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# Symmetry Properties and Exact Solutions of the Fokker-Planck Equation

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**Abstract**

Symmetry properties of some Fokker-Planck equations are studied. In the one-dimensional case, when symmetry groups turn out to be six-parameter ones, this allows to find changes of variables to reduce such Fokker-Planck equations to the one-dimensional heat equation. The symmetry and the family of exact solutions of the Kramers equation are obtained.

The one-dimensional Fokker-Planck (FP) equation has the form

$$\frac{\partial u}{\partial t} = -\frac{\partial}{\partial x}\Big[A(x,t)u\Big] + \frac{1}{2}\frac{\partial^2}{\partial x^2}\Big[B(x,t)u\Big], \tag{1}$$

where $u = u(x,t)$ is the probability density; $A$ and $B$ are differentiable functions. This is the basic equation in the theory of continuous Markovian processes. The following FP equations are of special interest [1, 2]:

(a) diffusion in a gravitational field

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\Big(gu\Big) + \frac{1}{2}D\frac{\partial^2 u}{\partial x^2}, \tag{2}$$

(b) the Ornstein-Uhlenbeck process

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\Big(kxu\Big) + \frac{1}{2}D\frac{\partial^2 u}{\partial x^2}, \tag{3}$$

(c) the Rayleigh-type process

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left[\left(\gamma x - \frac{\mu}{x}\right)u\right] + \frac{1}{2}\mu\frac{\partial^2 u}{\partial x^2}, \tag{4}$$

(d) models in population genetics [2]

$$\frac{\partial u}{\partial t} = \frac{\alpha}{2}\frac{\partial^2}{\partial x^2}\Big[(x-c)^2 u\Big] + \beta\frac{\partial}{\partial x}\Big[(x-c)u\Big], \tag{5}$$

$$\frac{\partial u}{\partial t} = \frac{\partial^2}{\partial x^2}\Big[\Big(1-x^2\Big)^2 u\Big], \tag{6}$$

$$\frac{\partial u}{\partial t} = \frac{\alpha}{2}\frac{\partial^2}{\partial x^2}\Big[x^2\Big(1-x^2\Big)^2 u\Big], \tag{7}$$

(c) the Rayleigh process

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left[\left(\gamma x - \frac{\mu}{x}\right)u\right] + \mu\frac{\partial^2 u}{\partial x^2}, \tag{8}$$

where $D, g, k, \gamma, \mu, \alpha, \beta, c$ are arbitrary constants.

Using Lie's method [3], one can make sure of that the maximal invariance group of equations (2)–(7) is a six-parameter one. The invariance group of the heat equation has the same dimension. It is to be pointed out that these six-parameter groups are different but they are locally isomorphic. That is why, one can reduce equations (2)–(7) to the heat equation.

**Theorem 1** *The change of variables*

$$u(x,t) = f(x,t)\omega(y(x,t),\tau(x,t)), \tag{9}$$

*where the function $f$ and new independent variables $y$ and $\tau$ are as follows:*

$$f = \exp\left\{-\frac{g}{D}x - \frac{g^2}{2D}t\right\}, \qquad y = x, \qquad \tau = \frac{D}{2}t, \tag{10}$$

$$f = \exp\Big\{kt\Big\}, \qquad y = \exp\Big\{kt\Big\}x, \qquad \tau = \frac{D}{4k}\exp\Big\{2kt\Big\}, \tag{11}$$

$$f = \exp\Big\{2\gamma t\Big\}x, \qquad y = \exp\Big\{\gamma t\Big\}x, \qquad \tau = \frac{\mu}{4\mu}\exp\Big\{2\gamma t\Big\}, \tag{12}$$

$$f = \exp\left\{-\left(\frac{\beta^2}{2\alpha} + \frac{\beta}{2} + \frac{\alpha}{8}\right)t\right\}(x-c)^{-(3/2+\beta/\alpha)}, \quad y = \sqrt{\frac{2}{\alpha}}\ln(x-c), \quad \tau = t, \tag{13}$$

$$f = \exp\Big\{-t\Big\}\left(1-x^2\right)^{-3/2}, \qquad y = \frac{1}{2}\ln\frac{1+x}{1-x}, \qquad \tau = t, \tag{14}$$

$$f = \exp\left\{-\frac{\alpha}{8}t\right\}x^{-3/2}(1-x)^{-3/2} \qquad y = \ln\frac{x}{1-x}, \qquad \tau = \frac{\alpha}{2}t, \tag{15}$$

*reduce equations (2)–(7), correspondingly, to the heat equation*

$$\omega_\tau = \omega_{yy}. \tag{16}$$

The proof can be easily obtained by inspection.

**Remark 1.** One can prove a more general statement. Equation (1) with coefficients

$$A(x,t) = A(x), \qquad B(x,t) = B = \mathrm{const} \tag{17}$$

is reduced to the heat equation if and only if

$$A^2 + B\frac{\partial A}{\partial x} = c_2x^2 + c_1x + c_0, \tag{18}$$

where $c_0, c_1, c_2$ are arbitrary constants. Note that equations (2)–(7) satisfy condition (18) and equation (8) does not. The general solution of equation (18), which is a Riccati one, cannot be obtained in quadratures [4].

**Remark 2.** The FP equation of the form

$$\frac{\partial u}{\partial t} = -\frac{\partial}{\partial x}\Big[(a(t)x + b(t))u\Big] + c(t)\frac{\partial^2 u}{\partial x^2} \tag{19}$$

was consider in [5, 6] and by means of algebraic methods a class of it was obtained. This result can be easily obtained if we note that equation (19) is reduced to the heat equation (16) by the substitution (9) with

$$\begin{aligned}
&f = \exp\Big\{-\int_0^t a(s)ds\Big\},\\
&y = \exp\Big\{-\int_0^t a(s)ds\Big\}x - \int_0^t b(s)\exp\Big\{-\int_0^s a(\xi)d\xi\Big\}ds,\\
&\tau = \int_0^t c(s)\exp\Big\{-2\int_0^s a(\xi)d\xi\Big\}ds.
\end{aligned} \tag{20}$$

Now consider the two-dimensional FP equation which describes the motion of a particle in a fluctuating medium (so-called Brownian movement)

$$\frac{\partial u}{\partial t} = -\frac{\partial}{\partial x}\Big(yu\Big) + \frac{\partial}{\partial y}\Big(V'(x)u\Big) + \gamma\frac{\partial}{\partial y}\left(yu + \frac{\partial u}{\partial y}\right), \tag{21}$$

where $u = u(t, x, y)$, $\gamma$ is a constant and $V(x)$ is the potential (its gradient defines the exterior force). Equation (21) is known as the Kramers equation [1].

**Theorem 2** *The maximal invariance group of the free Kramers equation*

$$\frac{\partial u}{\partial t} = -\frac{\partial}{\partial x}\Big(yu\Big) + \gamma\frac{\partial}{\partial y}\left(yu + \frac{\partial u}{\partial y}\right) \tag{22}$$

*is a six-dimensional Lie group generated by the following operators:*

$$\begin{aligned}
&P_0 = \partial_t, \qquad P_1 = \partial_x, \qquad I,\\
&G_1 = t\partial_x + \partial_y + \tfrac{1}{2}(y + \gamma x),\\
&S_1 = e^{\gamma t}\left(\tfrac{1}{\gamma}\partial_x + \partial_y + y\right), \qquad T_1 = e^{-\gamma t}\left(\tfrac{1}{\gamma}\partial_x - \partial_y\right)
\end{aligned} \tag{23}$$

*which satisfy the commutation relations*

$$\begin{aligned}
&\Big[P_0, G_1\Big] = P_1, \qquad \Big[P_0, S_1\Big] = \gamma S_1,\\
&\Big[P_0, T_1\Big] = -\gamma T_1, \quad \Big[P_1, G_1\Big] = -\tfrac{1}{2}\gamma, \quad \Big[T_1, S_1\Big] = I
\end{aligned} \tag{24}$$

*(the rest of the commutators are equal to zero).*

The proof can be obtained by Lie's method.

**Remark 3.** One can prove a more general statement: the widest symmetry group of equation (21) is achieved when $V'(x) = c_1 x + c$ ($c_1, c$ are arbitrary constants) and it is a six–parameter group.

**Remark 4.** The change of variables

$$u = \omega(\tau, \xi, \eta), \quad \tau = t, \quad \xi = x - \frac{c}{\gamma} t, \quad \eta = y - \frac{c}{\gamma} \tag{25}$$

reduces equation (21) with $V'(x) = c$ to the free Kramers equation (22).

Let us write down the final transformations generated by operators (23). Operators $P_0$ and $P_1$ generate translations on variables $t$ and $x$; $I$ generates the identical transformation; $G$ generates

$$\begin{aligned} &t' = t, \qquad x' = x + at, \qquad y' = y + a, \\ &u'(x') = \exp\left\{-\frac{1}{2}\left[ay + \frac{a^2}{2}(1 + \gamma t) + \gamma a x\right]\right\} u(x); \end{aligned} \tag{26}$$

$S_1$ generates

$$\begin{aligned} &t' = t, \qquad x' = x + \frac{b}{\gamma} e^{\gamma t}, \qquad y' = y + b e^{\gamma t}, \\ &u'(x') = \exp\left\{bye^{\gamma t} - \frac{b^2}{2} e^{2\gamma t}\right\} u(x); \end{aligned} \tag{27}$$

$T_1$ generates

$$t' = t, \qquad x' = x + \frac{\theta}{\gamma} e^{-\gamma t}, \qquad y' = y - \theta e^{-\gamma t}, \quad u'(x') = u(x), \tag{28}$$

where $a, b, \theta$ are the group parameters. It is appropriate to write here the corresponding formulae of generating solutions which follow from (26)–(28) (the general theory is contained in [7]):

$$u_{II}(t, x, y) = \exp\left\{\frac{a}{2}\left[y + \frac{a}{2}(1 + \gamma^t) + \gamma x\right]\right\} u_I(t', x', y'), \tag{29}$$

$$u_{II}(t, x, y) = \exp\left\{-bye^{\gamma t} + \frac{b^2}{2} e^{2\gamma t}\right\} u_I(t', x', y'), \tag{30}$$

$$u_{II}(t, x, y) = u_I(t', x', y'), \tag{31}$$

where $t', x', y'$ are given in (26)–(28), respectively.

It should be noted that transformations (26) are just the Galilean ones as soon as the variable $y$ in the Kramers equation is taken to be velocity of the particle.

A well-known solution of the Kramers equation (21) is the Boltzmann distribution

$$u(x, y) = \mathcal{N} \exp\left\{-v(x) - \frac{1}{2} y^2\right\} \tag{32}$$

($\mathcal{N}$ is a normalization constant). It is stationary solution. Applying this to (32) with $V = 0$, formulae (29)–(31), one can easily obtain a non-stationary solution of equation (22).

According to the algorithm of [8,7] and using the operator from (23), we find the ansatz

$$u(t,x,y) = \exp\left\{-\frac{y^2}{2}\right\}\varphi(\omega_1,\omega_2), \qquad \omega_1 = t, \quad \omega_2 = \gamma x - y. \tag{33}$$

Substitution of (33) into (22) gives rise to the heat equation

$$\frac{\partial\varphi}{\omega_1} - \gamma\frac{\partial^2\varphi}{\partial\omega_2} = 0. \tag{34}$$

The simplest solution of (34) is $\varphi =$ const, but it is the solution that leads, together with the ansatz (33), to the Boltzmann distribution (32). It is clear that by using solutions of the heat equation (34) and the ansatz (33), one can construct many partial solutions of equation (22). For example, the fundamental solution of (34) and (33) results in the following solution of equation (22):

$$u(t,x,y) = \frac{1}{\sqrt{4\pi\gamma t}}\exp\left\{-\frac{y^2}{2} - \frac{(\gamma x - y)^2}{4\gamma t}\right\}. \tag{35}$$

The operators $T_1$ from (23) lead to the ansatz

$$u = \tilde{\varphi}(\omega_1,\omega_2), \qquad \omega_1 = t, \quad \omega_2 = \gamma x + y \tag{36}$$

which reduces (23) to the heat equation (34), where $\varphi = e^{\gamma\omega_1}\tilde{\varphi}(\omega_1,\omega_2)$.

A great number of partial solutions of equation (23) can be found by means of the method described in [9](see also [7]). So, if $u_0 = u_0(t,x,y)$ is a solution of the equation, then the functions $Qu_0, Q^2u_0, \ldots$ will also be solutions with any symmetry operator $Q$.

For example, starting from $u_0 = e^{\gamma t}$, we find

$$u_1 = 2G_1e^{\gamma t} = e^{\gamma t}(\gamma x + y), \quad u_2 = G_1u_1 = e^{\gamma t}\Big[(\gamma t + 1) + \tfrac{1}{2}(\gamma x + y)^2\Big], \ \cdots \tag{37}$$

Analogously, by means of the operators $T_1$ from (23), we find, starting from $u_0 = e^{-y^2/2}$,

$$\begin{aligned} u_1 &= T_1\exp\left\{-\frac{y^2}{2}\right\} = y\exp\left\{-\left(\gamma t + \frac{y^2}{2}\right)\right\}, \\ u_2 &= T_1u_1 = (y^2-1)\exp\left\{-\left(2\gamma t + \frac{y^2}{2}\right)\right\}, \quad \cdots\cdots \end{aligned} \tag{38}$$

Solutions (35), (37) and (38) can be multiplied by the formuale of the generating solutions (29)–(31).

## References


[1] Gardiner C.W., Handbook of Stochastic Methods, Berlin, Springer, 1985.

[2] Kimura M., Diffusion models in population equations, *J. Appl. Prob.,* 1964, N 1, 177.

[3] Ovsyannikov L.V., Group Analysis of Differential Equations, Academic Press, New York, 1982.

[4] Kamke E., Differentialgleichungen. Lösungsmethoden und Lösungen, Teubner, Leipzig, 1959.

[5] Suzuki M., The decomposition formulas of exponential with some applications to quantum mechanics and statistical physics, *J. Math. Phys.,* 1985, V.26, 601.

[6] Wolf F., Lie algebric solutions of linear Fokker–Planck equations, *J. Math. Phys.,* 1988, V.29, 305.

[7] Fushchych W., Shtelen W. and Serov N., Symmetry Analysis and Exact Solutions of Equations of Nonlinear Mathematical Physics, Dordrecht, Kluwer Academic Publishers, 1993.

[8] Fushchych W.I., Algebraic-theoretical studies in mathematical physics, Kiev, Institute of Mathematics, 1981, p.6.

[9] Stelen W.M., Symmetry and exact solutions of nonlinear equations of mathematical physics, Kiev, Institute of Mathematics, 1987, p.31.