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Conformally Invariant Ansätze for the Maxwell Field

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Abstract

A general procedure for construction of conformally invariant Ansätze for the Maxwell field is suggested. Ansätze invariant with respect to inequivalent three-parameter subgroups of the conformal group are constructed.

1 Introduction

Since early seventies when W.I. Fushchych suggested a principally new (non-Lie) approach to study symmetry properties of the Maxwell equations [1]–[4], these equations are in the focus of his research activity. A number of fundamental results were obtained, such as the determination of Lie and non-Lie symmetries of the Maxwell equations [1]–[4], classification of equations of nonlinear electrodynamics and nonlinear representations of the Poincaré and Galilei algebras for the Maxwell field [5, 6], construction of invariant solutions of the Maxwell equations [7, 8] to mention only some of them. A complete review on this subject can be found in the monographs [9, 10, 11] which are recognized as the standard source of references in the field of symmetry analysis of equations of quantum mechanics.

The present paper is a continuation of our papers [7, 8]. Here we consider the problem of construction of conformally-invariant Ansätze for the vacuum Maxwell equations

$$\begin{aligned} &\mathrm{rot}\ \vec{E} = -\frac{\partial \vec{H}}{\partial x_0}, \qquad \mathrm{rot}\ \vec{H} = \frac{\partial \vec{E}}{\partial x_0}, \\ &\mathrm{div}\ \vec{E} = 0, \qquad\qquad \mathrm{div}\ \vec{H} = 0, \end{aligned} \tag{1}$$

which reduce system (1) to ordinary differential equations (ODE). It is well known that the above equations admit the conformal group $C(1,3)$ with the following basis generators [9]:

$$
\begin{aligned}
&P_\mu = \partial_{x_\mu}, \qquad J_{0a} = x_0\partial_{x_a} + x_a\partial_{x_0} + \varepsilon_{abc}(E_b\partial_{H_c} - H_b\partial_{E_c}),\\
&J_{ab} = x_b\partial_{x_a} - x_a\partial_{x_b} + E_b\partial_{E_a} - E_a\partial_{E_b} + H_b\partial_{H_a} - H_a\partial_{H_b},\\
&D = x_\mu\partial_\mu - 2(E_a\partial_{E_a} + H_a\partial_{H_a}),\\
&K_0 = 2x_0D - x_\mu x^\mu\partial_{x_0} + 2x_a\varepsilon_{abc}(E_b\partial_{H_c} - H_b\partial_{E_c}),\\
&K_a = -2x_aD - x_\mu x^\mu\partial_{x_0} - 2x_0\varepsilon_{abc}(E_b\partial_{H_c} - H_b\partial_{E_c}) -\\
&\qquad -2H_a(x_b\partial_{H_b}) - 2E_a(x_b\partial_{E_b}) + 2(x_bH_b)\partial_{H_a} + 2(x_bE_b)\partial_{E_a}.
\end{aligned} \tag{2}
$$

Here $\mu,\nu = 0,1,2,3; a,b,c = 1,2,3$. Henceforth we use the summation convention over repeated indices, those denoted by Greek letters are ranging from 0 to 3 and by Latin letters from 1 to 3. Lowering or rising indices is carried out by the metric tensor of the Minkowski space $g_{\mu\nu} : g_{00} = -g_{11} = -g_{22} = -g_{33} = 1,\ g_{\mu\nu} = 0, \mu \neq \nu$; ε_{abc} is the antisymmetric tensor of the third order; $\partial_{x_\mu} = \dfrac{\partial}{\partial x_\mu},\ \partial_{E_a} = \dfrac{\partial}{\partial E_a},\ \partial_{H_a} = \dfrac{\partial}{\partial H_a}$.

Using the fact that operators (2) realize a linear representation of the conformal algebra, we suggest a direct method for construction of the invariant Ansätze making it possible to avoid the awkward procedure of finding a basis of functional invariants of subalgebras of the algebra $AC(1,3)$.

2 Linear representation of the conformally invariant Ansätze

Let L be a nonzero subalgebra of the algebra $AC(1,3)$ with basis elements (2). An invariant linear in $E_1, E_2, E_3, H_1, H_2, H_3$ is called as a *linear invariant* of a subalgebra L. Suppose that L has six linear invariants

$$
f_{ma}(x)E_a + f_{m3+a}(x)H_a, \quad a = 1,2,3;\ m = 1,2,\ldots,6;
$$

which are functionally independent. They can be considered as components of the vector $F\vec{A}$, where $F = (f_{mn}(x)),\ m,n = 1,2,\ldots,6$ and $\vec{A}$ is a vector-column with components A_m, where $A_k = E_k$ $(k = 1,2,3)$ and $A_k = H_{k-3}$ $(k = 4,5,6)$ Furthermore, we suppose that the matrix F is nonsingular in some domain of $R_{1,3} = \{(x_0, x_1,\ x_2, x_3) : x_\mu \in R\}$. Providing the rank r of the subalgebra L is less or equal 3, there are additional $s = 4 - r$ invariants independent of components of $\vec{A}$. We denote these as $\omega_1,\ldots,\omega_s$.

According to the results of [12], the Ansatz $F\vec{A} = \vec{B}(\omega_1,\ldots,\omega_s)$ reduces the system of equations (1) to a system of differential equations which contains the independent variables

$\omega_1, \ldots, \omega_s$, dependent variables $B_1, B_2, \ldots, B_6$, and their first derivatives. This Ansatz can be written in the form

$$\vec{A} = Q(x)\vec{B}(\omega_1, \ldots, \omega_s), \qquad Q(x) = F^{-1}(x), \tag{3}$$

where $x = (x_0, x_1, x_2, x_3)$. If $s = 1$, then the reduced system is equivalent to a system of ODEs. Let us note that it was W.I. Fushchych who have noticed for the first time a possibility to look for solutions of differential equations of the form (3) [14, 15].

Let $L = \langle X_1, \ldots, X_c \rangle$, where

$$X_a = \xi_{a\mu}(x)\frac{\partial}{\partial x_\mu} + \rho_{amn}(x)A_n\frac{\partial}{\partial A_m} \qquad (a = 1, 2, \ldots, c).$$

Hereafter $m, n, k, l = 1, 2, \ldots, 6$. The function $f_{mn}(x)A_n$ is an invariant of the operator X_a if and only if

$$\xi_{a\mu}(x)\frac{\partial f_{mn}(x)}{\partial x_\mu}A_m + \rho_{akl}(x)A_l f_{mk}(x) = 0$$

or

$$\xi_{a\mu}(x)\frac{\partial f_{mn}(x)}{\partial x_\mu} + f_{mk}(x)\rho_{akm}(x) = 0 \tag{4}$$

for all n [13].

Let $F(x) = (f_{mn}(x))$, $\quad \Gamma_a(x) = (\rho_{akl}(x))$ be 6×6 matrices. Then the second term on the left-hand side of (4) is the entry (m, n) of the matrix $F(x)\Gamma_a(x)$. Whence, we get the following assertion.

Theorem 1. *The system of functions $f_{mn}(x)A_n$ is a system of functionally independent invariants if and only if the matrix $F = (f_{mn}(x))$ is nonsingular in some domain of the space $R_{1,3}$ and satisfies the system of partial differential equations*

$$\xi_{a\mu}(x)\frac{\partial F(x)}{\partial x_\mu} + F(x)\Gamma_a(x) = 0, \quad a = 1, \ldots, c.$$

In what follows we call an Ansatz of the form (3) linear.

3 Conformally invariant Ansätze for the Maxwell field

To construct conformally invariant Ansätze reducing (1) to systems of ODEs, one has to use three-dimensional subalgebras of the algebra $AC(1,3)$ having the basis elements (2). The complete list of nonequivalent subalgebras of the conformal algebra is known (see, e.g., [16]). Note that a similar problem for the spinor field is completely solved in [17].

As Ansätze for the Maxwell field corresponding to three-dimensional subalgebras of the extended Poincaré algebra $A\tilde{P}(1,3) = \langle P_\mu, J_{\mu\nu}, D|\mu,\nu = 0,1,2,3; \mu \neq \nu\rangle$ were found in [7, 8, 18], it is sufficient to consider only subalgebras nonconjugated to subalgebras of the algebra $A\tilde{P}(1,3)$.

We restrict our considerations to three-dimensional subalgebras of the algebra $AC(1,3)$ which belong to the third class according to notations of [16]

$$L_1 = \langle S + T + J_{12}, G_1 + P_2, M\rangle,$$

$$L_2 = \langle S + T + J_{12} + G_1 + P_2, G_2 - P_1, M\rangle,$$

$$L_3 = \langle J_{12}, S + T, M\rangle, \quad L_4 = \langle S + T, Z, M\rangle,$$

$$L_5 = \langle S + T + \alpha J_{12}, Z, M\rangle \ (\alpha > 0),$$

$$L_6 = \langle S + T + J_{12} + \alpha Z, G_1 + P_2, M\rangle \ (\alpha \neq 0),$$

$$L_7 = \langle S + T + J_{12}, Z, G_1 + P_2\rangle,$$

$$L_8 = \langle S + T + \beta Z, J_{12} + \alpha Z, M\rangle \ (\alpha \leq 0, \beta \in R, \alpha + \beta^2 \neq 0),$$

$$L_9 = \langle J_{12}, S + T, Z\rangle, \quad L_{10} = \langle R, S, T\rangle,$$

where $M = P_0 + P_3$, $G_a = J_{0a} - J_{a3}$, $R = D - J_{03}$, $Z = J_{03} + D$, $S = \frac{1}{2}(K_0 + K_3)$, $T = \frac{1}{2}(P_0 - P_3)$.

To construct invariant Ansätze, we make use of Theorem 1 and relate to each generator of the algebra L_j $(j = 1, \ldots, 10)$ some matrix Γ (see below).

As the operator P_μ is independent of $\dfrac{\partial}{\partial A_n}$ $(n = 1, 2, \ldots, 6)$, the corresponding matrix Γ is equal to zero.

Let $(-S_{\mu\nu})$ be the matrix Γ corresponding to $J_{\mu\nu}$. It is not diffucult to verify that

$$\begin{gathered}
S_{01} = \begin{pmatrix} 0 & -\tilde{S}_{23} \\ \tilde{S}_{23} & 0 \end{pmatrix}, \quad S_{02} = \begin{pmatrix} 0 & \tilde{S}_{13} \\ -\tilde{S}_{13} & 0 \end{pmatrix}, \quad S_{03} = \begin{pmatrix} 0 & -\tilde{S}_{12} \\ \tilde{S}_{12} & 0 \end{pmatrix}, \\
S_{12} = \begin{pmatrix} \tilde{S}_{12} & 0 \\ 0 & \tilde{S}_{12} \end{pmatrix}, \quad S_{13} = \begin{pmatrix} \tilde{S}_{13} & 0 \\ 0 & \tilde{S}_{13} \end{pmatrix}, \quad S_{23} = \begin{pmatrix} \tilde{S}_{23} & 0 \\ 0 & \tilde{S}_{23} \end{pmatrix},
\end{gathered} \tag{5}$$

where 0 is the 3×3 zero matrix and

$$S_{12} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad S_{13} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad S_{23} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}.$$

The matrix $(-2E)$, where E is the unit 6×6 matrix, corresponds to the operator D. Then, the matrices $(-4x_0 E - 2x_a S_{0a})$ and $(4x_a E + 2x_0 S_{0a} - 2x_b S_{ab})$ correspond to the operators K_0 and K_a, reprectively.

The matrices $S_{\mu\nu}, E$ realize a representation of the algebra $A\tilde{O}(1,3) = AO(1,3) \bigoplus \bigoplus \langle D\rangle$ inasmuch as

$$[S_{\mu\nu}, S_{\gamma\beta}] = g_{\mu\beta}D_{\nu\gamma} + g_{\nu\gamma}S_{\mu\beta} - g_{\mu\gamma}S_{\nu\beta} - g_{\nu\beta}S_{\mu\gamma}, \quad [E, S_{\mu\nu}] = 0,$$

where $\mu, \nu, \gamma, \beta = 0,1,2,3$; $g_{\mu\nu}$ is the metric tensor of the Minkowski space $R_{1,3}$.

The algebra $AO(1,3)$ having the generators $S_{\mu\nu}$ $(\mu, \nu = 0,1,2,3)$ contains as a subalgebra the algebra $A\tilde{E}(2)$ generated by $H_a = S_{0a} - S_{a3}$ $(a = 1,2)$, S_{12}, and S_{03}. Basis elements of $A\tilde{E}(2)$ fulfill the following commutation relations:

$$\begin{array}{lll} [H_1, S_{12}] = -H_2, & [H_2, S_{12}] = H_1, & [H_1, H_2] = 0, \\ [H_a, S_{03}] = H_a & (a = 1,2), & [S_{03}, S_{12}] = 0. \end{array}$$

Lemma 1. *Let*

$$\Lambda = \exp(\ln\theta E)\exp(\theta_0 S_{03})\exp(-\theta_3 S_{12})\exp(-2\theta_1 H_1)\exp(-2\theta_2 H_2), \tag{6}$$

where θ, θ_μ $(\mu = 0,1,2,3)$ *are functions of* $x = (x_0, x_1, x_2, x_3)$. *Then*

$$\begin{aligned} \xi_\mu \frac{\partial\Lambda}{\partial x_\mu} &= \Lambda\xi_\mu \left\{ \frac{\partial(\ln\theta)}{\partial x_\mu} E + \frac{\partial\theta_0}{\partial x_\mu}(S_{03} + 2\theta_1 H_1 + 2\theta_2 H_2) \right. \\ & \left. - \frac{\partial\theta_3}{\partial x_\mu}(S_{12} - 2\theta_1 H_2 + 2\theta_2 H_1) - 2\frac{\partial\theta_1}{\partial x_\mu} H_1 - 2\frac{\partial\theta_2}{\partial x_\mu} H_2 \right\}. \end{aligned}$$

The proof of the lemma is carried out by the Campbell-Hausdorff formula.

Theorem 2. *For each subagebra* $L_j = \langle X_a | a = 1,2,3\rangle$ $(j = (1,\ldots,10)$, *there exists a linear Ansatz (3) with* ω *being a solution of the system*

$$\xi_{a\mu}(x)\frac{\partial\omega}{\partial x_\mu} = 0, \quad a = 1,2,3; \tag{7}$$

$Q(x) = \Lambda^{-1}$, *where* Λ *has the form (6). Furthermore, the functions* $\theta, \theta_0, \theta_1, \theta_2, \theta_3, \omega$ *can be represented in the form*

$$\begin{aligned} L_1: \quad & \theta = (1 + (x_0 - x_3)^2), \ \theta_0 = -\frac{1}{2}\ln(1 + (x_0 - x_3)^2), \\ & \theta_1 = -\frac{1}{2}(x_2 + (x_0 - x_3)x_1)(1 + (x_0 - x_3)^2)^{-1}, \\ & \theta_2 = \frac{1}{2}(x_1 - (x_0 - x_3)x_2)(1 + (x_0 - x_3)^2)^{-1}, \\ & \theta_3 = -\arctan(x_0 - x_3), \ \omega = (x_1 - x_2(x_0 - x_3))(1 + (x_0 - x_3)^2)^{-1}; \\ L_2: \quad & \theta = [1 + (x_0 - x_3)^2], \ \ \theta_0 = -\frac{1}{2}\ln(1 + (x_0 - x_3)^2), \\ & \theta_1 = -\frac{1}{2}(x_2 + (x_0 - x_3)x_1)(1 + (x_0 - x_3)^2)^{-1}, \end{aligned}$$

$$\theta_2 = \frac{1}{2}(x_1 - (x_0 - x_3)x_2)(1 + (x_0 - x_3)^2)^{-1},\ \theta_3 = -\arctan(x_0 - x_3),$$
$$\omega = [(x_2 + (x_0 - x_3)x_1](1 + (x_0 - x_3)^2)^{-1}] - \arctan(x_0 - x_3);$$

$L_3:$
$$\theta = 1 + (x_0 - x_3)^2,\ \theta_0 = -\frac{1}{2}\ln[1 + (x_0 - x_3)^2],$$
$$\theta_1 = -\frac{1}{2}x_1(x_0 - x_3)[1 + (x_0 - x_3)^2]^{-1},$$
$$\theta_2 = -\frac{1}{2}x_2(x_0 - x_3)[1 + (x_0 - x_3)^2]^{-1},$$
$$\theta_3 = \arctan\frac{x_1}{x_2}, \omega = [1 + (x_0 - x_3)^2](x_1^2 + x_2^2)^{-1};$$

$L_4:$
$$\theta_1 = x_1^2,\quad \theta_0 = \ln|x_1| - \ln[1 + (x_0 - x_3)^2],$$
$$\theta_1 = -\frac{1}{2}x_1(x_0 - x_3)[1 + (x_0 - x_3)^2]^{-1},$$
$$\theta_2 = -\frac{1}{2}x_2(x_0 - x_3)[1 + (x_0 - x_3)^2]^{-1},\ \theta_3 = 0,\ \omega = \frac{x_1}{x_2};$$

$L_5:$
$$\theta = (x_1^2 + x_2^2)(1 + (x_0 - x_3)^2),\quad \theta_0 = \frac{1}{2}\ln[(x_1^2 + x_2^2)(1 + (x_0 - x_3)^2)^{-1}];$$
$$\theta_1 = -\frac{1}{2}x_1(x_0 - x_3)[1 + (x_0 - x_3)^2]^{-1},$$
$$\theta_2 = -\frac{1}{2}x_2(x_0 - x_3)[1 + (x_0 - x_3)^2]^{-1},$$
$$\theta_3 = \arctan\frac{x_2}{x_1},\quad \omega = \arctan\frac{x_2}{x_1} + \alpha\arctan(x_0 - x_3);$$

$L_6:$
$$\theta = (x_1 - (x_0 - x_3)x_2)^2(1 + (x_0 - x_3)^2)^{-1},$$
$$\theta_0 = \frac{1}{2}\ln[(x_1 - (x_0 - x_3)x_2)^2(1 + (x_0 - x_3)^2)^{-3}],$$
$$\theta_1 = -\frac{1}{2}(x_2 + (x_0 - x_3)x_1)(1 + (x_0 - x_3)^2)^{-1},$$
$$\theta_2 = \frac{1}{2}(x_1 - (x_0 - x_3)x_2)(1 + (x_0 - x_3)^2)^{-1},\ \theta_3 = -\arctan(x_0 - x_3),$$
$$\omega = \alpha\arctan(x_0 - x_3) - \ln[(x_1 - (x_0 - x_3)x_2)(1 + (x_0 - x_3)^2)^{-1}];$$

$L_7:$
$$\theta = (x_1 - (x_0 - x_3)x_2)^2(1 + (x_0 - x_3)^2)^{-1},$$
$$\theta_0 = \frac{1}{2}\ln[(x_1 - (x_0 - x_3)x_2)^2(1 + (x_0 - x_3)^2)^{-3}],$$
$$\theta_1 = -\frac{1}{2}(x_2 + (x_0 - x_3)x_1)(1 + (x_0 - x_3)^2)^{-1},$$
$$\theta_2 = \frac{1}{2}(x_1 - (x_0 - x_3)x_2)(1 + (x_0 - x_3)^2)^{-1},\ \theta_3 = -\arctan(x_0 - x_3),$$
$$\omega = [(x_0 + x_3)(1 + (x_0 - x_3)^2)^2 - 2x_1(x_2 + (x_0 - x_3)x_1) +$$
$$+(x_0 - x_3)(x_1^2(x_0 - x_3)^2 - x_2^2)](x_1 - (x_0 - x_3)x_2)^{-2} - x_0 + x_3;$$

$L_8:$
$$\theta = x_1^2 + x_2^2,\quad \theta_0 = \frac{1}{2}\ln[(x_1^2 + x_2^2)(1 + (x_0 - x_3)^2)^{-2}],$$
$$\theta_1 = -\frac{1}{2}x_1(x_0 - x_3)[1 + (x_0 - x_3)^2]^{-1},$$

$$\theta_2 = -\frac{1}{2}x_2(x_0 - x_3)[1 + (x_0 - x_3)^2]^{-1},\ \theta_3 = \arctan\frac{x_2}{x_1},$$
$$\omega = \ln(x_1^2 + x_2^2)(1 + (x_0 - x_3)^2)^{-1} + 2\alpha \arctan\frac{x_2}{x_1} - 2\beta \arctan(x_0 - x_3);$$

$$L_9: \quad \theta = x_1^2 + x_2^2,\quad \theta_0 = \frac{1}{2}\ln(x_1^2 + x_2^2) - \ln[1 + (x_0 - x_3)^2],$$
$$\theta_1 = -\frac{1}{2}x_1(x_0 - x_3)[1 + (x_0 - x_3)^2]^{-1},$$
$$\theta_2 = -\frac{1}{2}x_2(x_0 - x_3)[1 + (x_0 - x_3)^2]^{-1},$$
$$\theta_3 = \arctan\frac{x_2}{x_1},\quad \omega = (x_0 + x_3)[1 + (x_0 - x_3)^2](x_1^2 + x_2^2)^{-1} - x_0 + x_3;$$

$$L_{10}: \quad \theta = x_1^2 + x_2^2,\quad \theta_0 = -\frac{1}{2}\ln(x_1^2 + x_2^2),$$
$$\theta_1 = -\frac{1}{2}x_1(x_0 + x_3)(x_1^2 + x_2^2)^{-1},\quad \theta_2 = -\frac{1}{2}x_2(x_0 + x_3)(x_1^2 + x_2^2)^{-1},$$
$$\theta_3 = 0, \qquad \omega = \frac{x_2}{x_1};$$

To prove the assertion, we have to check that the functions θ, θ_μ $(\mu = 0, 1, 2, 3), \omega$ satisfy the system of equations (4), (7) for each subalgebra L_j. Consider in detail the case of the subalgebra L_4. System (7) has the form

$$(\partial_{x_0} - \partial_{x_3})\omega = 0, \qquad (x_0\partial_{x_3} + x_3\partial_{x_0} + x_\mu\partial_{x_\mu})\omega = 0,$$
$$\left[(x_0 - x_3)x_\mu\partial_{x_\mu} - \frac{1}{2}x_\mu x^\mu(\partial_{x_0} + \partial_{x_3}) + \frac{1}{2}(\partial_{x_0} - \partial_{x_3})\right]\omega = 0.$$

It is not difficult to verify that the rank of this system is equal 3 and its solution is the function $\omega = \frac{x_2}{x_1}$.

To obtain the matrix $\Lambda = F(x)$, we have to construct solutions of system (4). As the algebra L_4 contains the operator $M = P_0 + P_3$, it follows from the equation

$$(\partial_{x_0} + \partial_{x_3})\Lambda = 0$$

that $\Lambda = \Lambda(\xi, x_1, x_2)$ where $\xi = x_0 - x_3$. Taking into account the structure of the algebra L_4, we choose Λ in the form (6), where $\theta_3 = 0$. According to Lemma 1, we get the following system for the functions $\theta = \theta(x_1, x_2)$, $\theta_\nu = \theta_\nu(\xi, x_1, x_2)$, $\nu = 0, 1, 2$:

$$x_a Q_a - S_{03} - 2E = 0,$$
$$2\xi x_a Q_a + 2(1 + \xi^2)Q_\xi + 2\xi(S_{03} - 2E) - 2x_1H_1 - 2x_2H_2 = 0,$$

where $Q_a = \dfrac{\partial \ln\theta}{\partial x_a}E + \dfrac{\partial\theta}{\partial x_0}(S_{03} + 2\theta_1H_1 + 2\theta_2H_2) - 2\dfrac{\partial\theta_1}{\partial x_a}H_1 - 2\dfrac{\partial\theta_2}{\partial x_a}H_2$, $a = 1, 2$.

Having split the system by the matrices E, S_{03}, H_1, H_2, we come to a system of partial differential equations whose solutions can be represented in the form given above.

Thus, conformally invariant Ansätze corresponding to the algebras L_j $(j = 1, \ldots, 10)$ can be written in the linear form (3), where $Q(x) = \Lambda^{-1}(x)$, $\Lambda(x)$ is matrix (6) and the functions θ, θ_μ $(\mu = 0, 1, 2, 3)$, ω are given in Theorem 2.

As we have a specific representation of the matrices $S_{\mu\nu}, E$, we can calculate the matrix exponentials and represent the matrix $\Lambda^{-1}(x)$ as follows

$$\Lambda^{-1} = \theta \begin{Vmatrix} \Lambda_1 & \Lambda_2 \\ -\Lambda_2 & \Lambda_1 \end{Vmatrix},$$

where

$$\begin{aligned}
\Lambda_1 &= \begin{Vmatrix} \cos\theta_3 \cosh\theta_0 - \sigma_1 & -\sin\theta_3 \cosh\theta_0 + \sigma_2 & 2\theta_1 \\ \sin\theta_3 \cosh\theta_0 + \sigma_2 & \cos\theta_3 \cosh\theta_0 + \sigma_1 & 2\theta_2 \\ \sigma_3 & \sigma_4 & 1 \end{Vmatrix}, \\
\Lambda_2 &= \begin{Vmatrix} -\sin\theta_3 \sinh\theta_0 - \sigma_2 & -\cos\theta_3 \sinh\theta_0 - \sigma_1 & -2\theta_2 \\ \cos\theta_3 \sinh\theta_0 - \sigma_1 & -\sin\theta_3 \sinh\theta_0 + \sigma_2 & 2\theta_1 \\ -\sigma_4 & \sigma_3 & 0 \end{Vmatrix},
\end{aligned}$$

$$\begin{aligned}
\sigma_1 &= 2[(\theta_1^2 - \theta_2^2)\cos\theta_3 + 2\theta_1\theta_2 \sin\theta_3]e^{-\theta_0}, \\
\sigma_1 &= 2[(\theta_1^2 - \theta_2^2)\sin\theta_3 - 2\theta_1\theta_2 \cos\theta_3]e^{-\theta_0}, \\
\sigma_3 &= -2[\theta_1 \cos\theta_3 + \theta_2 \sin\theta_3]e^{-\theta_0}, \\
\sigma_4 &= 2[\theta_1 \sin\theta_3 - \theta_2 \cos\theta_3]e^{-\theta_0}.
\end{aligned}$$

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