eY_2^2 - 2a\{Y_2, Y_3\} - \frac{1}{2}bY_3^2 + fY_1, \quad \alpha = 2;$
c) $[Y_4, X] = -\frac{1}{2}X, [Y_2, X] = 2a\{Y_2, Y_3\} + bY_1 + cY_3$
 $[Y_3, X] = -aY_3^2 + dY_2, \quad \alpha = \frac{3}{2};$
d) $[Y_4, X] = -X, [Y_2, X] = Y_2^2, [Y_3, X] = -2\{Y_2, Y_3\} + aY_1, \quad \alpha \neq 2, 3;$
e) $[Y_4, X] = -\frac{1}{3}X, [Y_2, X] = Y_3^2, [Y_3, X] = aY_2, \quad a \neq 0, \quad \alpha = \frac{5}{2};$
f) $[Y_4, X] = X, [Y_1, X] = Y_3, [Y_2, X] = +2Y_4, \quad \alpha = \frac{1}{2};$

2. $[Y_1, Y_4] = Y_1, [Y_2, Y_3] = Y_2$

- a) $[Y_2, X] = Y_1, [Y_3, X] = X, [Y_4, X] = -X;$
b) $[Y_2, X] = Y_1^2, [Y_3, X] = X, [Y_4, X] = -2X;$
c) $[Y_1, X] = \{Y_2, Y_4\}, [Y_3, X] = -X, [Y_4, X] = X;$

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