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A Class of Representations of the $*$-Algebra of the Canonical Commutation Relations over a Hilbert Space and Instability of Embedded Eigenvalues in Quantum Field Models

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Dedicated to Professor Wilhelm Fushchych on the occasion of his sixtieth birthday

Abstract

In models of a quantum harmonic oscillator coupled to a quantum field with a quadratic interaction, embedded eigenvalues of the unperturbed system may be unstable under the perturbation given by the interaction of the oscillator with the quantum field. A general mathematical structure underlying this phenomenon is clarified in terms of a class of Fock space representations of the $*$-algebra of the canonical commutation relations over a Hilbert space. It is also shown that each of the representations is given as a composition of a proper Bogolyubov (canonical) transformation and a partial isometry on the Fock space of the representation.

1 Introduction

In the spectral analysis of models of an atom coupled to the quantized radiation field, one meets a difficult problem, i.e., a perturbation problem of embedded eigenvalues [1], to which the standard regular perturbation theory cannot be applied. To solve this problem with mathematical rigor is very important for a complete understanding of physical phenomena of atoms such as the Lamb shift and emission and absorption of light as well as for establishing a mathematically rigorous foundation of quantum electrodynamics. A traditional and informal picture is that the embedded eigenvalues of an atomic Hamiltonian except the lowest one (the ground state energy) should be unstable, i.e., they should disappear under the perturbation given by the interaction with the quantized radiation field, each forming a resonance pole whose real and imaginary parts explain, respectively, the Lamb shift and decay rate of the corresponding excited state. Partial solutions to the problem have been given so far [2, 3, 4, 18, 5, 6, 7, 8, 14, 15, 16, 9, 11, 17, 12, 10, 23],

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establishing a firm mathematical basis and a mathematically deeper understanding for the traditional picture.

In a series of papers [2, 3, 4, 5, 6], we considered models of a quantum harmonic oscillator coupled to a quantum scalar field or a quantized radiation field with a quadratic interaction, which may serve as simplified models of an atom coupled to the quantized radiation field, and showed that the instability phenomenon of embedded eigenvalues as described above occurs in those models, although it may depend on the size of parameters contained in their Hamiltonians. In [7], a general structure underlying the instability phenomenon of embedded eigenvalues in a class of models, which include the ones in [2, 3, 4, 5, 6], was analyzed in terms of the notion of *noninvertible Bogolyubov (canonical) transformation* (NIBT). The purpose of the present paper is to clarify the nature of the NIBT, which is not discussed in [7]. We show that the NIBT is a composition of a proper Bogolyubov transformation and a partial isometry on the relevant Fock space.

In Section 2, we review the NIBT from a representation theoretic viewpoint. We consider it as a representation, on the symmetric Fock space $\mathcal{F}_{\rm s}(\mathcal{H})$ over a Hilbert space $\mathcal{H}$, of the $*$-algebra of the canonical commutation relations (CCR) over *another* Hilbert space $\mathcal{K}$. In Section 3, we describe a connection of the representation of the $*$-algebra of the CCR introduced in Section 2 with the embedded eigenvalue problem in a class of quantum field models with quadratic interactions. The characterization of the NIBT mentioned above is given in Section 4.

2 A class of representations of the $*$-algebra of the CCR over a Hilbert space

For a densely defined linear operator T on a Hilbert space, we denote by $D(T)$ and T^* the domain and adjoint of T, respectively. Let a separable complex Hilbert space $\mathcal{H}$ be given. A triple $\{\mathcal{F}, \mathcal{D}, \{a(f)|f \in \mathcal{H}\}\}$ consisting of a complex Hilbert space $\mathcal{F}$, a dense subspace $\mathcal{D}$ of $\mathcal{F}$ and an antilinear mapping $a : f \to a(f)$ from $\mathcal{H}$ to the set of closed linear operators on $\mathcal{F}$ is called a representation of the $*$-algebra of the CCR over $\mathcal{H}$ if the following (i) and (ii) hold: (i) $\mathcal{D} \subset \cap_{f\in\mathcal{H}} D(a(f)) \cap D(a(f)^*)$, $a(f)\mathcal{D} \subset \mathcal{D}$, $a(f)^*\mathcal{D} \subset \mathcal{D}$ for all $f \in \mathcal{H}$; (ii) $\{a(f)|f \in \mathcal{H}\}$ fulfil the CCR over $\mathcal{H}$

$$[a(f), a(g)^*] = (f, g)_{\mathcal{H}}, \quad [a(f), a(g)] = 0, \quad f, g \in \mathcal{H}, \tag{2.1}$$

on $\mathcal{D}$, where $(\ \cdot\ ,\ \cdot\ )_{\mathcal{H}}$ denotes the inner product of $\mathcal{H}$.

Let

$$\mathcal{F}_{\rm s}(\mathcal{H}) = \bigoplus_{n=0}^{\infty} \otimes_{\rm s}^n \mathcal{H} \tag{2.2}$$

be the symmetric (boson) Fock space over $\mathcal{H}$, where $\otimes_{\rm s}^n \mathcal{H}$ denotes the n-fold symmetric tensor product Hilbert space of $\mathcal{H}$ with convention $\otimes_{\rm s}^0 \mathcal{H} = \mathbf{C}$. We denote by $\Omega_{\mathcal{H}}$ the Fock vacuum in $\mathcal{F}_{\rm s}(\mathcal{H})$ and by $a_{\mathcal{H}}(f), f \in \mathcal{H}$, the annihilation operators on $\mathcal{F}_{\rm s}(\mathcal{H})$ (antilinear in f), which are closed linear operators. We introduce a dense subspace

$$\mathcal{F}_{\rm fin}(\mathcal{H}) := \mathcal{L}\Big\{\Omega_{\mathcal{H}},\, a_{\mathcal{H}}(f_1)^* \cdots a_{\mathcal{H}}(f_n)^*\Omega_{\mathcal{H}}|\ n \geq 1,\, f_j \in \mathcal{H},\, j = 1, \ldots, n\Big\}, \tag{2.3}$$

where $\mathcal{L}\{\cdots\}$ denotes the subspace algebraically spanned by vectors in the set $\{\cdots\}$. Then $\{\mathcal{F}_{\mathrm{s}}(\mathcal{H}), \mathcal{F}_{\mathrm{fin}}(\mathcal{H}), \{a_{\mathcal{H}}(f)|f \in \mathcal{H}\}\}$ is a representation of the $*$-algebra of the CCR over $\mathcal{H}$, which is called the Fock representation of the $*$-algebra of the CCR over $\mathcal{H}$.

We denote by N_b the number operator on $\mathcal{F}_{\mathrm{s}}(\mathcal{H})$. It is well known that, for all $f \in \mathcal{H}$, $D(N_b^{1/2}) \subset D(a_{\mathcal{H}}(f)) \cap D(a_{\mathcal{H}}(f)^*)$ and, for all $f \in \mathcal{H}$ and $\Psi \in D(N_b^{1/2})$,

$$\|a_{\mathcal{H}}(f)^{\#}\Psi\|_{\mathcal{F}_{\mathrm{s}}(\mathcal{H})} \leq \|f\|_{\mathcal{H}} \|(N_b+1)^{1/2}\Psi\|_{\mathcal{F}_{\mathrm{s}}(\mathcal{H})}, \tag{2.4}$$

where $a_{\mathcal{H}}(\cdot)^{\#}$ denotes either $a_{\mathcal{H}}(\cdot)$ or $a_{\mathcal{H}}(\cdot)^*$.

In what follows we consider the case where $\mathcal{H}$ is given by the direct sum of two Hilbert spaces $\mathcal{K}$ and $\mathcal{M}$ with $\mathcal{K} \neq \{0\}$ and $\mathcal{M} \neq \{0\}$:

$$\mathcal{H} = \mathcal{K} \oplus \mathcal{M} = \Big\{(u,v)|u \in \mathcal{K}, v \in \mathcal{M}\Big\}. \tag{2.5}$$

Then we have the natural identification

$$\mathcal{F}_{\mathrm{s}}(\mathcal{H}) = \mathcal{F}_{\mathrm{s}}(\mathcal{K}) \otimes \mathcal{F}_{\mathrm{s}}(\mathcal{M}). \tag{2.6}$$

Remark 2.1 In applications to models of a quantum harmonic oscillator coupled to a quantum field, the Hilbert sapces $\mathcal{K}$ and $\mathcal{M}$ are taken as $\mathcal{K} = \oplus^m L^2(\mathbf{R}^d)$, $\mathcal{M} = \mathbf{C}^N$, with d, m, $N \in \mathbf{N}$. Then we have

$$\mathcal{F}_{\mathrm{s}}(\mathcal{H}) = \mathcal{F}_{\mathrm{s}}(\oplus^m L^2(\mathbf{R}^d)) \otimes \mathcal{F}_{\mathrm{s}}(\mathbf{C}^N) = \mathcal{F}_{\mathrm{s}}(\oplus^m L^2(\mathbf{R}^d)) \otimes L^2(\mathbf{R}^N).$$

Let $J_{\mathcal{K}}$ and $J_{\mathcal{M}}$ be conjugations on $\mathcal{K}$ and $\mathcal{M}$, respectively, and define

$$J_{\mathcal{H}} := J_{\mathcal{K}} \oplus J_{\mathcal{M}}, \tag{2.7}$$

which is a conjugation on $\mathcal{H}$. For a linear operator T on $\mathcal{H}$ and $f \in \mathcal{H}$, we set

$$T_c := J_{\mathcal{H}} T J_{\mathcal{H}}, \quad \bar{f} := J_{\mathcal{H}} f. \tag{2.8}$$

For two Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$, we denote by $\mathsf{B}(\mathcal{H}_1, \mathcal{H}_2)$ the space of bounded linear operators from $\mathcal{H}_1$ to $\mathcal{H}_2$ and set $\mathsf{B}(\mathcal{H}_1) = \mathsf{B}(\mathcal{H}_1, \mathcal{H}_1)$.

Let S and T be elements in $\mathsf{B}(\mathcal{K}, \mathcal{H})$ which satisfy

$$S^*S - T^*T = I_{\mathcal{K}}, \quad S^*T_c - T^*S_c = 0, \tag{2.9}$$

where $I_{\mathcal{K}}$ denotes the identity operator on $\mathcal{K}$. For each $u \in \mathcal{K}$, we define an operator $b(u)$ acting in $\mathcal{F}_{\mathrm{s}}(\mathcal{H})$ by

$$b(u) = a_{\mathcal{H}}(Su) + a_{\mathcal{H}}(T_c\bar{u})^*. \tag{2.10}$$

with $D(b(u)) = D(N_b^{1/2})$. It follows that $D(N_b^{1/2}) \subset D(b(u)^*)$ for all $u \in \mathcal{K}$. Hence $b(u)$ is closable. We denote its closure by the same symbol $b(u)$, so that $D(N_b^{1/2}) \subset D(b(u))$. We have

$$b(u)^* = a_{\mathcal{H}}(Su)^* + a_{\mathcal{H}}(T_c\bar{u}) \tag{2.11}$$

on $D(N_b^{1/2})$.

Proposition 2.1 *The triple* $\{\mathcal{F}_{\rm s}(\mathcal{H}), \mathcal{F}_{\rm fin}(\mathcal{H}), \{b(u)|u \in \mathcal{K}\}\}$ *is a representation of the* $*$*-algebra of the CCR over* $\mathcal{K}$.

The triple $\{\mathcal{F}_{\rm s}(\mathcal{H}), \mathcal{F}_{\rm fin}(\mathcal{H}), \{a(u,0)|u \in \mathcal{K}\}\}$ is a representation of the $*$-algebra of the CCR over $\mathcal{K}$. But, as the following proposition shows, this representation is not equivalent in general to the representation $\{\mathcal{F}_{\rm s}(\mathcal{H}), \mathcal{F}_{\rm fin}(\mathcal{H}), \{b(u)|u \in \mathcal{K}\}\}$.

Proposition 2.2 [7, Proposition 3.1] *Suppose that* $\dim \cap_{u\in\mathcal{K}} \ker b(u) < \infty$ *and there exist bounded linear operators* U, V *on* $\mathcal{F}_{\rm s}(\mathcal{H})$ *such that, for all* $u \in \mathcal{K}$, $b(u) = Ua(u,0)V$. *Then* V *is not invertible.*

Remark 2.2 The mapping $a(\cdot,0) \to b(\cdot)$ may be regarded as a Bogolyubov transformation in the Fock space $\mathcal{F}_{\rm s}(\mathcal{H})$. From this point of view, under the assumption of Proposition 2.2, the Bogolyubov transformation is *noninvertible*. This is a *different* type of Bogolyubov transformations from the usual ones as discussed in, e.g., [13], [21, 22].

Remark 2.3 Under identification (2.6), we have

$$a_{\mathcal{H}}(f)^{\#} = a_{\mathcal{K}}(u)^{\#} \otimes I_{\mathcal{F}_{\rm s}(\mathcal{M})} + I_{\mathcal{F}_{\rm s}(\mathcal{K})} \otimes a_{\mathcal{M}}(v)^{\#}, \quad f = (u,v) \in \mathcal{H}, \tag{2.12}$$

on $D(N_b^{1/2})$. There exist operators $W, V \in \mathsf{B}(\mathcal{K})$ and $P, Q \in \mathsf{B}(\mathcal{K}, \mathcal{M})$ such that

$$Su = (Wu, Qu), \quad Tu = (Vu, Pu), \quad u \in \mathcal{K}, \tag{2.13}$$

where W and Q (resp., V and P) are uniquely determined by S (resp. T). Hence, we have

$$\begin{aligned} b(u) &= a_{\mathcal{K}}(Wu) \otimes I_{\mathcal{F}_{\rm s}(\mathcal{M})} + I_{\mathcal{F}_{\rm s}(\mathcal{K})} \otimes a_{\mathcal{M}}(Qu) \\ &\quad + a_{\mathcal{K}}(V_c\bar{u})^* \otimes I_{\mathcal{F}_{\rm s}(\mathcal{M})} + I_{\mathcal{F}_{\rm s}(\mathcal{K})} \otimes a_{\mathcal{M}}(P_c\bar{u})^* \end{aligned} \tag{2.14}$$

on $D(N_b^{1/2})$. This is the original form of the NIBT discussed in [7][1].

Under additional conditions, one can express $a_{\mathcal{H}}(\cdot)$ in terms of $b(\cdot)$ and $b(\cdot)^*$:

Proposition 2.3 *Suppose that* S *and* T *satisfy, in addition to (2.9),*

$$SS^* - T_cT_c^* = I_{\mathcal{H}}, \quad T_cS_c^* - ST^* = 0. \tag{2.15}$$

Then, for all $f \in \mathcal{H}$,

$$a_{\mathcal{H}}(f) = b(S^*f) - b(T^*\bar{f})^*, \quad a_{\mathcal{H}}(f)^* = b(S^*f)^* - b(T^*\bar{f}). \tag{2.16}$$

on $D(N_b^{1/2})$.

The Segal field operators

$$\phi_{\mathcal{H}}(f) := \frac{1}{\sqrt{2}}(a_{\mathcal{H}}(f) + a_{\mathcal{H}}(f)^*), \quad f \in \mathcal{H}, \tag{2.17}$$

[1] In the paper [7], $b(\cdot)$, $a_{\mathcal{H}}(\cdot)$, $a_{\mathcal{M}}(\cdot)$ are denoted $B(\cdot), b(\cdot), a(\cdot)$, respectively.

are essentially self-adjoint on $\mathcal{F}_{\rm fin}(\mathcal{H})$[20, Theorem X.41]. We denote the closure of $\phi_{\mathcal{H}}(f)$ by $\overline{\phi_{\mathcal{H}}(f)}$. An analogue of the Segal field operator is defined in the representation $\{\mathcal{F}_{\rm s}(\mathcal{H}), \mathcal{F}_{\rm fin}(\mathcal{H}), \{b(u)|u \in \mathcal{K}\}$:

$$\Phi(u) := \frac{1}{\sqrt{2}}(b(u) + b(u)^*), \quad u \in \mathcal{K}. \tag{2.18}$$

By (2.10) and (2.11), we have $\Phi(u) = \phi_{\mathcal{H}}(Su + T_c\bar{u})$ on $D(N_b^{1/2})$. Hence, for all $u \in \mathcal{K}$, $\Phi(u)$ is essentially self-adjoint on $\mathcal{F}_{\rm fin}(\mathcal{H})$ and

$$\overline{\Phi(u)} = \overline{\phi_{\mathcal{H}}(Su + T_c\bar{u})}, \quad u \in \mathcal{K}. \tag{2.19}$$

For all $f \in \mathcal{H}$, $a_{\mathcal{H}}(f)^\#$ leaves the dense subspace

$$C^\infty(N_b) := \bigcap_{k=1}^{\infty} D(N_b^k) \tag{2.20}$$

invariant and so does $b(u)^\#$for all $u \in \mathcal{K}$.

We denote by $\mathcal{I}_2(\mathcal{K}, \mathcal{H})$ the space of Hilbert-Schmidt operators from $\mathcal{K}$ to $\mathcal{H}$.

Definition 2.4 Let $S, T \in \mathsf{B}(\mathcal{K}, \mathcal{H})$. We say that the pair $\langle S, T\rangle$ is in the set $\mathsf{S}(\mathcal{K}, \mathcal{H})$ if S and T satisfy (2.9), (2.15) and $T \in \mathcal{I}_2(\mathcal{K}, \mathcal{H})$.

Fundamental properties of the representation $\{\mathcal{F}_{\rm s}(\mathcal{H}), \mathcal{F}_{\rm fin}(\mathcal{H}), \{b(u)|u \in \mathcal{K}\}\}$ are summarized in the following theorem.

Theorem 2.5 *Let $\langle S, T\rangle \in \mathsf{S}(\mathcal{K}, \mathcal{H})$. Then there exist a unit vector $\Psi_0 \in \mathcal{F}_{\rm s}(\mathcal{H})$ and a unitary transformation $U : \mathcal{F}_{\rm s}(\mathcal{H}) \to \mathcal{F}_{\rm s}(\mathcal{K})$ such that the following* (a)–(d) *hold:*

(a) $\Psi_0 \in C^\infty(N_b)$ *and, for all $u \in \mathcal{K}$, $b(u)\Psi_0 = 0$.*

(b) *The subspace $\mathcal{L}\{\Psi_0, b(u_1)^* \cdots b(u_n)^*\Psi_0 | n \geq 1, u_j \in \mathcal{K}, j = 1, \cdots, n\}$ is dense in $\mathcal{F}_{\rm s}(\mathcal{H})$.*

(c) *$U\Psi_0 = \Omega_{\mathcal{K}}$ and $Ub(u_1)^* \cdots b(u_n)^*\Psi_0 = a_{\mathcal{K}}(u_1)^* \cdots a_{\mathcal{K}}(u_n)^*\Omega_{\mathcal{K}}$ for all $n \geq 1$, $u_j \in \mathcal{K}$, $j = 1, \cdots, n$.*

(d) *For all $u \in \mathcal{K}$, $U\overline{\Phi(u)}U^{-1} = \overline{\phi_{\mathcal{K}}(u)}$, $\quad Ub(u)U^{-1} = a_{\mathcal{K}}(u)$.*

Moreover, Ψ_0 is the only one (up to scalar multipliers) of vectors Ψ such that $b(u)\Psi = 0$ for all $u \in \mathcal{K}$.

Proof. Similar to the proof of [7, Theorem 3.4]. □

3 Hamiltonians

By using the representation $\{\mathcal{F}_{\rm s}(\mathcal{H}), \mathcal{F}_{\rm fin}(\mathcal{H}), \{b(u)|u \in \mathcal{K}\}\}$, we can construct a self-adjoint Hamiltonian acting in $\mathcal{F}_{\rm s}(\mathcal{H})$ whose spectrum can be exactly identified. In application to the embedded eigenvalue problem mentioned in the Introduction, this class of Hamiltonians gives a class of exactly soluble models [6, 7]. In this section we briefly review this aspect [7].

For every $K \in \mathcal{I}_2(\mathcal{H},\mathcal{H})$, there exists a unique closed linear operator $\langle a_{\mathcal{H}}|K|a_{\mathcal{H}}\rangle$ acting in $\mathcal{F}_{\rm s}(\mathcal{H})$ such that $\mathcal{F}_{\rm fin}(\mathcal{H})$ is a core of it, $\langle a_{\mathcal{H}}|K|a_{\mathcal{H}}\rangle\Omega_{\mathcal{H}} = 0, \langle a_{\mathcal{H}}|K|a_{\mathcal{H}}\rangle a(f)^*\Omega_{\mathcal{H}} = 0,\ f \in \mathcal{H}$, and, for all $n \geq 2, f_j \in \mathcal{H},\ j = 1,\cdots,n$,

$$\langle a_{\mathcal{H}}|K|a_{\mathcal{H}}\rangle a_{\mathcal{H}}(f_1)^*\cdots a_{\mathcal{H}}(f_n)^*\Omega_{\mathcal{H}}$$
$$= \sum_{i\neq j}^{n}(\bar{f}_i, Kf_j)_{\mathcal{H}} a_{\mathcal{H}}(f_1)^*\cdots a_{\mathcal{H}}\widehat{(f_i)}^*\cdots a_{\mathcal{H}}\widehat{(f_j)}^*\cdots a_{\mathcal{H}}(f_n)^*\Omega_{\mathcal{H}},$$

where $a\widehat{(f_i)}^*$ indicates the omission of $a\widehat{(f_i)}^*$. Also one can define a closed linear operator $\langle a^*_{\mathcal{H}}|K|a^*_{\mathcal{H}}\rangle$ acting in $\mathcal{F}_{\rm s}(\mathcal{H})$ such that $\mathcal{F}_{\rm fin}(\mathcal{H})$ is a core of it and $\langle a_{\mathcal{H}}|K|a_{\mathcal{H}}\rangle^* = \langle a^*_{\mathcal{H}}|K^*|a^*_{\mathcal{H}}\rangle$ on $\mathcal{F}_{\rm fin}(\mathcal{H})$ (for details, see [7, §II]).

For a self-adjoint operator A on $\mathcal{H}$, we denote by $d\Gamma_{\mathcal{H}}(A)$ the second quantization operator on $\mathcal{F}_{\rm s}(\mathcal{H})$ [19, p.302, Example 2].

We say that a densely defined linear operator on a Hilbert space $\mathcal{W}$ is Hilbert-Schmidt if it is uniquely extended to a Hilbert-Schmidt operator on $\mathcal{W}$. Let $\langle S,\, T\rangle \in \mathsf{S}(\mathcal{K},\mathcal{H})$ and h be a nonnegative self-adjoint operator on $\mathcal{K}$ such that $h = h_c$ and the following (h.1)–(h.3) hold:

(h.1) The subspace $\mathcal{H}_0 := \{f \in \mathcal{H}| S^*f, T_c^*f \in D(h)\}$ is dense in $\mathcal{H}$.

(h.2) The densely defined operators ThS^* and $Th^{1/2}$ are Hilbert-Schmidt on $\mathcal{H}$.

(h.3) The subspace $D_S(h) := \{u \in D(h)| S^*Su \in D(h)\}$ is a core of h.

It follows that $ShS^* + T_chT_c^*$ is densely defined, hence, it is a symmetric operator on $\mathcal{H}$ and ShT^* is Hilbert-Schmidt.

We define

$$H := d\Gamma_{\mathcal{H}}(\overline{ShS^* + T_chT_c^*}) + \langle a_{\mathcal{H}}|\overline{ThS^*}|a_{\mathcal{H}}\rangle + \langle a_{\mathcal{H}}|\overline{ThS^*}|a_{\mathcal{H}}\rangle^*, \tag{3.1}$$

and

$$E := -\|Th^{1/2}\|^2_{\rm HS} < 0, \tag{3.2}$$

where $\|\cdot\|_{\rm HS}$ denotes the Hilbert-Schmidt norm. The operator H gives an abstract form unifying Hamiltonians of models of a quantum harmonic oscillator coupled to a quantized field [2, 3, 4, 5, 6, 7, 8]. We can prove the following fact.

Theorem 3.1 [7, Theorem 4.2][2] *The operator H is essentially self-adjoint on the subspace*

$$\mathcal{F}_{\rm fin}(\mathcal{H}_0) := \mathcal{L}\{\Omega_{\mathcal{H}},\, a_{\mathcal{H}}(f_1)^*\cdots a_{\mathcal{H}}(f_n)^*\Omega_{\mathcal{H}}|\, n \geq 1, f_j \in \mathcal{H}_0,\, j = 1,\cdots,n\} \tag{3.3}$$

and its closure $\bar{H}$ is unitarily equivalent to $d\Gamma_{\mathcal{K}}(h) + E$ under the unitary transformation U given in Theorem 2.5: $U\bar{H}U^{-1} = d\Gamma_{\mathcal{K}}(h) + E$. In particular, $\bar{H}$ has a unique ground state given by the vector Ψ_0 (up to constant multipliers) with the ground state energy E.

[2]We would like to make a correction to the paper [7]: in the definition of the operator H in §IV of [7], the condition that $\{u \in D(h)|(W^*W + Q^*Q)u \in D(h)\}$ is a core of h also should be assumed, which corresponds to (h.3) in the present context, since we need this property for the proof of [7, Lemma 4.4].

In concrete models, the unperturbed Hamiltonian H_0 is of the form

$$H_0 = d\Gamma_{\mathcal{H}}(h \oplus \omega) = d\Gamma_{\mathcal{K}}(h) \otimes I_{\mathcal{F}_{\rm s}(\mathcal{M})} + I_{\mathcal{F}_{\rm s}(\mathcal{K})} \otimes d\Gamma_{\mathcal{M}}(\omega), \tag{3.4}$$

where ω is a nonnegative self-adjoint operator on $\mathcal{M}$. We write

$$H = H_0 + H_I \tag{3.5}$$

with

$$H_I = d\Gamma_{\mathcal{H}}(\overline{ShS^* + T_c h T_c^*}) - d\Gamma_{\mathcal{H}}(h \oplus \omega) + \langle a_{\mathcal{H}} | \overline{ThS^*} | a_{\mathcal{H}} \rangle + \langle a_{\mathcal{H}} | \overline{ThS^*} | a_{\mathcal{H}} \rangle^*. \tag{3.6}$$

For this form of H, Theorem 3.1 implies the following. For a self-adjoint operator T, we denote by $\sigma(T)$ (resp., $\sigma_{\rm p}(T)$) the spectrum (resp., the point spectrum) of T. Consider the case where $\sigma(h)$ is purely continuous with $\sigma(h) = [m, \infty)$ ($m \geq 0$: a constant), $\sigma_{\rm p}(h) = \emptyset$, and $\sigma(\omega)$ is purely discrete with $\sigma(\omega) = \{\omega_n\}_{n=1}^{\infty}$ such that $0 \leq \omega_1 < \omega_2 < \cdots < \omega_n \to \infty$ ($n \to \infty$). Then we have

$$\begin{aligned} \sigma(d\Gamma_{\mathcal{K}}(h)) &= \{0\} \cup [m, \infty), \quad \sigma_{\rm p}(d\Gamma_{\mathcal{K}}(h)) = \{0\}, & (3.7)\\ \sigma(d\Gamma_{\mathcal{M}}(\omega)) &= \sigma_{\rm p}(d\Gamma_{\mathcal{M}}(\omega)) = \{E_n\}_{n=0}^{\infty} & (3.8)\end{aligned}$$

with $E_0 = 0$ and $E_n > 0$, $n \geq 1$, where each E_n with $n \geq 1$ is determined by ω_j, $j \geq 1$. Hence,

$$\sigma(H_0) = \{E_n\}_{n=0}^{\infty} \cup [m, \infty), \quad \sigma_{\rm p}(H_0) = \{E_n\}_{n=0}^{\infty}, \tag{3.9}$$

which mean that each E_n is an eigenvalue of H_0 and the eigenvalues $E_n \geq m$ are embedded in the continuous spectrum of H_0. On the other hand, Theorem 3.1 implies that

$$\sigma(\bar{H}) = \{E\} \cup [E + m, \infty), \quad \sigma_{\rm p}(\bar{H}) = \{E\}. \tag{3.10}$$

Hence, all the embedded eigenvalues $E_n \geq m$ turn out to disappear under the perturbation H_I, i.e., they are unstable under the perturbation H_I (we may regard $E_n < m$ as eigenvalues changing to E or $E + m$ under the perturbation H_I). Thus, $\bar{H}$ gives, in an abstract form, a class of self-adjoint operators acting in the Fock space $\mathcal{F}_{\rm s}(\mathcal{H})$, which describe the instability phenomenon of embedded eigenvalues.

Remark 3.1 In concrete realizations of $\bar{H}$, the operators S and T contain a parameter $\lambda \in \mathbf{R}$, which physically denotes a coupling constant, where $\lambda = 0$ corresponds to the case of no interaction. Write $S = S(\lambda)$, $T = T(\lambda)$ and let $H(\lambda)$ be the operator H with S and T replaced by $S(\lambda)$ and $T(\lambda)$, respectively. In the examples we know, we have

$$\lim_{\lambda \to 0} [S(\lambda) h S^*(\lambda) + T_c(\lambda) h T_c(\lambda)^*] = h \oplus \omega, \quad \lim_{\lambda \to 0} T(\lambda) h S^*(\lambda) = 0$$

strongly on a suitable dense domain, so that $\lim_{\lambda \to 0} (\Psi, H(\lambda)\Phi) = (\Psi, H_0 \Phi)$ for all Ψ, Φ in a suitable dense subspace of $\mathcal{F}_{\rm s}(\mathcal{H})$. In this sense, writing H as (3.5) is not artificial.

Remark 3.2 In concrete models, their Hamiltonians are given first. Hence, the following inverse problem may be interesting and important: Let a nonnegative self-adjoint operator A on $\mathcal{H}$ and $K_1, K_2 \in \mathcal{I}_2(\mathcal{H}, \mathcal{H})$ be given such that

$$H' := d\Gamma_{\mathcal{H}}(A) + d\Gamma_{\mathcal{H}}(K_1) + \langle a_{\mathcal{H}}|K_2|a_{\mathcal{H}}\rangle + \langle a_{\mathcal{H}}|K_2|a_{\mathcal{H}}\rangle^*$$

is the Hamiltonian of a model. Then, find a general solution $\{h, S, T\}$ to the operator equations

$$\overline{ShS^* + T_c h T_c^*} = A + K_1, \quad \overline{ThS^*} = K_2,$$

such that $\langle S, T\rangle \in \mathsf{S}(\mathcal{K}, \mathcal{H})$ and h is a nonnegative self-adjoint operator on $\mathcal{K}$. If this problem is solved with full generality, then one has a complete solution to embedded eigenvalue problems arising in Hamiltonians of quantum mechanical particles and quantum fields with *quadratic interactions*.

4 A characterization

In this section we give a characterization of the representation $\{b(u)|u \in \mathcal{K}\}$ as stated in the Introduction. This is a new result.

We write each vector $f \in \mathcal{H}$ as $f = (f_{\mathcal{K}}, f_{\mathcal{M}})$ with $f_{\mathcal{K}} \in \mathcal{K}$ and $f_{\mathcal{M}} \in \mathcal{M}$. For each $A \in \mathsf{B}(\mathcal{K}, \mathcal{H})$, we can define an operator $\widetilde{A} \in \mathsf{B}(\mathcal{H})$ by

$$\widetilde{A}f = Af_{\mathcal{K}}, \quad f \in \mathcal{H}. \tag{4.1}$$

Then it is easy to see that, for all $f \in \mathcal{H}$,

$$\widetilde{A}^* f = (A^* f, 0) \tag{4.2}$$

and, for all $A, B \in \mathsf{B}(\mathcal{K}, \mathcal{H})$,

$$\widetilde{A}\widetilde{B}^* = AB^*, \tag{4.3}$$

$$\widetilde{B}^*\widetilde{A}f = (B^* A f_{\mathcal{K}}, 0), \quad f \in \mathcal{H}. \tag{4.4}$$

Let $\langle S, T\rangle \in \mathsf{S}(\mathcal{K}, \mathcal{H})$ and $P_{\mathcal{K}}$ be the orthogonal projection from $\mathcal{H}$ onto $\mathcal{K}$. Then, by direct computations, using (4.3), (4.4), (2.9) and (2.15), we can prove the following relations:

$$\widetilde{S}^*\widetilde{S} - \widetilde{T}^*\widetilde{T} = P_{\mathcal{K}}, \quad \widetilde{S}^*\widetilde{T}_c - \widetilde{T}^*\widetilde{S}_c = 0, \tag{4.5}$$

$$\widetilde{S}\widetilde{S}^* - \widetilde{T}_c\widetilde{T}_c^* = I_{\mathcal{H}}, \quad \widetilde{T}_c\widetilde{S}_c^* - \widetilde{S}\widetilde{T}^* = 0. \tag{4.6}$$

Let $L \in \mathsf{B}(\mathcal{H})$ be such that

$$L^*L = P_{\mathcal{K}}, \quad LL^* = I_{\mathcal{H}}, \tag{4.7}$$

i.e., L is a partial isometry on $\mathcal{H}$ with the initial space $\mathcal{K}$ and final space $\mathcal{H}$. We define $X, Y \in \mathsf{B}(\mathcal{H})$ by

$$X = \widetilde{S}L^*, \quad Y = \widetilde{T}L^*. \tag{4.8}$$

Lemma 4.1 *The following relations hold:*

$$X^*X - Y^*Y = I_{\mathcal{H}}, \quad X^*Y_c - Y^*X_c = 0, \tag{4.9}$$
$$XX^* - Y_cY_c^* = I_{\mathcal{H}}, \quad Y_cX_c^* - XY^* = 0. \tag{4.10}$$

Moreover, $Y \in \mathcal{I}_2(\mathcal{H})$.

Proof. For all $A \in \mathsf{B}(\mathcal{K}, \mathcal{H})$, $P_{\mathcal{K}}\widetilde{A}^* = \widetilde{A}^*$. Using this fact together with (4.5), (4.6), (4.7), and (4.8), one can prove (4.9) and (4.10). Let $\{f_n\}_{n=1}^{\infty}$ be a complete orthonormal system (CONS) of $\mathcal{H}$. Then $\{L^*f_n\}_{n=1}^{\infty}$ is a CONS of $\mathcal{K}$ as a closed subspace of $\mathcal{H}$. Since the range of L^* is equal to $\mathcal{K} \oplus \{0\}$, we can write $L^*f_n = (u_n, 0)$ with $u_n \in \mathcal{K}$. Then $\{u_n\}_{n=1}^{\infty}$ is a CONS of $\mathcal{K}$. We have $\sum_{n=1}^{\infty} \|Yf_n\|_{\mathcal{H}}^2 = \sum_{n=1}^{\infty} \|Tu_n\|_{\mathcal{H}}^2 < \infty$, since $T \in \mathcal{I}_2(\mathcal{K}, \mathcal{H})$. Hence, $Y \in \mathcal{I}_2(\mathcal{H})$. □

Remark 4.1 Let $X, Y \in \mathsf{B}(\mathcal{H})$. We say that the pair $\langle X, Y\rangle$ is in the set $\mathsf{S}(\mathcal{H})$ if X and Y have the properties stated in Lemma 4.1. Let $L \in \mathsf{B}(\mathcal{H})$ satisfying (4.7) be fixed. Define a mapping $F : \mathsf{S}(\mathcal{K}, \mathcal{H}) \to \mathsf{S}(\mathcal{H})$ by

$$F\langle S, T\rangle = \langle X, Y\rangle \tag{4.11}$$

with X and Y given by (4.8). Then it is easy to see that F is bijective with $F^{-1}\langle X, Y\rangle = \langle S', T'\rangle$, where S', $T' \in \mathsf{B}(\mathcal{K}, \mathcal{H})$ are defined by $S'u = XL(u, 0),\ T'u = YL(u, 0),\ u \in \mathcal{K}$.

For each $f \in \mathcal{H}$, we define an operator $c(f)$ by

$$c(f) = a_{\mathcal{H}}(Xf) + a_{\mathcal{H}}(Y_c\bar{f})^*, \tag{4.12}$$

with $D(c(f)) = D(N_b^{1/2})$, which is closable. We denote its closure by the same symbol. We have

$$c(f)^* = a_{\mathcal{H}}(Y_c\bar{f}) + a_{\mathcal{H}}(Xf)^*, \quad f \in \mathcal{H}, \tag{4.13}$$

on $D(N_b^{1/2})$.

Theorem 4.2 *The mapping* $\{a_{\mathcal{H}}, a_{\mathcal{H}}^*\} \to \{c, c^*\}$ *is a proper Bogolyubov transformation on* $\mathcal{F}_{\mathrm{s}}(\mathcal{H})$*. In particular, there exists a unitary operator* $U_{\mathcal{H}}$ *on* $\mathcal{F}_{\mathrm{s}}(\mathcal{H})$ *such that, for all* $f \in \mathcal{H}$*,*

$$c(f) = U_{\mathcal{H}}a_{\mathcal{H}}(f)U_{\mathcal{H}}^{-1}, \quad c(f)^* = U_{\mathcal{H}}a_{\mathcal{H}}(f)^*U_{\mathcal{H}}^{-1}, \tag{4.14}$$

Proof. This follows from Lemma 4.1 and a well-known fact (cf. [13, Chapter 2, §4, Theorem 4.1]). □

Remark 4.2 It is possible to represent $U_{\mathcal{H}}$ in an explicit form (cf. [13, Chapter 2, §4, Theorem 4.3]).

Corollary 4.3 *For all $u \in \mathcal{K}$,*

$$b(u) = U_{\mathcal{H}} a_{\mathcal{H}}(L(u,0)) U_{\mathcal{H}}^{-1}, \quad b(u)^* = U_{\mathcal{H}} a_{\mathcal{H}}(L(u,0))^* U_{\mathcal{H}}^{-1}. \tag{4.15}$$

Proof. By (2.10) and (4.12), we have

$$b(u) = c(L(u,0)), \quad u \in \mathcal{K}. \tag{4.16}$$

which, together with (4.14), implies (4.15). □

To express $a_{\mathcal{H}}(L(\,\cdot\,,0))$ as a transformation of $a_{\mathcal{H}}(\,\cdot\,,0)$, we recall a general notion. Let $\mathcal{H}_1, \mathcal{H}_2$ be Hilbert spaces and $C \in \mathsf{B}(\mathcal{H}_1, \mathcal{H}_2)$ be a contraction operator. Then we define a contraction linear opeartor $\Gamma_{\mathcal{H}_1,\mathcal{H}_2}(C) : \mathcal{F}_{\rm s}(\mathcal{H}_1) \to \mathcal{F}_{\rm s}(\mathcal{H}_2)$ by

$$\Gamma_{\mathcal{H}_1,\mathcal{H}_2}(C) = \oplus_{n=0}^{\infty}(\otimes^n C) \tag{4.17}$$

with convention $\otimes^0 C = 1$, where $\otimes^n C$ denotes the n-fold tensor product of C. In the case where C is a contraction operator on a single Hilbert space $\mathcal{H}_1$, we set

$$\Gamma_{\mathcal{H}_1}(C) = \Gamma_{\mathcal{H}_1,\mathcal{H}_1}(C). \tag{4.18}$$

Lemma 4.4 *Let $C \in \mathsf{B}(\mathcal{H}_1, \mathcal{H}_2)$ and $B \in \mathsf{B}(\mathcal{H}_2, \mathcal{H}_1)$ be contraction operators such that $CB = I_{\mathcal{H}_2}$. Then, for all $f \in \mathcal{H}_1$,*

$$\begin{aligned} \Gamma_{\mathcal{H}_1,\mathcal{H}_2}(C) a_{\mathcal{H}_1}(f)^* \Gamma_{\mathcal{H}_2,\mathcal{H}_1}(B) &= a_{\mathcal{H}_2}(Cf)^*, & (4.19)\\ \Gamma_{\mathcal{H}_1,\mathcal{H}_2}(B^*) a_{\mathcal{H}_1}(f) \Gamma_{\mathcal{H}_2,\mathcal{H}_1}(C^*) &= a_{\mathcal{H}_2}(Cf). & (4.20) \end{aligned}$$

Proof. By direct computations, one first proves (4.19) and (4.20) on $\mathcal{F}_{\rm fin}(\mathcal{H}_2)$ and then uses a limiting argument to obtain (4.19) and (4.20) as operator equalities. □

By (4.7), we have

$$\Gamma_{\mathcal{H}}(L)\Gamma_{\mathcal{H}}(L)^* = I_{\mathcal{F}_{\rm s}(\mathcal{H})}, \quad \Gamma_{\mathcal{H}}(L)^*\Gamma_{\mathcal{H}}(L) = \Gamma_{\mathcal{H}}(P_{\mathcal{K}}). \tag{4.21}$$

Note that $\Gamma_{\mathcal{H}}(P_{\mathcal{K}})$ is the orthogonal projection onto the closed subspace $\mathcal{F}_{\rm s}(\mathcal{K} \oplus \{0\}) = \mathcal{F}_{\rm s}(\mathcal{K}) \otimes \mathbf{C}$. Hence, $\Gamma_{\mathcal{H}}(L)$ is a partial isometry on $\mathcal{F}_{\rm s}(\mathcal{H})$.

Let

$$V_{\mathcal{H}} = U_{\mathcal{H}}\Gamma_{\mathcal{H}}(L). \tag{4.22}$$

Then, by (4.21), we can show that

$$V_{\mathcal{H}}V_{\mathcal{H}}^* = I_{\mathcal{F}_{\rm s}(\mathcal{H})}, \quad V_{\mathcal{H}}^*V_{\mathcal{H}} = \Gamma_{\mathcal{H}}(P_{\mathcal{K}}). \tag{4.23}$$

Hence, $V_{\mathcal{H}}$ is a partial isometry on $\mathcal{F}_{\rm s}(\mathcal{H})$ with the initial space $\mathcal{F}_{\rm s}(\mathcal{K}) \otimes \mathbf{C}$ and final space $\mathcal{F}_{\rm s}(\mathcal{H})$.

Corollary 4.5 *For all $u \in \mathcal{K}$,*

$$b(u) = V_{\mathcal{H}} a_{\mathcal{H}}(u,0) V_{\mathcal{H}}^*, \quad b(u)^* = V_{\mathcal{H}} a_{\mathcal{H}}(u,0)^* V_{\mathcal{H}}^*. \tag{4.24}$$

Proof. Applying Lemma 4.4 to the case $C = L$ and $B = L^*$, we obtain

$$\Gamma_{\mathcal{H}}(L)a_{\mathcal{H}}(f)^*\Gamma_{\mathcal{H}}(L^*) = a_{\mathcal{H}}(Lf)^*, \quad \Gamma_{\mathcal{H}}(L)a_{\mathcal{H}}(f)\Gamma_{\mathcal{H}}(L^*) = a_{\mathcal{H}}(Lf), \tag{4.25}$$

which, together with Corollary 4.3 and (4.22), imply (4.24). □

Corollary 4.5 clarifies the structure of the NIBT $\{a(\cdot, 0), a(\cdot, 0)^*\} \to \{b(\cdot), b(\cdot)^*\}$: it is implementable by the partial isometry $V_{\mathcal{H}}$, which is a composition of the partial isometry $\Gamma_{\mathcal{H}}(L)$ and the proper Bogolyubov transformation $U_{\mathcal{H}}$. Taking Remark 4.1 into account, this gives a complete characterization of the representation $\{b(u)|u \in \mathcal{K}\}$ of the CCR over $\mathcal{K}$.

The unitary transformation U given in Theorem 2.5 can be expressed in terms of $U_{\mathcal{H}}$ and operators $\Gamma_{\#}(\cdot)$. To see this, let $i_{\mathcal{K}}$ be the embedding operator of $\mathcal{K}$ into $\mathcal{H}$:

$$i_{\mathcal{K}}u := (u, 0), \quad u \in \mathcal{K}. \tag{4.26}$$

It is obvious that

$$i_{\mathcal{K}}^* i_{\mathcal{K}} = I_{\mathcal{K}}, \quad i_{\mathcal{K}} i_{\mathcal{K}}^* = P_{\mathcal{K}}. \tag{4.27}$$

In the same way as in the proof of Lemma 4.4, we can show that

$$\Gamma_{\mathcal{K},\mathcal{H}}(i_{\mathcal{K}})a_{\mathcal{K}}(u)^*\Gamma_{\mathcal{K},\mathcal{H}}(i_{\mathcal{K}})^* = a_{\mathcal{H}}(u, 0)^*\Gamma_{\mathcal{H}}(P_{\mathcal{K}}) \tag{4.28}$$

on $\mathcal{F}_{\rm fin}(\mathcal{H})$. Since $\Gamma_{\mathcal{H}}(P_{\mathcal{K}})\Gamma_{\mathcal{H}}(L)^* = \Gamma_{\mathcal{H}}(L)^*$ by (4.21), it follows that

$$b(u)^* = V_{\mathcal{H}}\Gamma_{\mathcal{K},\mathcal{H}}(i_{\mathcal{K}})a_{\mathcal{K}}(u)^*\Gamma_{\mathcal{K},\mathcal{H}}(i_{\mathcal{K}})^*V_{\mathcal{H}}^*, \quad u \in \mathcal{K}. \tag{4.29}$$

Thus, by Theorem 2.5, there exists a constant $\alpha \in \mathbf{C}$ such that $|\alpha| = 1$ and

$$U = \alpha\Gamma_{\mathcal{K},\mathcal{H}}(i_{\mathcal{K}})^*V_{\mathcal{H}}^* = \alpha\Gamma_{\mathcal{H},\mathcal{K}}(i_{\mathcal{K}}^*L^*)U_{\mathcal{H}}^{-1}. \tag{4.30}$$

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