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Periodic Soliton Solutions as Imbricate Series of Rational Solitons: Solutions to the Kadomtsev-Petviashvili Equation with Positive Dispersion

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Abstract

An inclined periodic soliton solution can be expressed as imbricate series of rational soliton solutions. A convenient form of the imbrication is given by using the bilinear form. A lattice soliton solution which propagaties in any direction can be also constructed by doubly imbricating rational solitons.

1 Introduction

The recent development of the nonlinear wave theory clarifies the role of a soliton in various systems. The inverse scattering theory shows that the time-asymptotic state of any initial conditions consists of solitons and ripples under the boundary condition that amplitudes tend to zero as $x \to \pm\infty$. Solitons are stable and the interaction between them affects only phase shifts [1]. Therefore, solitons are regarded as fundamental structures in nonlinear integrable systems. Spatial structures of solitons are usually solitary waves whose amplitudes tend to zero as $x \to \pm\infty$. It is known that soliton equations often allow an exact nonlinear superposition principle [2]-[8].

Let $U(x - c_s t)$ be a solitary wave solution of a nonlinear equation which is invariant under the group of translations, then the function

$$u(x - ct) = \sum_{n=-\infty}^{\infty} a\, U(x - ct - nh), \tag{1}$$

is periodic with a period of h. If this series of solitons is substituted into the equation and values of the velocity c (usually $c \neq c_s$) and the constant a can be determined in a consistent way, expression (1) is an exact solution of the given equation. Such a decomposition was first found by Toda [5] for the case of a cnoidal wave of the Toda lattice and the Korteweg-de Vries (K-dV) equations. A double cnoidal wave is biperiodic in space, which is regarded as the superposition of two line solitons of different sizes and is expressed in terms of two-dimensional Riemann theta functions [3, 4].

The two-dimensional generalization of the K-dV equation was given by Kadomtsev and Petviashvili [9] to discuss the stability of a one-dimensional soliton or line soliton against long transverse perturbation, which is known as the Kadomtsev-Petviashvili (K-P) equation

$$(u_t + 6uu_x + u_{xxx})_x + 3\,s\,u_{yy} = 0, \qquad s = \pm 1, \tag{2}$$

which corresponds to the cases of negative and positive dispersions when $s = +1$ and $s = -1$, respectively. They have shown that the line soliton of the K-dV equation is stable in the case of negative dispersion and is unstable for positive dispersion. This leads to the conjecture that localized solitons should be formed in the positive dispersion case since the line soliton is unstable. Such solitons have been found by Manakov et al. [10] and Ablowitz and Satsuma [11], which are no longer exponential in character but take the form of rational functions in space variables. Kuznetsov and Turitsyn [12] have shown that rational solitons are stable to any infinitesimal disturbances.

Another kind of a two-dimensional localized soliton is a periodic soliton, which was found by Zaitsev [2] at first. He obtained the x-periodic soliton solution from the imbrication of rational solitons in the x-direction. Periodic soliton solutions which describe the multisoliton interactions have been obtained by Tajiri and Murakami [13, 14] by using the bilinear transformation method. The stability of y-periodic and x-periodic solitons has been discussed by Zhdanov [15], using the inverse scattering transformation method. He has shown that they are unstable to the transverse disturbances. He also obtained the solution which describes the nonlinear stage of disturbances of a line soliton propagating in the x-direction and pointed out that a line soliton decays into a lower amplitude line soliton and y-periodic solitons. Although a line soliton and a periodic soliton are both linearly unstable to long wave disturbances, we can not rule out the possibility of observation of these solitons.

Interactions between a line soliton and a rational soliton have been investigated by Johnson and Thompson [16] and Freeman [17]. In the previous papers [14], [18]-[20], the interactions between two y-periodic solitons, between the y-periodic soliton and the line soliton and between the y-periodic soliton and the rational soliton were investigated. We found periodic soliton resonances in each case, which are qualitatively different from the resonant interactions between line solitons of K-P equation with negative dispersion [21]. These results lead to the conjecture that a close relation exists between the existence of a periodic soliton resonance and soliton instability. The emission of the periodic soliton or the rational soliton from the line soliton (periodic soliton) means the instability of the line soliton (periodic soliton). The absorption of the periodic soliton or the rational soliton in the line soliton (periodic soliton) corresponds to the resonance of the line soliton (periodic soliton) with the periodic soliton or the rational soliton. From these facts, the rational soliton can be regarded as a fundamental constituent together with the line soliton in unstable systems. Then, it is hoped to show that an inclined periodic soliton is also constructed by the imbricate series of rational solitons. Recently, the lattice soliton solution that has doubly a periodic array of the localized structure in the x-y plane was presented to the K-P equation with positive dispersion from doubly imbricate series of rational solitons [22]. Unfortunately, this lattice soliton propagates only in the x-direction. In this paper, it is shown that an inclined periodic soliton solution can be constructed as imbricate series of rational solitons. The lattice soliton solution that shows propagation in arbitrary direction is also obtained.

2 Inclined Periodic Soliton Solutions as Imbricate Series of Rational Solitons

The inclined periodic solution [13] to equation (2) with $s=-1$ (positive dispersion) is given by

$$u = 2\left[\alpha^2 - \frac{\beta^2}{K} - \frac{\alpha^2-\beta^2}{\sqrt{K}}\cosh(\alpha x+\gamma y-\Omega_r t+\sigma)\cos(\beta x+\delta y-\Omega_i t+\theta)\right.$$
$$\left.-\frac{2\alpha\beta}{\sqrt{K}}\sinh(\alpha x+\gamma y-\Omega_r t+\sigma)\sin(\beta x+\delta y-\Omega_i t+\theta)\right] \tag{3}$$
$$\Big/\left[\cosh(\alpha x+\gamma y-\Omega_r t+\sigma) - \frac{1}{\sqrt{K}}\cos(\beta x+\delta y-\Omega_i t+\theta)\right]^2,$$

where

$$\begin{cases} \Omega_r = \alpha^3 - 3\alpha\beta^2 - 3\dfrac{\alpha\gamma^2-\alpha\delta^2+2\beta\gamma\delta}{\alpha^2+\beta^2}, \\ \Omega_i = 3\alpha^2\beta - \beta^3 - 3\dfrac{2\alpha\gamma\delta-\beta\gamma^2+\beta\delta^2}{\alpha^2+\beta^2}, \\ K = \dfrac{\beta^2(\alpha^2+\beta^2)^2+(\beta\gamma-\alpha\delta)^2}{-\alpha^2(\alpha^2+\beta^2)^2+(\beta\gamma-\alpha\delta)^2}. \end{cases} \tag{4}$$

The existence condition for the nonsingular solution (3) is given by $K>1$. Solution (3) is rewritten in the convenient form by using the bilinear form,

$$u = 2\frac{\partial^2}{\partial x^2}\ln f$$

with

$$f = \sqrt{K}\cosh(\alpha x+\gamma y-\Omega_r t+\sigma) - \cos(\beta x+\delta y-\Omega_i t+\theta). \tag{5}$$

The rational soliton solution to the K-P equation with positive dispersion is given by

$$u = 4\frac{1/(L+L^*)^2 - \xi^2 + \eta^2}{\left[1/(L+L^*)^2+\xi^2+\eta^2\right]^2}, \tag{6}$$

where

$$\begin{cases} \xi = \mathrm{Re}[x - 2iLy - 12L^2 t] + \xi^0, \\ \eta = \mathrm{Im}[x - 2iLy - 12L^2 t] + \eta^0, \end{cases} \tag{7}$$

where ξ^0 and η^0 are arbitrary constants and * indicates the complex conjugate. This is a localized solution with velocity $\mathbf{c} = (c_x, c_y)$, $c_x = 12|L|^2$, $c_y = -12\mathrm{Im}(L)$ and decays like $(x^2+y^2)^{-1}$ as $(x^2+y^2)^{1/2} \to \infty$. It is interesting to note that the rational soliton solution can be rewritten as the following expression

$$u = -\frac{\partial^2}{\partial x^2}\ln\left[\frac{1}{(\xi+ir_0(\eta))^2}\cdot\frac{1}{(\xi-ir_0(\eta))^2}\right], \tag{8}$$

where

$$r_0 = \left[\eta^2 + \frac{1}{(L+L^*)^2}\right]^{1/2}. \tag{9}$$

Then, we assume the form of the imbricate series for the inclined periodic soliton solution as follows:

$$\begin{aligned} u = -\frac{\partial^2}{\partial x^2} \ln \Bigg[& \left\{ \sum_{n=-\infty}^{\infty} \frac{1}{[(\varphi(x,y,t) + i\psi(x,y,t))/2\pi - n]^2} \right\} \\ & \times \left\{ \sum_{n=-\infty}^{\infty} \frac{1}{[(\varphi(x,y,t) - i\psi(x,y,t))/2\pi - n]^2} \right\} \Bigg], \end{aligned} \tag{10}$$

where $\varphi(x,y,t)$ and $\psi(x,y,t)$ are functions of x, y and t to be determined. It is important to note that (10) is rewritten in the following form,

$$u = 2\frac{\partial^2}{\partial x^2} \ln\left[\cosh\psi(x,y,t) - \cos\varphi(x,y,t)\right]. \tag{11}$$

Comparing (11) with (5), we find

$$\left\{ \begin{array}{l} \cosh\psi(x,y,t) = \sqrt{K}\cosh(\alpha x + \gamma y - \Omega_r t + \sigma), \\ \cos\varphi(x,y,t) = \cos(\beta x + \delta y - \Omega_i t + \theta), \end{array} \right. \tag{12}$$

or

$$\left\{ \begin{array}{l} \cosh\psi(x,y,t) = \cosh(\alpha x + \gamma y - \Omega_r t + \sigma), \\ \cos\varphi(x,y,t) = \dfrac{1}{\sqrt{K}}\cos(\beta x + \delta y - \Omega_i t + \theta). \end{array} \right. \tag{13}$$

Eqs. (12) and (13) are readily solved to give

$$\left\{ \begin{array}{l} \psi = \ln\left[\sqrt{K}\cosh(\alpha x + \gamma y - \Omega_r t + \sigma) + \sqrt{K\cosh^2(\alpha x + \gamma y - \Omega_r t + \sigma) - 1}\right], \\ \varphi = \beta x + \delta y - \Omega_i t + \theta, \end{array} \right. \tag{14}$$

and

$$\left\{ \begin{array}{l} \psi = \alpha x + \gamma y - \Omega_r t + \sigma, \\ \varphi = \arccos\left[\dfrac{1}{\sqrt{K}}\cos(\beta x + \delta y - \Omega_i t + \theta)\right], \end{array} \right. \tag{15}$$

respectively. Substituting (14) or (15) into (10), we have the inclined periodic soliton solution as imbricate series of rational soliton solution.

Taking $\alpha \to 0$ and $\delta \to 0$ in (4) and (14), we have

$$\left\{ \begin{array}{l} \psi = \ln\left[\sqrt{1 + \dfrac{\beta^4}{\gamma^2}}\cosh(\gamma y + \sigma) + \sqrt{\left(1 + \dfrac{\beta^4}{\gamma^2}\right)\cosh^2(\gamma y + \sigma) - 1}\right] = \psi_0, \\ \varphi = \beta x - \left(-\beta^3 + 3\dfrac{\gamma^2}{\beta}\right)t + \theta = \varphi_0, \end{array} \right. \tag{16}$$

and solution (10) is expressed by

$$\begin{aligned} u &= 2\frac{\partial^2}{\partial x^2}\ln\left[\sin\left(\frac{\varphi_0+i\psi_0}{2}\right)\sin\left(\frac{\varphi_0-i\psi_0}{2}\right)\right] \\ &= -\frac{\beta^2}{2}\left\{\mathrm{cosec}^2\left(\frac{\varphi_0+i\psi_0}{2}\right)+\mathrm{cosec}^2\left(\frac{\varphi_0-i\psi_0}{2}\right)\right\}, \end{aligned} \tag{17}$$

which is equal to the following equation

$$u = -2\left\{\sum_{n=-\infty}^{\infty}\left[\frac{1}{\left(x-\omega t+\tilde{\theta}+i\tilde{\psi}_0-2\pi n/\beta\right)^2}+\frac{1}{\left(x-\omega t+\tilde{\theta}-i\tilde{\psi}_0-2\pi n/\beta\right)^2}\right]\right\}, \tag{18}$$

where $\omega = -\beta^2+3\gamma^2/\beta^2$, $\tilde{\psi}_0 = \psi_0/\beta$, which is given by Zaitsev [2]. Thus, solution (10) is in agreement with the result of [2] taking the limit $\alpha \to 0$ and $\delta \to 0$.

Now, we consider the asymptotic formulas of the solution. As an example, taking the limit α, β, γ and $\delta \to 0$ with $\alpha/\beta \to 0$, $\delta/\gamma \to 0$, $\delta/\alpha \to l_1 \simeq O(1)$ and $\gamma/\beta \to l_2 \simeq O(1)$, we have

$$\begin{aligned} u = -\frac{\partial^2}{\partial x^2}\ln &\left[\left\{\sum_{n=-\infty}^{\infty}\frac{1}{\left(x-3{l_2}^2t+i\sqrt{1/{l_2}^2+{l_2}^2y^2}-2\pi n/\beta\right)^2}\right\}\right. \\ &\left.\times\left\{\sum_{n=-\infty}^{\infty}\frac{1}{\left(x-3{l_2}^2t-i\sqrt{1/{l_2}^2+{l_2}^2y^2}-2\pi n/\beta\right)^2}\right\}\right], \end{aligned} \tag{19}$$

which is a simple summation of rational solitons with $L = l_2/2$. Therefore, the hump which constitutes the inclined periodic soliton with small α, β, γ and δ resembles a rational soliton. In this sense, it is proper to regard the inclined periodic soliton as a nonlinear superposition of rational solitons.

3 Lattice Soliton Solution

The lattice soliton solution given by reference [22] has only propagation in the x- direction. In this section, the lattice soliton solution which shows propagation in an arbitrary direction is constructed as double imbricate series of rational solitons.

We assume the form of the solution as follow

$$u(x,y,t) = -2\left[\sum_{m,n}\left\{\frac{1}{[\xi+ir(\eta)-(mh+2\pi in)]^2}+\frac{1}{[\xi-ir(\eta)-(mh-2\pi in)]^2}\right\}+\tilde{c}\right], \tag{20}$$

where

$$\left\{\begin{array}{l} \xi = x+\beta y-\gamma t+\xi^0, \\ \eta = y-\delta t+\eta^0. \end{array}\right. \tag{21}$$

The summation ranges over all integer pair m and n; β, γ, δ and $\tilde{c}$ are some real constants, $r(\eta)$ is a function of η to be determined and h is an interval. As this superposition is not a simple summation in the sense of linear theory, the function $r(\eta)$ is different from $r_0(\eta)$ given by (9), owing to the nonlinearity. If we take the order of summation over m and n in (21) the same as the order of the definition of elliptic functions, equation (20) is rewritten by using Weierstrass' $\wp$ function [23] as follows

$$u = -2\left[\wp(\xi + ir(\eta)|\omega_1, \omega_3) + \wp(\xi - ir(\eta)|\omega_1, \omega_3) + c\right] , \tag{22}$$

where periods $2\omega_1 = h$, $2\omega_3 = 2\pi i$; and

$$c = \tilde{c} + 2 \sum_{\substack{m,n \\ m^2+n^2\neq 0}}{}' \frac{1}{(mh + 2\pi in)^2}$$

where the summation ranges over all integer pairs (m, n) except for $(0, 0)$. We must determine $r(\eta)$ for (22) to satisfy (2) with $s = -1$.

By using the relations [23]

$$\left\{[\wp(\xi \pm ir(\eta))]^2\right\}' = \frac{1}{6}\wp'''(\xi \pm ir(\eta)) \tag{23}$$

and

$$\begin{vmatrix} \wp(\xi + ir(\eta)) & \wp'(\xi + ir(\eta)) & 1 \\ \wp(\xi - ir(\eta)) & -\wp'(\xi - ir(\eta)) & 1 \\ \wp(2ir(\eta)) & -\wp'(2ir(\eta)) & 1 \end{vmatrix} = 0, \tag{24}$$

we can show that

$$\begin{aligned}(3u^2 + u_{xx})_{xx} = 24\frac{\partial}{\partial x}\Big[&\{\wp(2ir(\eta)) + c\}\{\wp'(\xi + ir(\eta)) + \wp'(\xi - ir(\eta))\} \\ &+ \wp'(2ir(\eta))\{\wp(\xi + ir(\eta)) - \wp(\xi - ir(\eta))\}\Big].\end{aligned} \tag{25}$$

Substituting (22) into (2) and taking account of (25), we have

$$\begin{aligned}&\frac{\partial}{\partial x}\Bigg\{\left[i\frac{d^2r}{d\eta^2} + 4\wp'(2ir|\omega_1, \omega_3)\right][\wp(\xi + ir|\omega_1, \omega_3) - \wp(\xi - ir|\omega_1, \omega_3)] \\ &+ \left[\frac{\gamma}{3} + 4H + 4\wp(2ir|\omega_1, \omega_3) + i\,\frac{\delta}{3}\frac{dr}{d\eta} + \left(\beta + i\,\frac{dr}{d\eta}\right)^2\right]\wp'(\xi + ir|\omega_1, \omega_3) \\ &+ \left[\frac{\gamma}{3} + 4H + 4\wp(2ir|\omega_1, \omega_3) - i\,\frac{\delta}{3}\frac{dr}{d\eta} + \left(\beta - i\,\frac{dr}{d\eta}\right)^2\right]\wp'(\xi - ir|\omega_1, \omega_3)\Bigg\} = 0.\end{aligned} \tag{26}$$

We find that coefficients of $\wp'(\xi + ir)$ and $\wp'(\xi - ir)$ are complex conjugate each other and, if $\delta = -6\beta$, the coefficient of the term $\{\wp(\xi + ir(\eta)) - \wp(\xi - ir(\eta))\}$ coincides with the derivative of that of the terms $\wp'(\xi + ir)$ and $\wp'(\xi - ir)$, and have only to impose the following condition for $r(\eta)$

$$\left(\frac{dr}{d\eta}\right)^2 = \frac{\gamma}{3} + 4H + \beta^2 + 4\wp(2ir|\omega_1, \omega_3) \tag{27}$$

which is a sufficient condition for (22) to be a solution of the K-P equation with positive dispersion (2). By using the relation

$$\wp\left(2ir(\eta)\;\middle|\;\omega_1 = \frac{\alpha h}{2}, \omega_3 = \pi i\right) = -\frac{1}{4}\wp\left(r(\eta)\;\middle|\;\tilde{\omega_1} = \frac{\pi}{2}, \tilde{\omega_3} = \frac{ih}{4}\right), \tag{28}$$

Eq. (27) is rewritten as follows

$$\left(\frac{dr(\eta)}{d\eta}\right)^2 = \frac{\gamma}{3} + 4H + \beta^2 - \wp(r(\eta)|\tilde{\omega_1}, \tilde{\omega_3}). \tag{29}$$

As the minimum value of the function $\wp(r|\tilde{\omega_1}, \tilde{\omega_3})$ is $e_1 = \wp(\tilde{\omega_1}|\tilde{\omega_1}, \tilde{\omega_3}) > 0$, the reality condition for $r(\eta)$ gives the following condition

$$\frac{\gamma}{3} + 4H + \beta^2 - e_1 > 0, \tag{30}$$

whi ch may be regarded as the existence condition for a bounded solution.

Following [22], we have the solution of equation (29) as follows:

$$r(\eta) = \frac{1}{\sqrt{e_1 - e_3}} \operatorname{cn}^{-1}\left[E \operatorname{cn}(\kappa\eta + \theta, k_2), k_1\right], \tag{31}$$

where $e_3 = \wp(\tilde{\omega_3}|\tilde{\omega_1}, \tilde{\omega_3}) < 0$ and $k_1 = \sqrt{-(2e_3 + e_1)/(e_1 - e_3)}$ which is the modulus of the elliptic function,

$$E^2 = \frac{A - B}{A}, \tag{32}$$

$$k_2 = E \Bigg/ \sqrt{\frac{1 - {k_1}^2}{{k_1}^2} + E^2}, \tag{33}$$

$$\kappa = k_1 \sqrt{A} \sqrt{\frac{1 - {k_1}^2}{{k_1}^2} + E^2}, \tag{34}$$

with

$$A = (e_1 - e_3)\left(\frac{\gamma}{3} + 4H + \beta^2 - e_3\right), \qquad B = (e_1 - e_3)^2.$$

It should be noted that the r.h.s. of (32) is always positive under condition (30) and less than 1. Therefore, $r(\eta)$ takes a bounded real value for all η. Equation (34) is rewritten as follows

$$\gamma = 3\left[\frac{\kappa^2}{e_1 - e_3} - 4H - \beta^2 - (e_1 + e_3)\right], \tag{35}$$

which may be regarded as the dispersion relation of a lattice soliton. The substitution of (31) into (22) gives the exact solution of the K-P equation with positive dispersion. Thus, we obtain the lattice soliton solution that has propagation in an arbitrary direction.

4 Concluding Remarks

We have shown that the inclined periodic soliton solution is expressed as exact imbricate series of rational soliton solutions. A convenient form of the imbricate series is given by using the bilinear form. The lattice soliton solution can be also constructed by doubly imbricating rational solitons, which show a doubly periodic array of localized structures in the x-y plane and propagation in an arbitrary direction. As this superposition is not a simple summation in the sense of linear theory, the shape of constituent humps of these periodic waves is a little different from that of the rational soliton. However, according as the interval between humps increases more and more, the form of constituent humps approaches to the exact rational soliton solution. In this sense, it is proper to regard periodic solitons as the imbrication of rational solitons.

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