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Proper-Time Formulation of Classical Electrodynamics †

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Abstract

We show that Maxwell's equations have a *generalization associated with the proper-time of the source* and a new invariance group which leaves this variable fixed for all observers. *We show that the second postulate (of Einstein) depends on the anthropocentric view that the only clock to use is the proper-clock of the observer.* Our work is motivated by the results of Fushchych and Shtelen (F-S), who showed that the free Maxwell equations have an additional invariance group which is Galilean. This work makes it clear that our group is distinct from but closely related to the Lorentz group, and is not Galilean. Since our present (constructive) approach requires a source, it does not apply to the free field case. In an earlier paper, we showed that the F-S transformation is an element of the proper-time (canonical) group which includes the group constructed here. This work is also related to the work of Wegener who showed that use of the proper time allows the construction of Galilean transformations from Lorentz transformations.

0. Introduction

Background

In the 40 years between Maxwell's theory of electrodynamics [13] in 1865 and the introduction of the special theory of relativity by Einstein [3], Lorentz [12], and Poincaré [16] in 1905, a scientific revolution had taken firm root. Maxwell's theory had provided answers to almost all major questions in electromagnetism and optics. The economic and social benefits of the new science of electromagnetism was being felt all over the world. Almost a hundred years had elapsed since Lagrange boasted (1788) of making mechanics a branch of (mathematical) analysis, "No diagrams will be found in this work". From this point of view, it comes as no surprise that, when problems arose in the interface between mechanics

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and electromagnetism, there was a natural tendency to keep the Maxwell theory intact and seek modifications of the Newtonian theory.

When Einstein and his contemporaries began to study the issues associated with the foundations of electrodynamics, they had a number of options open to them in addressing the fact that the Newtonian and Maxwell theories were invariant under different transformation groups:

1. both theories were incorrect and the proper theory was yet to be found;
2. the Maxwell theory was incorrect and the proper theory would be invariant under the Galilean group;
3. the Maxwell theory was correct and a proper Newtonian theory would be invariant under the Lorentz group; and
4. the assumption of an ether for electromagnetic propagation was correct so that Galilean relativity applied to mechanics while electromagnetism had a preferred reference frame.

At the time it was unthinkable that the Maxwell theory had any serious flaws. Lorentz [12] had recently (1903) shown that all the macroscopic phenomena of electrodynamics and optics could be accounted for based on the analysis of the microscopic behavior of electrons and ions.

Einstein rejected the fourth possibility and proposed that all physical theories should satisfy the (now well-known) postulates of special relativity:

1. the physical laws of nature and the results of all experiments are independent of the particular inertial frame of the observer (in which the experiment is performed); and
2. the speed of light is independent of the motion of the source.

The first postulate abandons the notion of absolute space, while the second abandons absolute time. It is important to note that another postulate is required in order to implement the above two postulates:

3. The correct *implementation* of postulates 1 and 2 is to represent time as a fourth coordinate and constrain the relationship between components so as to satisfy the natural invariance induced by the Lorentz group (of electromagnetism). As is well known, this procedure leads to the concept of Minkowski space.

This third postulate was made by Minkowski [14, 15], a well-known mathematician, and was embraced by many. Others, including Einstein, Lorentz and Poincaré regarded it as a mathematical abstraction lacking a physical content and maintained that space and time have distinct in physical properties. Although Einstein demurred, the feeling among many of the leading physicists at that time was that an alternate implementation should be possible which preserved some remnant of an "absolute time" variable while still allowing for the constancy of the speed of light. The works of Ritz [17] and Tolman [18] are notable in this direction. The inability to obtain a viable alternative forced acceptance of the current implementation.

Purpose

In a recent series of papers [9-11], the physical efficiency and redundancy of the Minkowski postulate was addressed by constructing an alternate implementation of the first two postulates without assuming that time be treated as a fourth coordinate. Our approach is based on the observation that we may use the *invariant* proper-time associated with the system of physical interest in place of the observer "proper-time" which is not an invariant concept. The use of such a variable is not new and dates back to Tetrode and Fock (for a recent review see Fanchi [4]). *Our approach is distinct in that we do not view it as a fifth variable but as a real dynamical variable representing a possible clock available to all observers.* As such, we treat the transformation from the observer proper-time to the system proper-time as a canonical (contact) transformation on the extended phase space. This approach forces the identification of the canonical Hamiltonian which generates the Lie Algebra (Poisson) brackets and thereby leads to a conceptually and technically much simpler implementation of the special theory. This work is distinct from but similar to and in the same spirit as the work of Wegener [19], who also recognized that the proper-time could and should be used in the representation of physical systems.

This work is dedicated to Professor Fushchych on the occasion of his sixtieth birthday. It is a continuation of [11] which was motivated by the work of Professor Fushchych and Shtelen on the invariance of Maxwell's equations [5]. It was proved by them that the free Maxwell equations have an additional invariance group which is Galilean (e.g., it fixes time). Since our approach is new, we give a review of the basic ideas in Section 1. In Section 2, we show that the corresponding *proper-time generalization of Maxwell's equations* has a new transformation theory which fixes the proper-time of the source. *We show in passing that the second postulate depends on the anthropocentric view that the only clock to use is the proper-clock of the observer.*

1. Proper-Time Theory

For the dynamics of a classical observable, the Poisson bracket is defined by:

$$\{A(p,q),\ B(p,q)\} \equiv \frac{\partial A \partial B}{\partial p \partial q} - \frac{\partial A \partial B}{\partial q \partial p}.$$

The Hamiltonian equations

$$\frac{\partial H}{\partial p} = \dot{q}, \qquad \frac{\partial H}{\partial q} = -\dot{p}$$

insure that the time development of an arbitrary classical function $W(q,p,t)$ is given by:

$$\frac{dW(q,p,t)}{dt} = \{H, W(q,p,t)\} + \frac{\partial W}{\partial t}.$$

Next, define the proper time τ through the relation:

$$dt = \frac{H}{mc^2} d\tau.$$

The time evolution of the function W is given by the chain rule:

$$\frac{dW}{d\tau} = \frac{dW}{dt}\frac{dt}{d\tau} = \frac{H}{mc^2}\{H, W\} + \frac{\partial W}{\partial \tau}.$$

An energy functional K conjugate to the time τ will be defined to satisfy:

$$\{K, W\} = \frac{H}{mc^2}\{H, W\}$$

$$p = 0 \Longrightarrow H = K = mc^2.$$

If the mass m remains invariant during the dynamics, the form of the functional K can be directly determined as

$$K = \frac{H^2}{2mc^2} + \frac{mc^2}{2},$$

or more generally, if m is a well-defined mass point, we can define K by:

$$K = mc^2 + \int_{mc^2}^{H} \frac{dt}{d\tau} dH' = mc^2 + \int_{mc^2}^{H} \frac{H'}{mc^2} dH'.$$

The evolution of the function W in terms of τ can be expressed now as follows:

$$\frac{dW}{d\tau} = \{K, W\} + \frac{\partial W}{\partial \tau}.$$

Consider the behavior of a single noninteracting particle of mass m, with momentum $\mathbf{p}$ as measured in some inertial frame. The usual form of the Hamiltonian representing this system is $H = \sqrt{c^2\mathbf{p}^2 + m^2c^4}$. For this example, the conjugate proper energy is given by: $K = \dfrac{\mathbf{p}^2}{2mc^2} + mc^2$.

Several interesting points should be noted:

a. The functional form of the energy K is the same as that of the nonrelativistic energy of the system, even though the system is fully relativistic (see Theorem 1.2 below).

b. The momentum parameter in the functional form of the energy K is the momentum as measured in the original inertial frame, not the proper frame of the particle (which, of course, would measure zero momentum). This reemphasizes the form of the transformation as a canonical time transformation, and not a Lorentz transformation (see Theorems 1.1 and 1.2 below).

c. If the particle were to interact with external influences, the proper frame would not be an inertial frame but the proper time is always defined and, in fact, is the only true time relevant to the particle itself and available to all observers.

Transformation Group

The following result relates the phase flows for the $(\mathbf{p}, \mathbf{q}, t)$ and $(\mathbf{p}, \mathbf{q}, \tau)$ variables.

Theorem 1.1 *There exists a function $S = S(\mathbf{p}, \mathbf{q})$ such that:*

$$\mathbf{p} \cdot d\mathbf{q} - H dt = \mathbf{p} \cdot d\mathbf{q} - K d\tau + dS.$$

As was noted earlier, the proper-time is invariant for all inertial observers. However, different observers will use different Hamiltonians to describe the phase flow of the system. Consider two inertial observers in frames X and X' with (extended) phase space coordinates $(\mathbf{p}, \mathbf{q}, t)$ and $(\mathbf{p}', \mathbf{q}', t')$, respectively (for the dynamics of some system). We let P denote the set of Poincaré transformations on space-time reference frames. In particular, $P(X, X^{'})$ indicates the Poincaré map from X to $X^{'}$. We let C_τ indicate the set of canonical proper-time transformations defined on extended phase space and denote the map from $(\mathbf{p}, \mathbf{q}, t)$ to $(\mathbf{p}, \mathbf{q}, \tau)$ by $C[\mathbf{q}, t, \tau]$. The following result is proved in [11]:

Theorem 1.2. *The proper-time coordinates on X are related to those on X' by the transformation:*

$$S_m[\mathbf{q}, \mathbf{q}', \tau] = C[\mathbf{q}', t', \tau]P(X, X')C^{-1}[\mathbf{q}, t, \tau].$$

Theorem 1.1 proves that C_τ is a group (Canonical Proper-Time group), while Theorem 1.2 proves that S_m is in the group S formed via a similarity action on the Poincaré group by C_τ. It is shown in [10] that, in the many-particle interaction case, S exists but is not related to the Poincaré group. This is to be expected because of the No-Interaction theorem of Currie [1,2].

2. Proper-Time Maxwell Theory

In this section, we construct a direct representation of the group S associated with proper-time transformations between observers. In order to make our approach transparent, we follow the original approach of Einstein [3]. Let us consider two inertial observers X and $X^{'}$, with $X^{'}$ moving along the positive x-axis with velocity v as seen by X. Let the source of an electromagnetic field also move along the x-axis with velocity w_x as seen by X and velocity $w'_{x'}$ as seen by $X^{'}$. We also assume that the (proper) clocks of X and $X^{'}$ both begin when their origins coincide (Einstein synchronization). It follows as in [3] that:

$$\begin{aligned} &x' = \gamma(v)(x - vt), \qquad y' = y, \qquad z' = z, \qquad t' = \gamma(v)(t - vx/c^2), \text{ and} \\ &x = \gamma(v)(x' + vt'), \qquad y' = y, \qquad z' = z, \qquad t = \gamma(v)(t' + vx'/c^2), \end{aligned}$$

with $\gamma(v) = 1/\sqrt{1 - v^2/c^2}$, represent the Lorentz transformations between our two observers X and $X^{'}$.

The two observers X and $X^{'}$ can compute the proper-time for the source via:

$$dt = \delta^{-1}(u)d\tau \qquad \text{and} \qquad dt' = \delta^{-1}(u')d\tau, \ \delta(h) = 1/\sqrt{1 + h^2/c^2}, \tag{1}$$

where $u_x = dx/d\tau$ is the velocity of the source as seen by X and $u'_{x'} = dx'/d\tau$ is the source velocity as seen in the $X^{'}$ system. If we use the proper-clocks of the observers, $\mathbf{w}$ and $\mathbf{w}^{'}$ are related to $\mathbf{u}$ and $\mathbf{u}^{'}$ by:

$$\mathbf{u}' = \gamma(w')\mathbf{w}', \qquad \mathbf{w}' = \delta(u')\mathbf{u}'; \qquad \text{and} \tag{2a}$$

$$\mathbf{u} = \gamma(w)\mathbf{w}, \qquad \mathbf{w} = \delta(u)\mathbf{u}. \tag{2b}$$

From $w'_{x'} = \dfrac{(w_x - v)}{(1 - vx_x/c^2)}$, $w'_{y'} = \dfrac{w_y}{\gamma(v)(1 - vw_x/c^2)}$ and $w'_{z'} = \dfrac{w_z}{\gamma(v)(1 - vw_xc^2)}$, we have:

$$\delta(u')u'_{x'} = \frac{\delta(u)u_x - v}{(1 - v\delta(u)u_x/c^2)}, \qquad \delta(u')u'_{y'} = \frac{\delta(u)u_y}{\gamma(v)(1 - v\delta(u)u_x/c^2)} \qquad \text{and}$$
$$\delta(u')u'_{z'} = \frac{\delta(u)u_x}{\gamma(v)(1 - v\delta(u)u_x/c^2)}. \tag{2c}$$

Setting $\Lambda(h) = \dfrac{1}{\tau}\displaystyle\int_0^\tau \delta(h)ds$ and $\Delta(h) = \displaystyle\int_0^\tau \delta(h)ds$, our transformations between observers become:

$$x' = \gamma(v)(x - v\Lambda(u)\tau), \qquad \Delta(u') = \gamma(v)(\Delta(u) - vx/c), \tag{3a}$$

$$x = \gamma(v)(x' + v\Lambda(u')\tau), \qquad \Delta(u) = \gamma(v)(\Delta(u') + vx'/c). \tag{3b}$$

It is clear that our transformations are nonlocal in time if $\mathbf{u}$ is not constant (as to be expected from the work of Fushchych and Shtelen [5]). We do not require that the source have a constant velocity so that τ is not a linear variable. In the case where u is constant, $\Lambda(u) = \delta(u)$ and the transformations are local.

From (1), it is easy to see that:

$$\partial_t = \gamma(v)(\delta(u')\partial_\tau - v\partial_{x'}), \qquad \partial_x = \gamma(v)(\partial_{x'} - v/c^2\delta(u')\partial_\tau),$$
$$\partial_{t'} = \gamma(v)(\delta(u)\partial_\tau + v\partial_x), \qquad \partial_x = \gamma(v)(\partial_{x'} + v/c^2\delta(u)\partial_\tau).$$

Let us now consider Maxwell's equations as seen from X for the field at the source. Assuming that the current density $\mathbf{J}$ can be written as $c\mathbf{J} = \rho\mathbf{w}$ and using (1) and (2), these equations can be written in the form (*generalized to depend on the proper-time of the source*):

$$\nabla \cdot \mathbf{H} = 0,$$
$$\nabla \cdot \mathbf{E} = 4\pi\rho,$$
$$\nabla \times \mathbf{E} = -\delta(u)/c\frac{\partial \mathbf{H}}{\partial \tau},$$
$$\nabla \times \mathbf{H} = \delta(u)/c\left[\frac{\partial \mathbf{E}}{\partial \tau} + 4\pi\rho\mathbf{u}\right].$$

Following the same calculations as in Einstein [3], we find that:

$$E'_{x'} = E_x, \qquad H'_{x'} = H_x,$$
$$E'_{y'} = \gamma(E_y - v/c\, H_z), \qquad H'_{y'} = \gamma(H_y + v/c\, E_z),$$
$$E'_{z'} = \gamma(E_z + v/c\, H_y), \qquad H'_{z'} = \gamma(H_z - v/c\, E_y),$$

and

$$\rho' = \rho(1 - vu\delta(u)/c^2),$$

leading to:

$$\begin{aligned}
&\nabla' \cdot \mathbf{H}' = 0, \\
&\nabla' \cdot \mathbf{E}' = 4\pi\rho, \\
&\nabla' \times \mathbf{E}' = -\delta(u')/c\frac{\partial \mathbf{H}'}{\partial \tau}, \\
&\nabla' \times \mathbf{H}' = \delta(u')/c\left[\frac{\partial \mathbf{E}'}{\partial \tau} + 4\pi\rho'\mathbf{u}'\right].
\end{aligned}$$

It follows that Maxwell's equations are invariant under transformations (3). We have chosen this approach so that we need not assume any special properties for the source. The use of Einstein's synchronization makes it possible for us to make the lower limit in the definition of $\Lambda(\Delta)$ the same for each observer. The use of zero as the lower limit is a convention.

If another observer X'' is also present, then using $x'' = \gamma(v')(x' - v'\Lambda(u')\tau)$ and (3a), we easily obtain:

$$x'' = \gamma(v')\gamma(v)(1 + vv'/c^2)[x - \frac{v + v'}{(1 + vv'/c^2)}\Lambda(u)\tau] \qquad \text{and}$$
$$\gamma(v'') = \gamma(v')\gamma(v)(1 + vv'/c^2).$$

It follows that we get the same (Lorentz) velocity addition law for (inertial) observers:

$$v'' = \frac{v + v'}{(1 + vv'c^2)},$$

as in conventional treatments of special relativity.

Conclusion

It is easy to see that our approach does not violate gauge invariance and leads to a version of the wave equation which depends on the source. It follows that (in our approach) the speed of light will depend on the motion of the source.

We see also from this approach that the second postulate rests on the assumption that the proper-clock of the observer is the correct one. This raises the question of creating experiments to decide which clock is physically justified. There have been many experiments, but they can be interpreted to favor both approaches. These issues will be explored at a later time. It should also be clear that these results could have been obtained at any time after Minkowski discovered the proper-time and recognized its importance. These results may appear paradoxical. However, we should remember that there may be more than one representation of a given set of postulates. Minkowski assumed that we should make time a fourth coordinate on philosophical and mathematically pleasing grounds. His approach is both elegant and mathematically consistent, but lacks a physical content.

Our approach is based on the observation (of Feynman) that *time is both a physical quantity and an index which identifies the order of physical events*. The successful construction of the Feynman Time-Ordered Operator Calculus [8], its intuitive physical content, and our ability to use it to prove the Dyson conjecture [8], convinced us that the use of time as a fourth coordinate may well be a major cause of problems in the merging of relativity with quantum mechanics.

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