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On New Representations of Galilei Groups

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Abstract

We have constructed new realizations of the Galilei group and its natural extensions by Lie vector fields. These realizations together with the ones obtained by Fushchych & Cherniha (*Ukr. Math. J.*, 1989, **41**, N 10, 1161; N 12, 1456) and Rideau & Winternitz (*J. Math. Phys.*, 1993, **34**, 558) give a complete description of inequivalent representations of the Galilei, extended Galilei, and generalized Galilei groups in the class of Lie vector fields with two independent and two dependent variables.

1. Introduction

As is well known, the problem of classification of linear and nonlinear partial differential equations (PDEs) admitting a given Lie transformation group G is closely connected to the one of describing inequivalent representations of its Lie algebra AG in the class of Lie vector fields (LVFs) [1]–[3]. Given a representation of the Lie algebra AG, one can, in principle, construct all PDEs admitting the group G by means of the infinitesimal Lie method [1, 2, 4].

In the present paper we study representations of the Lie algebra of the Galilei group $G(1,1)$ (which will be called in the sequel the Galilei algebra $AG(1,1)$) and its natural extensions in the class of LVFs

$$Q = \xi^1(t,x,u,v)\partial_t + \xi^2(t,x,u,v)\partial_x + \eta^1(t,x,u,v)\partial_u + \eta^2(t,x,u,v)\partial_v, \tag{1}$$

where $t,\ x$ and u, v are considered as independent and dependent variables, correspondingly, and $\xi^1,\ldots,\eta^2$ are some sufficiently smooth real-valued functions.

Representations of the Galilei group with basis generators (1) are realized on the set of solutions of the linear and nonlinear $(1+1)$-dimensional heat, Schrödinger, Hamilton-Jacobi, Burgers and KdV equations to mention only a few PDEs (for more details, see [4]).

We say that operators $P_0,\ P_1,\ M,\ G,\ D,\ A$ of the form (1) realize a representation of the generalized Galilei algebra $AG_2(1,1)$ (called also the Schrödinger algebra $ASch(1,1)$) if

- they are linearly independent,
- they satisfy the following commutation relations:

$$\begin{array}{lll}
[P_0,\ P_1]=0, & [P_0,\ M]=0, & [P_1,\ M]=0, \\
[P_0,\ G]=P_1, & [P_1,\ G]=\frac{1}{2}M, & [P_0,\ D]=2P_0, \\
[P_1,\ D]=P_1, & [P_0,\ A]=D, & [P_1,\ A]=G, \\
[M,\ G]=0, & [M,\ D]=0, & [M,\ A]=0, \\
[M,\ G]=0, & [M,\ D]=0, & [M,\ A]=0, \\
[G,\ D]=-G, & [G,\ A]=0, & [D,\ A]=2A.
\end{array} \tag{2}$$

In the above formulae, $[Q_1,\ Q_2]\equiv Q_1Q_2-Q_2Q_1$ is the commutator.

The subalgebra of the above algebra spanned by the operators P_0, P_1, M, G, is the Galilei algebra. The Lie algebra having the basis elements P_0, P_1, M, G, D is called the extended Galilei algebra $AG_1(1,1)$.

It is straightforward to verify that relations (2) are not altered by an arbitrary invertible transformation of the independent and dependent variables

$$\begin{array}{ll}
t\to t'=f_1(t,x,u,v), & x\to x'=f_2(t,x,u,v), \\
u\to u'=g_1(t,x,u,v), & v\to v'=g_2(t,x,u,v),
\end{array} \tag{3}$$

where $f_1,\ldots,g_2$ are sufficiently smooth functions. Invertible transformations of the form (3) form a group (called diffeomorphism group) which establishes a natural equivalence relation on the set of all possible representations of the algebra $AG(1,1)$. Two representations of the Galilei algebra are called equivalent if the corresponding basis operators can be transformed one into another by a change of variables (3).

In the papers by Fushchych and Cherniha [5, 6] different linear representations of the Galilei group and of its generalizations were used to classify Galilei-invariant nonlinear PDEs in n dimensions with an arbitrary $N\in\mathbf{N}$ (see also [7]). The next paper in this direction was the one by Rideau and Winternitz [8]. It gives a description of inequivalent representations of the algebras $AG(1,1)$, $AG_1(1,1)$, $AG_2(1,1)$ under supposition that commuting operators P_0, P_1, M can be reduced to the form

$$P_0=\partial_t,\quad P_1=\partial_x,\quad M=\partial_u \tag{4}$$

by transformation (3).

The results of [8] can be summarized as follows. The basis elements P_0, P_1, M are given formulae (4) and the remaining basis elements are adduced below

1. Inequivalent representations of the Galilei algebra

$$\begin{array}{ll}
\text{(a)} & G=t\partial_x+\frac{1}{2}x\partial_u+f(v)\partial_t, \\
\text{(b)} & G=t\partial_x+\frac{1}{2}x\partial_u+v\partial_v.
\end{array} \tag{5}$$

2. Inequivalent representations of the extended Galilei algebra

$$\begin{array}{ll}
\text{(a)} & G=t\partial_x+\frac{1}{2}x\partial_u,\quad D=2t\partial_t+x\partial_x+f(v)\partial_u, \\
\text{(b)} & G=t\partial_x+\frac{1}{2}x\partial_u,\quad D=2t\partial_t+x\partial_x-\frac{1}{2}v\partial_v, \\
\text{(c)} & G=t\partial_x+\frac{1}{2}x\partial_u+v\partial_t,\quad D=2t\partial_t+x\partial_x+3v\partial_v, \\
\text{(d)} & G=t\partial_x+\frac{1}{2}x\partial_u+\partial_v,\quad D=2t\partial_t+x\partial_x+\varepsilon\partial_u-v\partial_v.
\end{array} \tag{6}$$

3. Inequivalent representations of the generalized Galilei algebra

$$
\begin{aligned}
&\text{(a)}\ G = t\partial_x + \tfrac{1}{2}x\partial_u, \quad D = 2t\partial_t + x\partial_x + f(v)\partial_u,\\
&\qquad A = t^2\partial_x + tx\partial_x + \left(\tfrac{1}{4}x^2 + f(v)t\right)\partial_u,\\
&\text{(b)}\ G = t\partial_x + \tfrac{1}{2}x\partial_u, \quad D = 2t\partial_t + x\partial_x + 2v\partial_v,\\
&\qquad A = t^2\partial_x + tx\partial_x + \left(\tfrac{1}{4}x^2 + \varepsilon v\right)\partial_u + (2t+\alpha v)v\partial_v,\\
&\text{(c)}\ G = t\partial_x + \tfrac{1}{2}x\partial_u + \partial_v, \quad D = 2t\partial_t + x\partial_x + \varepsilon\partial_u - v\partial_v,\\
&\qquad A = t^2\partial_t + tx\partial_x + \left(\tfrac{1}{4}x^2 + \varepsilon t\right)\partial_u + (x - tv)\partial_v.
\end{aligned}
\tag{7}
$$

Here

$$
f(v) = \begin{cases} \alpha, \\ v, \end{cases}
$$

α is an arbitrary constant and $\varepsilon = 0, 1$.

Remark 1. Representation (7b) with ($\varepsilon = 0,\ \alpha = 0$) were obtained for the first time in [5, 6].

Remark 2. The forms of basis operators of the extended Galilei and generalized Galilei algebras are slightly simplified as compared to those given in [8]. For example, the operators $D,\ A$ from (7c) read as

$$
\begin{aligned}
\tilde{D} &= 2t\partial_t + x\partial_x + \varepsilon\partial_u - \frac{1}{2}(1 + 2\ln\tilde{v})\tilde{v}\partial_{\tilde{v}},\\
\tilde{A} &= t^2\partial_t + tx\partial_x + \left(\frac{1}{4}x^2 + \varepsilon t\right)\partial_u + \left(x - \frac{1}{2}t(1 + 2\ln\tilde{v})\right)\tilde{v}\partial_{\tilde{v}}.
\end{aligned}
$$

It is readily seen that the operators $\{D, A\}$ and $\{\tilde{D}, \tilde{A}\}$ are related to each other by the transformation $v = \ln(\tilde{v}e^{\frac{1}{2}})$.

Generally speaking, basis elements $P_0,\ P_1,\ M$ have not to be reducible to the form (4). The requirement of reducibility imposes an additional constraint on the choice of basis elements of the algebras $AG(1,1)$, $AG_1(1,1)$, $AG_2(1,1)$, thus narrowing the set of all possible inequivalent representations. This is the reason why formulae (5) – (7) give no complete description of representations of the Galilei, extended Galilei, and generalized Galilei algebras. As established in the present paper, there are five more classes of representations of $AG(1,1)$, six more classes of representations of $AG_1(1,1)$ and one new representation of the generalized Galilei algebra $AG_2(1,1)$.

2. Principal results.

Before formulating the principal assertion we prove an auxiliary lemma.

Lemma 1 *Let $P_0,\ P_1,\ M$ be mutually commuting linearly independent operators of the form (1). Then there exists transformation (3) reducing these operators to one of the forms*

$$
P_0 = \partial_t, \quad P_1 = \partial_x, \quad M = \partial_u; \tag{8}
$$

$$P_0 = \partial_t, \quad P_1 = \partial_x, \quad M = \alpha(u,v)\partial_t + \beta(u,v)\partial_x; \tag{9}$$
$$P_0 = \partial_t, \quad P_1 = x\partial_t, \quad M = 2\partial_u; \tag{10}$$
$$P_0 = \partial_t, \quad P_1 = x\partial_t, \quad M = \gamma(x)\partial_t; \tag{11}$$
$$P_0 = \partial_t, \quad P_1 = x\partial_t, \quad M = 2u\partial_t, \tag{12}$$

where $\alpha,\ \beta,\ \gamma$ *are arbitrary smooth functions of the corresponding arguments.*

Proof. Let R be a 2×4 matrix whose entries are coefficients of the operators $P_0,\ P_1$.

Case 1. $\operatorname{rank} R = 2$. It is a common knowledge that any nonzero operator Q of the form (1) having smooth coefficients can be transformed by the change of variables (3) to become $Q' = \partial_{t'}$ (see, e.g. [1]). Consequently, without loosing generality, we can suppose that the relation $P_0 = \partial_t$ holds (hereafter we skip the primes). As the operator P_1 commutes with P_0, its coefficients do not depend on t, i.e.,

$$P_1 = \xi^1(x,u,v)\partial_t + \xi^2(x,u,v)\partial_x + \eta^1(x,u,v)\partial_u + \eta^2(x,u,v)\partial_v.$$

By assumption, one of the coefficients $\xi^2,\ \eta^1,\ \eta^2$ is not equal to zero. Without loss of generality, we can suppose that $\xi^2 \neq 0$ (if this is not the case, we make a change $x\to u,\ u\to x$ or $x\to v,\ v\to x$). Performing the transformation

$$t' = t + F^1(x,u,v), \qquad x' = F^2(x,u,v), \qquad u' = G^1(x,u,v), \qquad v' = G^2(x,u,v),$$

where the functions $F^1,\ F^2$ are solutions of PDEs

$$P_1F^2 + \xi^1 = 0, \quad P_1F^2 = 1$$

and $G^1,\ G^2$ are functionally independent first integrals of the PDE $P_1F = 0$, we reduce the operators P_0, P_1 to become $P_0 = \partial_t$, $P_1 = \partial_x$.

Next, as the operator M commutes with $P_0,\ P_1$, its coefficients do not depend on $t,\ x$. Consequently, it has the form

$$M = \xi^1(u,v)\partial_t + \xi^2(u,v)\partial_x + \eta^1(u,v)\partial_u + \eta^2(u,v)\partial_v.$$

Suppose first that $(\eta^1)^2 + (\eta^2)^2 \neq 0$. Then, the change of variables

$$t' = t + F^1(u,v), \qquad x' = x + F^2(u,v), \qquad u' = G^1(u,v), \qquad v' = G^2(u,v),$$

where $F^1,\ F^2,\ G^1$ are solutions of PDEs

$$MF^1 + \xi^1 = 0, \quad MF^2 + \xi^2 = 0, \quad MG^1 = 1$$

and G^2 is a first integral of the PDE $MF = 0$, reduces the operators $P_0,\ P_1,\ M$ to the form (8).

If $\eta^1 = 0,\ \eta^2 = 0$, then formulae (9) are obtained.

Case 2. $\operatorname{rank} R = 1$. If we make transformation (3) reducing the operator P_0 to the form $P_0 = \partial_t$, then the operator P_1 becomes $P_1 = \xi(x,u,v)\partial_t$ (the function ξ does not depend on t because P_0 and P_1 commute). As $\xi \neq$ *const* (otherwise the operators P_0 and P_1 are linearly dependent), making the change of variables

$$t' = t, \quad x' = \xi(x,u,v), \quad u' = u, \quad v' = v$$

transforms the operator P_1 to be $P_1 = x\partial_t$.

It follows from the commutation relations $[P_0,\ M] = 0$, $[P_1,\ M] = 0$ that

$$M = \tilde{\xi}(x,u,v)\partial_t + \tilde{\eta}^1(x,u,v)\partial_u + \tilde{\eta}^2(x,u,v)\partial_v.$$

Subcase 2.1. $\tilde{\eta}^1 = \tilde{\eta}^2 = 0$. Provided the equalities $\tilde{\xi}_u = \tilde{\xi}_v = 0$ hold, formulae (11) are obtained. If $(\tilde{\xi}_u)^2 + (\tilde{\xi}_v)^2 \neq 0$, then making the transformation

$$t' = t, \quad x' = x, \quad u' = \tilde{\xi}(x,u,v), \quad v' = v$$

we arrive at formulae (12).

Subcase 2.2. $(\tilde{\eta}^1)^2 + (\tilde{\eta}^2)^2 \neq 0$. Performing the change of variables

$$t' = t + F(x,u,v), \qquad x' = x, \qquad u' = G^1(x,u,v), \qquad v' = G^2(x,u,v),$$

where $F,\ G^1, G^2$ satisfy PDEs

$$MF + \tilde{\xi} = 0, \quad MG^1 = 2, \quad MG^2 = 0,$$

we rewrite the operators $P_0,\ P_1,\ M$ in the form (10). The lemma is proved.

Theorem 1 *Inequivalent representations of the Galilei algebra by LVFs (1) are exhausted by those given in (6) and by the following ones:*

$$\begin{aligned} 1.\quad & P_0 = \partial_t, \quad P_1 = \partial_x, \quad M = 2u\partial_x, \\ & G = (t + xu)\partial_x + u^2\partial_u; \end{aligned} \tag{13}$$

$$\begin{aligned} 2.\quad & P_0 = \partial_t, \quad P_1 = \partial_x, \quad M = 2\left(u\partial_t \pm x\sqrt{\lambda u^2 - 2u}\,\partial_x\right), \\ & G = xu\partial_t + \left(t \pm x\sqrt{\lambda u^2 - 2u}\right)\partial_x \pm u\sqrt{\lambda u^2 - 2u}\,\partial_u; \end{aligned} \tag{14}$$

$$\begin{aligned} 3.\quad & P_0 = \partial_t, \quad P_1 = \partial_x, \quad M = 2(u\partial_t + v\partial_x), \\ & G = xu\partial_t + (t + xv)\partial_x + uv\partial_u + (u + v^2)\partial_v; \end{aligned} \tag{15}$$

$$\begin{aligned} 4.\quad & P_0 = \partial_t, \quad P_1 = x\partial_t, \quad M = -\tfrac{2}{\lambda}\left(1 \pm \sqrt{1 + \lambda x^2}\right)\partial_t, \\ & G = tx\partial_t + \left(x^2 + \tfrac{1}{\lambda}\left(1 \pm \sqrt{1 + \lambda x^2}\right)\right)\partial_x + \varepsilon\partial_u; \end{aligned} \tag{16}$$

$$\begin{aligned} 5.\quad & P_0 = \partial_t, \quad P_1 = x\partial_t, \quad M = 2u\partial_t, \\ & G = tx\partial_t + (x^2 - u)\partial_x + xu\partial_u, \end{aligned} \tag{17}$$

where λ is an arbitrary real parameter, $\varepsilon = 0, 1$.

Proof. To prove the theorem it suffices to solve the commutation relations for the basis operators P_0, P_1, M, G of the Galilei algebra in the class of LVF (1) within diffeomorphisms (3). All inequivalent realizations of the three-dimensional commutative algebra having the basis operators P_0, P_1, M are given by formulae (8) – (12). What is left is to solve the commutation relations for the generator of Galilei transformations $G = \xi^1(t,x,u,v)\partial_t + \xi^2(t,x,u,v)\partial_x + \eta^1(t,x,u,v)\partial_u + \eta^2(t,x,u,v)\partial_v$

$$[P_0,\ G] = P_1, \quad [P_1,\ G] = \tfrac{1}{2}M, \quad [M,\ G] = 0 \tag{18}$$

for each set of operators $P_0,\ P_1,\ M$ listed in (8) – (12). Since case (8) has been studied in detail in [8] and shown to yield representations (5), we will restrict ourselves to considering cases (9) – (12).

Case 1. Operators $P_0,\ P_1,\ M$ have the form (9). It is easy to establish that, using transformations (3), it is possible to reduce the operator M from (9) to one of the forms

$$M = 2(\lambda\partial_t + u\partial_x), \quad M = 2(u\partial_t + \beta(u)\partial_x), \quad M = 2(u\partial_t + v\partial_x),$$

where β is an arbitrary smooth function and λ is an arbitrary real constant.

Subcase 1.1. $M = 2(\lambda\partial_t + u\partial_x)$. Inserting the formulae for $P_0,\ P_1,\ M$ into (18) and equating the coefficients of linearly independent operators ∂_t, ∂_x, ∂_u, ∂_v yield the following over-determined system of PDEs for coefficients of the operator G:

$$\begin{aligned}&\xi^1_t = 0, \quad \xi^2_t = 1, \quad \eta^1_t = 0, \quad \eta^2_t = 0, \quad \xi^1_x = \lambda, \quad \xi^2_x = u,\\ &\eta^1_x = 0, \quad \eta^2_x = 0, \quad \lambda\xi^1_t + u\xi^1_x = 0, \quad \lambda\xi^2_t + u\xi^2_x - \eta^1 = 0\end{aligned}$$

As a compatibility condition of the above system, we get $\lambda = 0$ and what is more

$$\xi^1 = F^1(u, v), \quad \xi^2 = t + xu + F^2(u, v), \quad \eta^1 = u^2, \quad \eta^2 = F^3(u, v),$$

where $F^1,\ F^2,\ F^3$ are arbitrary smooth functions.

Making the change of variables

$$t' = t + T(u, v), \quad x' = x + X(u, v), \quad u' = u, \quad v' = V(u, v), \tag{19}$$

where $t,\ X,\ V$ are solutions of the system of PDEs

$$u^2T_u + F^3T_v + F^1 = 0, \quad u^2T_u + F^3T_v + F^1 = 0, \quad u^2V_u + F^3V_v = 0.$$

we transform the operator G to become

$$G = (t + xu)\partial_x + u^2\partial_u,$$

thus getting formulae (13).

Subcase 1.2. $M = 2(u\partial_t + \beta(u)\partial_x)$. Substituting the expressions for $P_0,\ P_1,\ M$ into (18) and equating the coefficients of the linearly-independent operators ∂_t, ∂_x, ∂_u, ∂_v give the following over-determined system of PDEs for coefficients of the operator G:

$$\begin{aligned}&\xi^1_t = 0, \quad \xi^2_t = 1, \quad \eta^1_t = 0, \quad \eta^2_t = 0, \quad \xi^1_x = u, \quad \xi^2_x = \beta(u), \quad \eta^1_x = 0, \quad \eta^2_x = 0,\\ &u\xi^1_t + \beta(u)\xi^1_x - \eta^1 = 0, \quad u\xi^2_t + \beta(u)\xi^2_x - \dot\beta(u)\eta^1 = 0.\end{aligned}$$

The general solution of the above system reads

$$\xi^1 = xu + F^1(u, v), \quad \xi^2 = t + x\beta(u) + F^2(u, v), \quad \eta^1 = u\beta(u), \quad \eta^2 = F^3(u, v),$$

where

$$\beta(u) = \pm\sqrt{\lambda u^2 - 2u},$$

$F^1,\ F^2,\ F^3$ are arbitrary smooth functions and λ is an arbitrary real parameter.

Performing, if necessary, the change of variables (19), we can put the functions F^1, F^2, F^3 equal to zero. Thus, the operator G is of the form

$$G = xu\partial_t + \left(t \pm \sqrt{\lambda u^2 - 2u}\right)\partial_x \pm u\sqrt{\lambda u^2 - 2u}\,\partial_u$$

and we arrive at representation (14).

Subcase 1.3 $M = 2(u\partial_t + v\partial_x)$. With this choice of M, the commutation relations (18) give the following system of PDEs for coefficients of the operator G:

$$\xi^1_t = 0, \quad \xi^2_t = 1, \quad \eta^1_t = 0, \quad \eta^2_t = 0, \quad \xi^1_x = u, \quad \xi^2_x = v, \quad \eta^1_x = 0, \quad \eta^2_x = 0,$$
$$u\xi^1_t + v\xi^1_x - \eta^1 = 0, \quad u\xi^2_t + v\xi^2_x - \eta^2 = 0,$$

which general solution reads

$$\xi^1 = xu + F^1(u,v), \quad \xi^2 = t + xv + F^2(u,v), \quad \eta^1 = uv, \quad \eta^2 = u + v^2.$$

Here F^1, F^2 are arbitrary smooth functions.

Making the transformation (19) with $V \equiv v$, we reduce the operator G to the form

$$G = xu\partial_t + (t + xv)\partial_x + uv\partial_u + (u + v^2)\partial_v,$$

thus getting representation (15).

Case 2. Operators P_0, P_1, M have the form (10). An easy check shows that the system of PDEs obtained by substitution of P_0, P_1, M from (10) into (18) is incompatible.

Case 3. Operators P_0, P_1, M have the form (11). In this case, the commutation relations (18) give rise to the following system of PDEs for the coefficients of the operator G:

$$\xi^1_t = x, \quad \xi^2_t = 0, \quad \eta^1_t = 0, \quad \eta^2_t = 0, \quad x\xi^1_t - \xi^2 = \gamma(x), \quad \gamma(x)\xi^1_t + \dot\gamma(x)\xi^2 = 0.$$

Solving it, we have

$$\xi^1 = xu + F^1(x,u,v), \quad \xi^2 = x^2 - \gamma(x), \quad \eta^1 = F^2(x,u,v), \quad \eta^2 = F^3(x,u,v),$$

where

$$\gamma(x) = -\tfrac{1}{\lambda}\left(1 \pm \sqrt{1 + \lambda x^2}\right),$$

F^1, F^2, F^3 are arbitrary smooth functions and λ is an arbitrary real constant.

Making the change of variables

$$t' = t + T(x,u,v), \quad x' = x, \quad u' = U(x,u,v), \quad v' = V(x,u,v) \tag{20}$$

transforms the operator G as follows

$$G = tx\partial_t + \left(x^2 + \tfrac{1}{\lambda}\left(1 \pm \sqrt{1 + \lambda x^2}\right)\right) + \varepsilon\partial_u, \quad \varepsilon = 0, 1.$$

Consequently, representation (16) is obtained.

Case 4. Operators P_0, P_1, M have the form (12). Inserting these into commutation relations (18) we get the system of PDEs for coefficients of the operator G

$$\xi^1_t = x, \quad \xi^2_t = 0, \quad \eta^1_t = 0, \quad \eta^2_t = 0, \quad x\xi^1_t - \xi^2 = u, \quad u\xi^1_t + \eta^1 = 0$$

having the following general solution:

$$\xi^1 = tx + F^1(x,u,v), \quad \xi^2 = x^2 - u, \quad \eta^1 = xu, \quad \eta^2 = F^2(x,u,v),$$

where F^1, F^2 are arbitrary smooth functions.

The change of variables (20) with $U \equiv u$ reduces the operator G to the form $G = tx\partial_t + (x^2-u)\partial_x + xu\partial_u$, which yields representation (17). The theorem has been proved.

Below we give without proof the assertions describing extensions of the Galilei algebra in the class of LVFs (1).

Theorem 2 *Inequivalent representations of the extended Galilei algebra $AG_1(1,1)$ by LVFs (1) are exhausted by those given in (7) and by the following ones:*

1. $P_0 = \partial_t, \quad P_1 = \partial_x, \quad M = 2u\partial_x,$
   $G = (t+xu)\partial_x + u^2\partial_u, \quad D = 2t\partial_t + x\partial_x + u\partial_u + \varepsilon\partial_v;$
2. $P_0 = \partial_t, \quad P_1 = \partial_x, \quad M = 2\left(-u\partial_t \pm \sqrt{2u}\,\partial_x\right),$
   $G = -xu\partial_t + \left(t \pm x\sqrt{2u}\right)\partial_x \pm \sqrt{2}\,u^{3/2}\partial_u,$
   $D = 2t\partial_t + x\partial_x + 2u\partial_u + \varepsilon\partial_v;$
3. $P_0 = \partial_t, \quad P_1 = \partial_x, \quad M = 2(u\partial_t + v\partial_x),$
   $G = xu\partial_t + (t+xv)\partial_x + uv\partial_u + (u+v^2)\partial_v,$
   $D = 2t\partial_t + x\partial_x + 2u\partial_u + v\partial_v;$
4. $P_0 = \partial_t, \quad P_1 = x\partial_t, \quad M = x^2\partial_t,$
   $G = tx\partial_t + \frac{1}{2}x^2\partial_x, \quad D = 2t\partial_t + x\partial_x + \varepsilon\partial_u;$
5. $P_0 = \partial_t, \quad P_1 = x\partial_t, \quad M = x^2\partial_t,$
   $G = tx\partial_t + \frac{1}{2}x^2\partial_x + \partial_u, \quad D = 2t\partial_t + x\partial_x - u\partial_u;$
6. $P_0 = \partial_t, \quad P_1 = x\partial_t, \quad M = 2u\partial_t,$
   $G = tx\partial_t + (x^2-u)\partial_x + xu\partial_u, \quad D = 2t\partial_t + x\partial_x + 2u\partial_u,$

where $\varepsilon = 0, 1$.

Theorem 3 *Inequivalent representations of the generalized Galilei algebra $AG_2(1,1)$ by LVFs (1) are exhausted by those given in (8) and by the following one:*

$$\begin{aligned}
&P_0 = \partial_t, \quad P_1 = \partial_x, \quad M = 2\left(-u\partial_t \pm \sqrt{2u}\,\partial_x\right),\\
&G = -xu\partial_t + \left(t \pm x\sqrt{2u}\right)\partial_x \pm \sqrt{2}\,u^{3/2}\partial_u, \quad D = 2t\partial_t + x\partial_x + 2u\partial_u,\\
&A = \left(t^2 - \tfrac{1}{2}ux^2\right)\partial_t + \left(tx \pm \tfrac{1}{2}x^2\sqrt{2u}\right)\partial_x + \left(2tu \pm x\sqrt{2}\,u^{3/2}\right)\partial_u.
\end{aligned}$$

Proof of Theorems 2,3 is analogous to that of Theorem 1 but computations are much more involved.

Let us note that the list of inequivalent representations of the Lie algebra of the Poincaré group $P(1,1)$ and its natural extensions in the class of LVF with two independent and one dependent variables given in [9] is also not complete. The reason is that these representations are constructed under assumption that the generators of time and space translations can be reduced to the form $P_0 = \partial_{x_0},\ P_1 = \partial_{x_1}$, which is not always possible. If we skip the above constraint, one more representation of the Lie algebra of the Poincaré group is obtained

$$P_0 = \partial_{x_0}, \quad P_1 = x_1\partial_{x_0}, \quad J_{01} = x_0x_1\partial_{x_0} + (x_1^2 - 1)\partial_{x_1}. \tag{21}$$

And what is more, there is one new representation of the Lie algebra of the extended Poincaré group $AP(1,1)$, where the basis operators $P_0,\ P_1,\ J_{01}$ are of the form (21) and the generator of dilations reads $D = x_0\partial_{x_0} + \varepsilon\partial_u,\ \varepsilon = 0, 1$.

In [10], we have studied realizations of the Poincaré algebras $AP(n,m)$ with $n+m \geq 2$ by LVFs in the space with $n+m$ independent and one dependent variables. It was established, in particular, that, provided the generators of translations $P_\mu,\ \mu = 0, 1, \ldots, n+m-1$ can be reduced to the form $P'_\mu = \partial_{x'_\mu}$, each representation of the algebra $AP(n,m)$ with $n+m > 2$ is equivalent to the standard linear representation

$$P_\mu = \partial_{x_\mu}, \quad J_{\mu\nu} = g_{\mu\alpha}x_\alpha\partial_{x_\nu} - g_{\nu\alpha}x_\alpha\partial_{x_\mu},$$

where

$$g_{\mu\nu} = \begin{cases} 1, & \mu = \nu = 1, \ldots, n, \\ -1, & \mu = \nu = n+1, \ldots, m, \\ 0, & \mu \neq \nu \end{cases}$$

and the summation over the repeated indices from 0 to $n+m$ is understood. In view of the results obtained in the present paper, it is not but natural to assume that if there will be no additional constraints on basis elements P_μ, then new representations will be obtained. Investigation of this problem is in progress now and will be reported elsewhere.

3. Conclusions

Our search for new representations of the Galilei algebra and its extensions was motivated not only by an aspiration to a completeness (which is very important) but also by a necessity to have new Galilei-invariant equations. Since the representations of the groups $G(1,1)$, $G_1(1,1)$, $G_2(1,1)$ obtained in the present paper are in most cases nonlinear in the field variables $u,\ v$, PDEs admitting these will be principally different from the standard Galilei-invariant models used in quantum theory. Nevertheless, being invariant under the Galilei group and, consequently, obeying the Galilei relativistic principle, they fit into the general scheme of selecting admissible quantum mechanics models.

Furthermore, (1+1)-dimensional PDEs having extensive symmetries are the most probable candidates to the role of integrable models. A peculiar example is the seven-parameter family of the nonlinear Schrödinger equations suggested by Doebner and Goldin [11]. As established in [12] in the case when the number of space variables is equal to one, *all* subfamilies with exceptional symmetry are either linearizable or integrable by quadratures. Another example is the Eckhaus equation which is invariant under the generalized Galilei group (see, e.g., [8]) and is linearizable by a contact transformation [13].

But even in the case where a Galilei-invariant equation can not be linearized or integrated in some way, one can always utilize the symmetry reduction procedure [1, 2, 4] to obtain its exact solutions. And the wider is a symmetry group admitted by the PDE considered, the more efficient is an application of the mentioned procedure (for more details see [4]).

Thus, PDEs invariant under the Galilei group $G(1,1)$ and its extensions possess a number of attractive properties and certainly deserve a detailed study. We intend to devote one of our future publications to construction and investigation of PDEs invariant under the groups $G(1,1)$, $G_1(1,1)$, $G_2(1,1)$ having the generators given in Theorems 1–3.

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