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Lorentz Transformations for the Schrödinger Equation

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Dedicated to Wilhelm Fushchych – Inspirer, Mentor, Friend and Pioneer in non–Lie symmetry methods – on the occasion of his sixtieth birthday

Abstract

We show that the free Schrödinger equation admits Lorentz space-time transformations when corresponding transformations of the ψ-function are nonlocal. Some consequences of this symmetry are discussed.

It is well known that the three–dimensional Schrödinger equation

$$S\psi = 0, \qquad \text{where} \qquad S \equiv i\partial_t + \frac{1}{2m}\Delta, \tag{1}$$

which is one of the basic equations of nonrelativistic quantum mechanics, is invariant under the Galilean transformations and it is not invariant under Lorentz transformations. The purpose of this paper is to show that it is possible to consider (1) to be invariant under the Lorentz space–time transformations. A new possibility arises when one adopts a non–Lie approach initiated in [1, 2] and further developed in [3–6].

The symmetry operators of (1)

$$G_a = t\partial_a - imx_a, \qquad a = 1, 2, 3 \tag{2}$$

generate the Galilean transformations ($v_a = \text{const}$)

$$\begin{aligned} & x'_a = x_a + v_a t, \qquad t' = t, \\ & \psi'(x') = \exp\left\{ im\left(\vec{v}\cdot\vec{x} + \frac{1}{2}\vec{v}^2 t\right)\right\}\psi(x), \end{aligned} \tag{3}$$

which leave equation (1) invariant (for more detail see, for example, [6]).

Let us consider the operators

$$J_{oa} = t\partial_a + x_a\partial_t + imx_a\left(\frac{1}{2m^2}\Delta - 1\right). \tag{4}$$

It is easy to see that on the manifold of solutions of equation (1) operators (4) coincide with those given in (2) and, hence, the former ones are symmetries of (1). Unlike (2), operators (4) are of non-Lie type, and to find final transformations generated by (4), one can use the method suggested in [4, 6]. In particular, one obtains

$$x'_\mu = \exp\Big\{v_a(t\partial_a + x_a\partial_t)\Big\} x_\mu \exp\Big\{v_a(t\partial_a + x_a\partial_t)\Big\}, \qquad \mu = 0,1,2,3; \quad x_0 \equiv t; \tag{5}$$

$$\psi'(x') = \exp\Big\{v_a(t\partial_a + x_a\partial_t)\Big\} \exp\Big\{-v_a J_{oa}\Big\}\psi(x). \tag{6}$$

Expression (5) leads to the Lorentz space-time transformations (for more detail see [6], Appendix 3). Transformations which follow from (6) are nonlocal.

This example of Lorentz space-time transformations for the Schrödinger equation as well as examples of Galilean space–time transformations for Dirac, Maxwell, and some other equations considered in [4-6] demonstrate that the Galilei and Lorentz space-time symmetries are not mutually exclusive in a physical theory. There is a bridge between them and this bridge rests on nonlocal transformations.

Remark 1. It should be noted that operators (4) together with the generators of rotations $J_{ab} = x_a\partial_b - x_b\partial_a$ do not form a Lorentz algebra. They satisfy the following commutation relations (the operator S is given in (1))

$$[J_{0a}, J_{0b}] = -J_{ab}S/m, \qquad [J_{ab}, J_{0c}] = \delta_{bc}J_{0a} - \delta_{ac}J_{0b},$$

that is, they form a Galilei algebra on the set of solutions of (1).

Remark 2. Solutions of the Schödinger equation (1) invariant under (4) coincide with those invariant under (2), but using transformations (5), (6) they can be given different interpretation.

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References

[1] Fushchych W.I. On additional invariance of relativistic equations of motion, *Teoret. i Mat. Fizika*, 1971, V.7, N 11, 3–12.

[2] Fushchych W.I. On additional invariance of the Klein–Gordon–Fock equation, *Dokl. Akad. Nauk SSSR*, 1976, V.230, N 3, 570–572.

[3] Fushchych W.I. and Nikitin A.G., Symmetries of Maxwell's Equations, D. Reidel, 1987.

[4] Shtelen V.M. Non-Lie symmetry and nonlocal transformations, Preprint, Institute of Mathematics, Ukrainian Academy of Sciences, Kiev, 1987.

[5] Fushchych W.I. and Shtelen V.M. Are Maxwell's equations invariant under the Galilean transformations? *Dokl. Akad. Nauk Ukrainian SSR*, 1991, N 3, 23–27.

[6] Fushchych W.I., Shtelen V.M. and Serov N.I. Symmetry Analysis and Exact Solutions of Equations of Nonlinear Mathematical Physics, Kluwer Academic Publishers, Dordrecht, 1993.