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Lie Superalgebra Valued Self-Dual Yang-Mills Fields and Symmetry Reduction

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Abstract

Self-dual Yang-Mills fields with values in a Lie superalgebra on the four-dimensional Euclidean space and pseudo-Euclidean space of signature (2,2) can be reduced by subgroups of the corresponding conformal group to integrable systems with anticommuting degrees of freedom. Examples of reductions are presented.

1 Introduction

The self-dual Yang-Mills (hereafter abbreviated: SDYM) equations with fields taking values in an (ordinary) Lie algebra have previously been investigated using the method of symmetry reduction [1-4]. From their complete integrability through the twistor construction [5-7], whether these equations are defined on E^4, the four-dimensional Euclidean space, or $E^{(2,2)}$, R^4 endowed with the diagonal metric of zero signature: $diag(+1,+1,-1,-1)$, reductions to known integrable systems have been obtained under subgroups of their space invariance group: $SO(5,1)(E^4)$ and $SO(3,3)(E^{(2,2)})$. For example, the Nahm [8-10], Euler [11], Chazy [11], (modified, matrix) Korteweg-de Vries (KdV) [12-17], (generalized) nonlinear Schrödinger (NLS) [12-14, 17], Boussinesq [15, 16], N-wave [18], (non)-periodic Toda lattice [17], static non-Abelian Chern-Simons [19], Kadomtsev-Petviashvili (KP) and Davey-Stewartson equations [18] have been retrieved with translational symmetries as well as the Ernst [20, 21] and Painlevé (P_I to P_{VI}) [22] equations with symmetries involving dilations and/or rotations. Accordingly, the reduced SDYM equations can be seen as the compatibility conditions of similarly reduced corresponding linear systems, also called Lax pairs [5, 6, 17, 23].

Comparable results have been obtained for self-dual gravity, where the $sl(\infty)$-Toda and Gibbons-Hawking equations [24-26] have been recovered via symmetry reductions, along with a set of hidden symmetries [27]. Supersymmetric extensions of various integrable models, such as super-Liouville, super-KdV, and super-Toda, have also been derived with the same method from supersymmetric generalizations of the SDYM equations, some with the help of added differential constraints [28, 29]. Let us mention that versions of heterotic $N=2$ strings (with gauge symmetry) possess, as a consistent background, a (null) translational symmetry reduction of co-dimension 2 (residual dimension equal to 2) of the SDYM equations [30, 31], let alone that $N = 2$ strings in four dimensions could be integrable systems [32], as suggested by scattering amplitudes calculations.

In addition, it has been shown that hierarchies of well-known integrable systems, such as KdV, NLS, and KP, could be recovered through reduction via translations of a universal hierarchy of commuting flows with the SDYM flow as an initial flow [33]. On a similar note, a supersymmetric version of the AKNStheory based on loop algebras of Lie superalgebras has been found to generate supersymmetric hierarchies of known integrable systems (KdV, NLS,...) under appropriate reductions [34, 35].

Recently, a connection between the Wilsonian low-energy limit of $N = 2$ super Yang-Mills with or without adjoint matter and finite-dimensional integrable systems involving ordinary Lie algebras was exhibited [36, 37]. This led to a conjecture [38] that integrable systems based on Lie superalgebras could arise as an equivalent limit of $N = 2$ super Yang-Mills systems coupled to general matter fields. A use of Lie superalgebras in integrable systems could also be as gauge algebras of SDYM fields. The SDYM equations can then be reduced via symmetry to integrable systems involving anticommuting degrees of freedom. However, these reduced systems will not be necessarily invariant under space(-time) supersymmetries.

In what follows, the reduction by symmetry of the SDYM equations with values in Lie superalgebras is considered. Section 2 begins with a review of such equations and their linear systems. Section 3 pursues with the invariance conditions imposed on the fields and introduces the notation for the classes of subalgebras of interest. Section 4 provides two examples of reduction under simple subgroups. Finally, Section 5 summarizes the results and indicates future directions of this work.

2 SDYM Equations and Linear Systems

In order to set our notation, let us recall the SDYM equations in R^4:

$$F = *F, \qquad \text{or} \qquad F_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}, \tag{2.1}$$

with $F = d\omega + \omega \wedge \omega$, the curvature 2-form of the field strength, and ω, a 1-form of connection with values in a Lie superalgebra (super Lie algebra) $\mathcal{H}$ of even dimensions m and odd dimensions n [39-41] on a gauge bundle. In component form, $\epsilon_{\mu\nu\rho\sigma}$ is the completely antisymmetric 4d tensor normalized to unity, and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$, where A_μ are the Lie superalgebra valued components of the gauge field $(\mu, \nu, \rho, \sigma = 1, ..., 4)$:

$$A_\mu = A_\mu^a M_a + \xi_\mu^\alpha N_\alpha, \tag{2.2}$$

where $\{M_a, a = 1, ..., m\}$, and $\{N_\alpha, \alpha = 1, ..., n\}$ stand, respectively, for the bases of the even and odd part of $\mathcal{H}$. The fields A_μ^a and ξ_μ^α denote, respectively, functions on E^4 (or$E^{(2,2)}$) with values in the even and odd parts of a Grassmann algebra of suitable dimensions (i.e., CB_L, where $2^{L-1} \geq m,\ 2^{L-1} \geq n$) [41].

The Lax pairs, or linear systems related to (2.1), can be written as [17, 23]:

$$\begin{array}{ll} E^4 \qquad (2.3a) & E^{(2,2)} \qquad (2.3b) \\ [D_1 + iD_2 - \lambda(D_3 + iD_4)]\Psi = 0, & [D_1 + iD_2 + \lambda(D_3 - iD_4)]\Psi = 0, \\ [D_3 - iD_4 + \lambda(D_1 - iD_2)]\Psi = 0, & [D_3 + iD_4 + \lambda(D_1 - iD_2)]\Psi = 0, \end{array}$$

where the covariant derivative is defined as: $D_\mu \equiv \partial_\mu + A_\mu$. The multiplet Ψ is composed of even Grassmann valued functions on $E^4 \times CP^1$ (or $E^{(2,2)}\times$ (upper (or lower) sheet hyperboloid of H^2)) holomorphic with respect to λ, i.e., $\partial_{\bar\lambda}\Psi = 0$, where the parameter $\lambda \in CP^1$ (or upper (or lower) sheet hyperboloid of H^2) and x^μ are Cartesian coordinates on E^4 (or $E^{(2,2)}$), $\mu = 1, ..., 4$. The integrability of the linear system(s) (2.3) gives rise to the SDYM equations (2.1). Both systems (2.1) and (2.3) are left invariant under the conformal transformations $SO(5,1)$ (E^4) (or $SO(3,3)$ ($E^{(2,2)}$)) as well as under the gauge transformations generated by the Lie superalgebra $\mathcal{H}$. A lift of the conformal transformations to CP^1 (or H^2-sheet), to be specified below, has to be introduced in order to preserve the holomorphy of Ψ on $E^4 \times CP^1$ (or $E^{(2,2)} \times H^2$-sheet) with respect to the complex structures induced by the linear systems. Let us add that the twistor construction [5-7] is still valid when a (linear) Lie supergroup [39,41] based on a Lie superalgebra ($\mathcal{H}$) of dimensions (m/n) is selected as gauge group, since it could be interpreted as an extended (linear) Lie group of $(m+n)\,2^{L-1}$ dimensions [41].

3 Invariance Conditions and Lift to Parameter Space: a Reminder

The extension of a gauge algebra to a Lie superalgebra does not alter the formulation of invariance conditions and the lift of the group action to the space of the parameter λ, as defined for a Lie gauge algebra [17,23]. Briefly, two types of fields are involved in the SDYM (2.1) and linear (2.3) equations: Yang-Mills fields (A_μ) and a multiplet (Ψ) of scalar fields transforming under the fundamental representation of the gauge group.

The reduction under a subgroup G (with a Lie algebra denoted by $\mathcal{G}$) of the space symmetry group is effected via the substitution of the corresponding G-invariant fields.

The G-invariant Yang-Mills fields respect the following infinitesimal conditions [42-45]:

$$\mathcal{L}_X A_\mu = D_\mu W \equiv \partial_\mu W + [A_\mu, W], \tag{3.1}$$

for all $X \in \mathcal{G}$, where $\mathcal{L}_X$ is the Lie derivative along the vector field associated to X, and W is a function of $\mathcal{G} \times E^4$ (or $\mathcal{G} \times E^{(2,2)}$), into the Lie superalgebra $\mathcal{H}$, the gauge algebra characterizing the lift of the G-action to the gauge bundle. Moreover, the $(m+n)$-multiplet Ψ will be infinitesimally G-invariant if the following (infinitesimal) conditions are met:

$$\mathcal{L}_{\tilde X}\Psi = -W\Psi, \tag{3.2}$$

for all $X \in \mathcal{G}$, where $\tilde X$ denotes the lift of the G-action to the 6-dimensional space: $E^4 \times CP^1$ (or $E^{(2,2)} \times H^2$).

These (infinitesimal) invariance constraints still allow an invariance up to a gauge transformation, unless $W = 0$.

Let us restrict ourselves to isometry subgroups: i.e., $SO(4) \otimes T^4(E^4)$ and $SO(2,2) \otimes T^4(E^{(2,2)})$. A basis for the $so(4)$ algebra acting on R^4 can be decomposed into the direct sum of two $so(3)$ algebras:

$$X_1 = -\frac{1}{2}(M_{23} + M_{14}), \qquad X_2 = \frac{1}{2}(M_{13} - M_{24}), \qquad X_3 = -\frac{1}{2}(M_{12} + M_{34}), \tag{3.3a}$$

$$Y_1 = -\frac{1}{2}(M_{23} - M_{14}), \qquad Y_2 = \frac{1}{2}(M_{13} + M_{24}), \qquad Y_3 = -\frac{1}{2}(M_{12} - M_{34}), \tag{3.3b}$$

where $[T_a, T_b] = \epsilon_{abc} T_c$, with either $T_a = X_a$ or $Y_a, a = 1, 2, 3$.

The matrices

$$[M_{\alpha\beta}]_{\mu\nu} = \delta_{\mu\alpha}\delta_{\nu\beta} - \delta_{\mu\beta}\delta_{\nu\alpha}, \tag{3.4}$$

$\alpha, \beta, \mu, \nu = 1, ..., 4$, generate rotations in the $x^\alpha x^\beta$-plane.

Since the $so(2,2)$ algebra is split into two $so(2,1)$ subalgebras, it can then be represented in R^4 as:

$$\mathcal{A}_1 = \frac{1}{2}(N_{23} - N_{14}), \qquad \mathcal{A}_2 = \frac{1}{2}(N_{13} + N_{24}), \qquad \mathcal{A}_3 = -\frac{1}{2}(N_{12} - N_{34}), \tag{3.5a}$$

$$\mathcal{B}_1 = \frac{1}{2}(N_{23} + N_{14}), \qquad \mathcal{B}_2 = -\frac{1}{2}(N_{13} - N_{24}), \qquad \mathcal{B}_3 = \frac{1}{2}(N_{12} + N_{34}), \tag{3.5b}$$

where $[T_a, T_b] = f^c_{ab} T_c$, with $T_a = \mathcal{A}_a$ or $\mathcal{B}_a, a = 1, 2, 3, f^3_{12} = 1, f^1_{23} = -1$, and $f^2_{31} = -1$.

Here the matrices:

$$\begin{aligned} &[N_{\alpha\beta}]_{\mu\nu} = \delta_{\mu\alpha}\delta_{\nu\beta} + \delta_{\mu\beta}\delta_{\nu\alpha}, \qquad \text{if} \quad \mu\nu \neq 12 \text{ and } 34, \\ &[N_{\alpha\beta}]_{\mu\nu} = \delta_{\mu\alpha}\delta_{\nu\beta} - \delta_{\mu\beta}\delta_{\nu\alpha}, \qquad \text{if} \quad \mu\nu = 12 \text{ or } 34, \end{aligned} \tag{3.6}$$

produce rotations or boosts in the $x^\alpha x^\beta$-plane.

Each subalgebra of the isometry (conformal) algebra can be used to reduce the SDYM equations and their linear system bythe corresponding symmetry subgroup. However, reduced systems are equivalent if symmetry subalgebras belong to the same conjugacy classes of subalgebras under the adjoint action of the isometry (conformal) group itself [1, 2, 46]. Such a classification for $so(4) \triangleright t^4$ can be found in Ref. 47.

Through their vector parts, which form a basis of antiholomorphic vectors, the linear systems (2.3) induce complex structures ($\mathcal{J}$) on $E^4 \times CP^1$ or $E^{(2,2)} \times H^2$-sheet [17, 23]:

$$\begin{array}{ll} \qquad E^4 \times CP^1 & \qquad E^{(2,2)} \times H^2 \\ \mathcal{J} = (s_a(2Y_{a\,\rho\mu}), \epsilon^j_i), & \mathcal{J} = (-s_a(2\mathcal{B}_a{}^\rho{}_\mu), \epsilon^j_i), \end{array} \tag{3.7}$$

$$\text{with} \quad s_a s_a = 1, \qquad\qquad \text{with}: \quad (h_1)^2 + (h_2)^2 - (h_3)^2 = -1, \tag{3.8}$$

where s_a and h_a are, respectively, coordinates which describe λ on CP^1 and a sheet of H^2.

The main point of the procedure is that the holomorphy of the multiplet Ψ with respect to the antiholomorphic vector field bases (related to $\mathcal{J}$) is preserved by the lift $\tilde{X}$ of the group action if

$$\mathcal{L}_{\tilde{X}} \mathcal{J} = 0, \tag{3.9}$$

for all $X \in \mathcal{G} \subseteq so(5,1)$ or $so(3,3)$, where $\mathcal{L}_{\tilde{X}}$ stands for the Lie derivative along $\tilde{X}$.

For the above mentioned isometry algebras, the lifted vector field bases which obeys to (3.9) are:

$$\begin{array}{ll} \qquad e(4) & \qquad e(2,2) \\ \tilde{\mathbf{X}}_a = (X_{a\,\mu\nu}) x_\mu \partial_\nu, & \tilde{\mathcal{A}}_a = -(\mathcal{A}_a{}^\mu{}_\nu) x^\nu \partial_\mu, \\ \tilde{\mathbf{Y}}_a = (Y_{a\,\mu\nu}) x_\mu \partial_\nu - \epsilon_{abc} s^b \partial_{s^c}, & \tilde{\mathcal{B}}_a = -(\mathcal{B}_a{}^\mu{}_\nu) x^\nu \partial_\mu - f^c_{ab} h^b \partial_{h^c}, \\ \tilde{\mathbf{P}}_\mu = \partial_\mu, & \tilde{\mathbf{P}}_\mu = \partial_\mu. \end{array} \tag{3.10}$$

4 Reduced SDYM Equations and Linear Systems

The method is identical to the procedure presented in Refs. 17 and 23. The orbit variables $\{\chi_m, m = 1, ..., d \leq n\}$ and invariant variables $\{\xi_A, A = 1, ..., 6 - d\}$ are determined using the lifted n-dimensional vector field basis of the chosen symmetry algebra ($\mathcal{G}$). A (new) spectral parameter (ζ) can be selected with the condition: $\mathcal{L}_{X_i}\zeta \neq 0$, for some $i = 1, ..., n$. The G-reduced SDYM equations and G-reduced linear systems are obtained by substitution of the coordinates χ_m and ξ_A, as well as the G-invariant Yang-Mills fields (A_μ) and multiplet (Ψ) in equations (2.1) and (2.3). Let us note that the reduced systems will be left invariant by residual gauge transformations generated by a residual Lie superalgebra, if the centralizer of the image of the homomorphism of the isotropy subalgebra into $\mathcal{H}$ with respect to $\mathcal{H}$ is itself a Lie superalgebra [36].

Two examples showing how the above method is applied are given below.

1. E^4, $\mathcal{G}$-basis $= \{P_1, P_2, P_3\}$, $\mathcal{H} = osp(1/2, R)$

According to (3.9), the lifted translations are not modified by a parameter contribution:

$$\tilde{\mathbf{P}}_\mathbf{1} = \partial_1, \qquad \tilde{\mathbf{P}}_\mathbf{2} = \partial_2, \qquad \tilde{\mathbf{P}}_\mathbf{3} = \partial_3. \tag{4.1}$$

Therefore, the orbits coordinates are:

$$x^1, \quad x^2, \quad x^3, \quad \text{and} \quad \bar{\lambda}. \tag{4.2}$$

For a simpler treatment, the coordinate $\bar{\lambda}$ can be seen as an orbit variable if the holomorphy condition on Ψ: $\partial_{\bar{\lambda}}\Psi = 0$, is equivalently read as an invariance constraint with respect to translations along this coordinate.

We are then left with the invariant coordinates:

$$x^4, \qquad \lambda. \tag{4.3}$$

The G-invariant Yang-Mills fields A_μ can be written as:

$$A_\mu(x^4) = A_\mu^a(x^4)M_a + \xi_\mu^\alpha N_\alpha, \tag{4.4}$$

with:

$$M_1 = \begin{pmatrix} 0 & \vec{0} \\ \vec{0}^T & \sigma_1 \end{pmatrix}, \qquad M_2 = \begin{pmatrix} 0 & \vec{0} \\ \vec{0}^T & -i\sigma_2 \end{pmatrix}, \qquad M_3 = \begin{pmatrix} 0 & \vec{0} \\ \vec{0}^T & \sigma_3 \end{pmatrix},$$
$$N_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad N_2 = \begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{4.5}$$

while the G-invariant multiplet of scalar fields Ψ has the form:

$$\Psi = \psi(x^4, \lambda). \tag{4.6}$$

The reduced linear system (2.3) follows from the substitution of (4.2)-(4.6):

$$[i\lambda\partial_4 - (A_1 + iA_2) + \lambda(A_3 + iA_4)]\psi(x^4, \lambda) = 0, \tag{4.7a}$$

$$[-i\partial_4 + \lambda(A_1 - iA_2) + (A_3 - iA_4)]\psi(x^4, \lambda) = 0, \tag{4.7b}$$

where the parameter of the reduced equations is the original variable λ.

The involution of the Lax pair (4.7) produces the $\{P_1, P_2, P_3\}$-reduced SDYM equations (2.1), where the field strengths $F_{\mu\nu}$ depend only on the coordinate x^4. This system constitutes an integrable set of 15 equations for A^a_μ and ξ^α_μ. It can be further simplified with the gauge choice $A^a_4 = \xi^\alpha_4 = 0$, for all $a = 1,2,3$, and $\alpha = 1,2$, and the algebraic condition $A^a_i = \delta^a_i \omega_i$, with $i = 1,2,3$, which then lead to the extension with anticommuting fields of the $so(2,1)$- Nahm's equations:

$$\begin{aligned}
&\partial_4\omega_1 + 2\omega_2\omega_3 + \xi^1_2\xi^1_3 - \xi^2_2\xi^2_3 = 0,\\
&\partial_4\omega_2 - 2\omega_3\omega_1 + \xi^1_3\xi^1_1 + \xi^2_3\xi^2_1 = 0,\\
&\partial_4\omega_3 - 2\omega_1\omega_2 - \xi^1_1\xi^1_2 - \xi^2_1\xi^2_2 = 0,
\end{aligned}\tag{4.8}$$

and the linear equations for odd components ξ_μ:

$$\begin{aligned}
&\partial_4\xi^1_1 - \omega_2\xi^2_3 + \omega_3\xi^1_2 = 0, && \xi^1_1\xi^1_2 - \xi^2_1\xi^2_2 = 0,\\
&\partial_4\xi^2_1 + \omega_2\xi^1_3 - \omega_3\xi^2_2 = 0, && \xi^1_1\xi^1_2 + \xi^2_1\xi^2_2 = 0,\\
&\partial_4\xi^1_2 + \omega_1\xi^2_3 - \omega_3\xi^1_1 = 0, && \xi^1_2\xi^1_3 + \xi^2_2\xi^2_3 = 0,\\
&\partial_4\xi^2_2 + \omega_1\xi^1_3 + \omega_3\xi^2_1 = 0, && \xi^1_2\xi^2_3 + \xi^2_2\xi^1_3 = 0,\\
&\partial_4\xi^1_3 - \omega_1\xi^2_2 + \omega_2\xi^2_1 = 0, && \xi^1_3\xi^1_1 - \xi^2_3\xi^2_1 = 0,\\
&\partial_4\xi^2_3 - \omega_1\xi^1_2 - \omega_2\xi^1_1 = 0, && \xi^1_3\xi^2_1 + \xi^2_3\xi^1_1 = 0.
\end{aligned}\tag{4.9}$$

However, equations (4.9) for ξ^α_μ are rather stringent and force these fields to vanish for nontrivial configurations: $\omega_i, i = 1,2,3$. A similar set of reduced equations has been obtained for $\mathcal{H} = su(2/1)$ when algebraically constrained to recover extended $so(3)$-Nahm's equations. Let us add that integrable reductions leading to 2d or 3d residual differential systems can be carried out via, for example, translations .

In order to illustrate the method in the case of a symmetry subalgebra bearing a nontrivial lift, the nilpotent Lie superalgebra $su(1/1)$ was chosen for simplicity, even if it gives rise to a system of reduced equations that still can be solved without the use of a linear system.

2. $E^{(2,2)}$, $\mathcal{G}$-basis $= \{\mathcal{B}_3, P_3, P_4\}$,$\mathcal{H} = su(1/1)$

$$\tilde{\mathcal{B}}_\mathbf{3} = \frac{1}{2}(x^1\partial_2 - x^2\partial_1 + x^3\partial_4 - x^4\partial_3) + i(\lambda\partial_\lambda - \bar{\lambda}\partial_{\bar{\lambda}}), \quad \tilde{\mathbf{P}}_\mathbf{3} = \partial_3, \quad \tilde{\mathbf{P}}_\mathbf{4} = \partial_4. \tag{4.10}$$

Including $\mathbf{P}_{\bar{\lambda}}$-invariance, the orbit coordinates are given by:

$$\theta = -\arctan\left(\frac{x^2}{x^1}\right), \qquad x^3, \quad x^4, \quad \bar{\lambda}. \tag{4.11}$$

The invariant variables are then:

$$r = \sqrt{(x^1)^2 + (x^2)^2}, \quad \zeta = e^{2i\theta}\lambda, \tag{4.12}$$

where ζ corresponds to a new parameter.

The G-invariant A_μ can be written as:

$$\begin{aligned}
(A_1, A_2, A_3, A_4)^T = e^{-2\theta\mathcal{B}_3}\Big[&(u_1, u_2, u_3, u_4)^T M_1 +\\
&(v_1, v_2, v_3, v_4)^T N_1 + (w_1, w_2, w_3, w_4)^T N_2\Big],
\end{aligned}\tag{4.13}$$

where $u_\mu = u_\mu(r), v_\mu = v_\mu(r), w_\mu = w_\mu(r)$, and the matrices M_1, N_1, N_2 are defined as:

$$M_1 = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}; \qquad N_1 = \begin{pmatrix} 0 & i \\ 1 & 0 \end{pmatrix}; \qquad N_2 = \begin{pmatrix} 0 & 1 \\ i & 0 \end{pmatrix}. \tag{4.14}$$

The G-invariant multiplet is simply a function of the invariant coordinates:

$$\Psi = \psi(r, \zeta). \tag{4.15}$$

Substitution of (4.11)-(4.15) into (2.3) leads to the G-reduced linear system:

$$\nabla_1 \psi \equiv \left[\partial_r + \frac{2\zeta}{r}\partial_\zeta + a^+_{12} + \zeta a^-_{34}\right]\psi = 0, \tag{4.16a}$$

$$\nabla_2 \psi \equiv \left[\zeta(\partial_r - \frac{2\zeta}{r}\partial_\zeta + a^-_{12}) + a^+_{34}\right]\psi = 0, \tag{4.16b}$$

where:

$$\begin{aligned} a^\pm_{12} &= (u_1 \pm iu_2)M_1 + (v_1 \pm iv_2)N_1 + (w_1 \pm iw_2)N_2, \\ a^\pm_{34} &= (u_3 \pm iu_4)M_1 + (v_3 \pm iv_4)N_1 + (w_3 \pm iw_4)N_2. \end{aligned} \tag{4.17}$$

Using the holonomic version of (4.16) with the operators $r\nabla_1$ and $\frac{r}{\zeta}\nabla_2$, the integrability of the Lax pair (4.7) provides the corresponding G-reduced SDYM equations.

Let us rapidly recall that for Grassmann valued equations, the solution starts with solving the "body" or "level 0" part, which in the case of SDYM equations amounts to solve gauge Lie algebra reduced integrable systems. This solution is then inserted as background to the "level 1" equations for the anticommuting fields, which are followingly substituted in the field equations of "level 2", and so on.

5 Conclusion

In the above, the method of symmetry reduction has been applied to the self-dual Yang-Mills equations and their linear systems, or Lax pairs, extended to Lie superalgebras as gauge algebras, on the Euclidean space and pseudo-Euclidean space of signature $(2, 2)$ in four dimensions. The reduced systems are left with anticommuting degrees of freedom. As for the case of a Lie gauge algebra, a lift of the group action to a 6-dimensional space formed of the product of E^4 (or$E^{(2,2)}$) by the parameter space CP^1 (or H^2-sheet) preserving the holomorphy of the multiplet of scalar fields (Ψ) was essential in the reduction of the linear systems. Two examples were added to illustrate the method. It was found that the anticommuting fields are strongly restricted for a residual dimension equal to 1.

Further reductions by representatives of conjugacy classes of the associated conformal group could be carried on E^4 or $E^{(2,2)}$. One could also investigate higher-dimensional and/or self-dual spaces versions of these models with the same methods as well as reductions with Yang-Mills fields and multiplets of scalar fields invariant up to a gauge transformation. Finally, embeddings of known superintegrable systems via algebraic conditions could be searched among the reduced systems.

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