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Representations of the Q-deformed Euclidean Algebra $U_q(iso_3)$ and Spectra of their Operators

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Abstract

Representations of the q-deformed Euclidean algebra $U_q(\mathrm{iso}_3)$, which at $q \to 1$ gives the universal enveloping algebra $U(\mathrm{iso}_3)$ of the Lie algebra iso_3 of the Euclidean Lie group $ISO(3)$, are studied. Explicit formulas for operators of irreducible $*$-representations defined by two parameters $\rho \in \mathbb{R}$ and $s \in \frac{1}{2}\mathbb{Z}$ are given. At $q \to 1$, these representations exhaust all irreducible infinite-dimensional $*$-representations of $U(\mathrm{iso}_3)$. The spectrum of the operator $T_{\rho,s}(I_3)$ corresponding to a q-analogue of the infinitesimal operator of shifts along the third axis is given. Contrary to the case of the classical Euclidean algebra iso_3, this spectrum is discrete and has one point of accumulation.

1. Introduction

The aim of this paper is to study irreducible representations of the q-deformed Euclidean algebra $U_q(\mathrm{iso}_3)$ defined on the base of the algebra $U_q(\mathrm{so}_3)$ given in [1] (see also [2]). The classical Lie algebras so_3 and sl_2 are isomorphic. The algebra $U_q(\mathrm{so}_3)$ from [1] differs from the quantum algebra $U_q(\mathrm{sl}_2)$ defined by Drinfeld [3] and Jimbo [4]. Namely, $U_q(\mathrm{sl}_2)$ is defined by means of the Cartan subalgebra and root subspaces of the Lie algebra sl_2. The q-deformed algebra $U_q(\mathrm{so}_3)$ is a q-deformation of the defining relations $[J_1, J_2] = J_3$, $[J_2, J_3] = J_1$, $[J_3, J_1] = J_2$.

Adding to $U_q(\mathrm{so}_3)$ the generator I_3 corresponding to infinitesimal shifts along the third axis and postulating commutation relations of I_3 with other generators, we obtain the q-deformed algebra $U_q(\mathrm{iso}_3)$. This algebra is a q-deformation of the Lie algebra iso_3 of the Euclidean group $ISO(3)$ which is the semidirect product of the rotation group $SO(3)$ and the translation group of a 3-dimensional Euclidean space. There are difficulties with definition of the Hopf algebra structure in this algebra. We do not consider this problem here and only note that our algebra $U_q(\mathrm{iso}_3)$ can be embedded into the quantum algebra $U_q(\mathrm{iu}_3)$ (the q-deformation of the Lie algebra of the inhomogeneous unitary group). The last quantum algebra is equipped with the structure of a Hopf algebra.

We construct infinite-dimensional irreducible representations of the algebra $U_q(\mathrm{iso}_3)$. They are given by two numbers $\rho \in \mathbb{R}$ and $s \in \frac{1}{2}\mathbb{Z}$. Unfortunately, we cannot state that they exhaust all irreducible $*$-representations of $U_q(\mathrm{iso}_3)$. But at $q \to 1$, they give all irreducible $*$-representations of the Lie algebra iso_3. Thus, we can state that we constructed

q-deformations of all irreducible $*$-representations of iso_3. Remark that irreducible $*$-representations of $U_q(\mathrm{iso}_3)$ of class 1 with respect to the subalgebra $U_q(\mathrm{so}_3)$ (that is, in the spaces of these representations, there exist vectors invariant with respect to this subalgebra) were constructed in [5].

We find the spectrum and spectral measure of the representation operator corresponding to the generator I_3 of $U_q(\mathrm{iso}_3)$. This operator is bounded and has a discrete spectrum. It is interesting that in the classical case (i. e., when $q = 1$) this operator is bounded and has a continuous spectrum. We could find this spectrum and the spectral measure by means of involving into consideration the theory of q-orthogonal polynomials [6, 7]. The operators $T_{\rho,s}(I_3)$ are representable by Jacobi matrices. Thus, we can employ the theory of such operators [8] and this leads to the theory of q-orthogonal polynomials.

Everywhere below we assume that the deformation parameter q lies in the finite interval $0 < q < 1$.

2. The q-deformed algebra $U_q(\mathrm{so}_3)$ and its representations

The algebra $U_q(\mathrm{so}_3)$ is a q-deformation of the universal enveloping algebra of the Lie algebra so(3) of the rotation group $SO(3)$. It is generated by three elements I_{21}, I_{32} and I_{31} satisfying the relations

$$[I_{21}, I_{32}]_{q^{1/4}} \equiv q^{1/4} I_{21} I_{32} - q^{-1/4} I_{32} I_{21} = I_{31}, \tag{1}$$

$$[I_{32}, I_{31}]_{q^{1/4}} = I_{21}, \qquad [I_{31}, I_{21}]_{q^{1/4}} = I_{32}. \tag{2}$$

Unfortunately, a Hopf algebra structure is not known on $U_q(\mathrm{so}_3)$. Nevertheless, it is shown [9] that we can consider tensor products of irreducible finite dimensional representations which are q-deformations of irreducible representations of the Lie algebra so_3.

Let us remark that according to (1), the element I_{31} is determined by I_{21} and I_{32}. Thus, the algebra $U_q(\mathrm{so}_3)$ can be defined by I_{21} and I_{32}, but now, instead of the quadratic relations (1) and (2), we must take the cubic relations [10]

$$\begin{aligned} &I_{21} I_{32}^2 - (q^{1/2} + q^{-1/2}) I_{32} I_{21} I_{32} + I_{32}^2 I_{21} = -I_{21}, \\ &I_{21}^2 I_{32} - (q^{1/2} + q^{-1/2}) I_{21} I_{32} I_{21} + I_{32} I_{21}^2 = -I_{32} \end{aligned}$$

which can be written down in the form

$$[[I_{21}, I_{32}]_{q^{1/4}}, I_{32}]_{q^{-1/4}} = -I_{21}, \quad [[I_{32}, I_{21}]_{q^{1/4}}, I_{21}]_{q^{-1/4}} = -I_{32}.$$

The formulas $I_{21}^* = -I_{21}$ and $I_{32}^* = -I_{32}$ determine the $*$-algebra structure on $U_q(\mathrm{so}_3)$. The formulas $I_{21}^* = -I_{21}$ and $I_{32}^* = I_{32}$ determine on $U_q(\mathrm{so}_3)$ the $*$-structure defining the q-deformed algebra $U_q(\mathrm{so}_{2,1})$.

We need below only those irreducible representations of $U_q(\mathrm{so}_3)$ which are q-deformations of irreducible representations of the Lie algebra so_3. These representations are given by nonnegative integers or half-integers l. The representation T_l, labeled by a number l, acts on the linear space V_l with the orthonormal basis

$$|m\rangle, \quad m = -l, -l+1, \ldots, l, \tag{3}$$

and is given in terms of q-numbers $[a]=(q^{a/2}-q^{-a/2})/(q^{1/2}-q^{-1/2})$ by the formulas

$$T_l(I_{21})|m\rangle = i[m]|m\rangle, \tag{4}$$

$$T_l(I_{32})|m\rangle = d(m)([l-m][l+m+1])^{1/2}|m+1\rangle - d(m-1)([l-m+1][l+m])^{1/2}|m-1\rangle, \tag{5}$$

$$\begin{aligned} T_l(I_{31})|m\rangle = \mathrm{i}q^{1/2}\{q^m d(m)([l-m][l+m+1])^{1/2}|m+1\rangle \\ +q^{-m}d(m-1)([l-m+1][l+m])^{1/2}|m-1\rangle\}, \end{aligned} \tag{6}$$

where $\mathrm{i}=\sqrt{-1}$ and

$$d(m)=\left(\frac{[m][m+1]}{[2m][2m+2]}\right)^{1/2}=\left(\frac{1}{(q^m+q^{-m})(q^{m+1}+q^{-m-1})}\right)^{1/2}.$$

Let us note that the operators $T_l(I_{21})$ and $T_l(I_{32})$ are anti-Hermitian. The operator $T_l(I_{31})$ is not anti-Hermitian. Relations (1) and (2) do not allow us to make both operators $T_l(I_{32})$ and $T_l(I_{31})$ anti-Hermitian since the element I_{31} is not invariant under the $*$-operation.

Note that T_l, $l=0,\frac{1}{2},1,\ldots$, do not exhaust all irreducible representations of $U_q(\mathrm{so}_3)$. The classification of irreducible $*$-representations of $U_q(\mathrm{so}_3)$ is given in [11].

3. The q-deformed algebra $U_q(\mathrm{iso}_3)$

In order to construct the q-deformed algebra $U_q(\mathrm{iso}_3)$, we add to the generating elements I_{21} and I_{32} of the algebra $U_q(\mathrm{so}_3)$ the element I_3 which satisfies the relations

$$[I_3,I_{21}]\equiv I_3I_{21}-I_{21}I_3=0, \tag{7}$$

$$I_{32}^2I_3-(q^{1/2}+q^{-1/2})I_{32}I_3I_{32}+I_3I_{32}^2=-I_3, \tag{8}$$

$$I_3^2I_{32}-(q^{1/2}+q^{-1/2})I_3I_{32}I_3+I_{32}I_3^2=0. \tag{9}$$

The associative algebra generated by the elements $I_{i,i-1}$, $i=2,3$, and I_3 obeying relations (1), (2) and (7)–(9) is denoted by $U_q(\mathrm{iso}(3,\mathbf{C}))$. Introducing the involution (antilinear antiautomorphism)

$$I_{i,i-1}^*=-I_{i,i-1}, \quad I_3^*=-I_3 \tag{10}$$

into $U_q(\mathrm{iso}(3,\mathbf{C}))$, we obtain the algebra $U_q(\mathrm{iso}_3)$. The involution

$$I_{21}^*=-I_{21}, \quad I_{32}^*=I_{32}, \quad I_3^*=-I_3 \tag{11}$$

determines the algebra $U_q(\mathrm{iso}_{2,1})$.

If $q\to 1$, then the algebra $U_q(\mathrm{iso}_3)$ tends to the universal enveloping algebra of the Lie algebra of the Euclidean group $ISO(3)$ (the group of motions of a 3-dimensional Euclidean space).

The element I_3 is a q-analogue of the infinitesimal operator for shifts along to the third axis. We may determine the elements in $U_q(\mathrm{iso}_3)$ which are q-analogues of infinitesimal operators for shifts along the first and second axes. They are given by

$$I_2=q^{1/4}I_{32}I_3-q^{-1/4}I_3I_{32}, \qquad I_1=q^{1/4}I_{31}I_3-q^{-1/4}I_3I_{31}\equiv q^{1/4}I_{21}I_2-q^{-1/4}I_2I_{21}.$$

However, the elements I_1 and I_2 are not invariant under the $*$-operation.

The algebra $U_q(\mathrm{iso}_3)$ can be obtained by means of the contraction from the algebra $U_q(\mathrm{so}_4)$. The last algebra is a generalization of the algebra $U_q(\mathrm{so}_3)$ and is generated by the elements $I_{i,i-1}$, $i = 1, 2, 3$, satisfying the defining relations

$$[I_{43}, I_{21}] \equiv I_{43}I_{21} - I_{21}I_{43} = 0,$$

$$I_{i,i-1}^2 I_{i+1,i} - (q^{1/2} + q^{-1/2}) I_{i,i-1} I_{i+1,i} I_{i,i-1} + I_{i+1,i} I_{i,i-1}^2 = -I_{i+1,i},$$

$$I_{i+1,i}^2 I_{i,i-1} - (q^{1/2} + q^{-1/2}) I_{i+1,i} I_{i,i-1} I_{i+1,i} + I_{i,i-1} I_{i+1,i}^2 = -I_{i,i-1},$$

where $i = 2, 3$. Replacing I_{43} by RI_3 in the last two relations taken for $i = 3$ and tending R to infinity, we obtain the defining relations for the algebra $U_q(\mathrm{iso}_3)$.

4. Representations of $U_q(\mathrm{iso}_3)$

We describe those irreducible infinite-dimensional $*$-representations of $U_q(\mathrm{iso}_3)$ which are q-deformation of irreducible infinite-dimensional $*$-representations of the Lie algebra iso_3 (they are infinitesimal forms of irreducible unitary representations of the group $ISO(3)$). The last representations $T'_{\rho,s}$ of iso_3 are given by two numbers $\rho \in \mathbb{R}$ and $s \in \frac{1}{2}\mathbb{Z}$. They act in the Hilbert space V_s with the orthonormal basis

$$|l, m\rangle, \quad l = |s|, |s|+1, |s|+2, \ldots, \quad m = -l, -l+1, \ldots, l.$$

In fact, this basis is the set of bases $|l, m\rangle$, $m = -l, -l+1, \ldots, l$, of irreducible representations of the subalgebra so(3) and the restriction of $T'_{\rho,s}$ onto the subalgebra so_3 decomposes into the sum of the irreducible representations T_l of this subalgebra, for which $l = |s|, |s|+1, \ldots, \infty$.

The corresponding irreducible representations of $U_q(\mathrm{iso}_3)$ are denoted by $T_{\rho,s}$, where ρ and s take the same values. The representation $T_{\rho,s}$ acts in the space V_s described above and is given in the basis $\{|l, m\rangle\}$ by (4) and (5) for the operators $T_{\rho,s}(I_{21})$ and $T_{\rho,s}(I_{32})$ and by the formula

$$\begin{aligned} T_{\rho,s}(I_3)|l, m\rangle &= \mathrm{i}\rho \frac{[s][m]}{[l][l+1]} |l, m\rangle - \rho \left(\frac{[l+s][l-s][l+m][l-m]}{[l]^2[2l-1][2l+1]} \right)^{1/2} |l-1, m\rangle \\ &+ \rho \left(\frac{[l+s+1][l-s+1][l+m+1][l-m+1]}{[l+1]^2[2l+1][2l+3]} \right)^{1/2} |l+1, m\rangle, \end{aligned} \tag{12}$$

where numbers in the square brackets are q-numbers.

It is proved by direct (but awkward) calculation that the operators $T_{\rho,s}(I_{21})$, $T_{\rho,s}(I_{32})$ and $T_{\rho,s}(I_3)$ satisfy the defining relations of the algebra $U_q(\mathrm{iso}_3)$. We omit this calculation.

Theorem 1 *The representations $T_{\rho,s}$, $\rho \neq 0$, are $*$-representations for $U_q(\mathrm{iso}_3)$. They are irreducible and pairwise nonequivalent.*

Proof. It is checked by direct calculation that the operators $T_{\rho,s}(I_{21})$, $T_{\rho,s}(I_{32})$ and $T_{\rho,s}(I_3)$ satisfy the conditions defining $*$-representations of $U_q(\mathrm{iso}_3)$. Irreducibility of $T_{\rho,s}$, $\rho \neq 0$, is proved by the standard method. Pairwise nonequivalence of these representations follows from the fact that the operator $T_{\rho,s}(I_3)$ has different spectra for different values of the pair (ρ, s). The spectrum of $T_{\rho,s}(I_3)$ will be found in the next section.

5. Spectrum of the operator $T_{\rho,s}(I_3)$

Let us find the spectrum of the operator $L_\rho = \mathrm{i}T_{\rho,s}(I_3)$, $\mathrm{i} = \sqrt{-1}$. The carrier space V_s of the representation $T_{\rho,s}$ can be represented as the direct sum $V_s = \sum_m \otimes V_{sm}$, $m = 0, 1, 2, \ldots$, where

$$V_{sm} = \sum_{l=\max\,\{s,m\}}^{\infty} \otimes \mathbf{C}|l,m\rangle.$$

The subspaces V_{sm} are invariant with respect to the operator L_ρ. We shall find spectra of L_ρ on each of these subspaces. The spectrum of L_ρ on V_s is obtained by uniting these spectra.

Further we consider the vectors $(-\mathrm{i})^{-l}|l,m\rangle$ instead of the vectors $|l,m\rangle$. In this case, the third summand in (12) must be multiplied by $-\mathrm{i}$ and the second one by i.

If $|x,m\rangle$ is an eigenvector of the operator L_ρ: $L_\rho|x,m\rangle = x|x,m\rangle$, then

$$|x,m\rangle = \sum_{l=k}^{\infty} P_{l-k}(x)|l,m\rangle, \quad k = \max\,(|m|,\ |s|). \tag{13}$$

Formula (12) is symmetric with respect to permutation of s and m and to change of signs at m and s. Therefore, we may assume, without loss of generality, that s and m are positive and that $s \geq m$.

Substituting expression (13) for $|x,m\rangle$ into the relation $L_\rho|x,m\rangle = x|x,m\rangle$ and acting by L_ρ upon $|l,m\rangle$, we easily find that the vector $|x,m\rangle$ is an eigenvector of L_ρ with the eigenvalue x if P_{l-k} satisfy the recurrence relation

$$\begin{aligned}
&\left(\frac{[n+2s+1][n+1][n+s+m+1][n+s-m+1]}{[n+s+1]^2[2n+2s+1][2n+2s+3]}\right)^{1/2} P_{n+1}(x)\\
&\quad+\left(\frac{[n+2s][n][n+s+m][n+s-m]}{[n+s]^2[2n+2s-1][2n+2s+1]}\right)^{1/2} P_{n-1}(x)\\
&\quad-\frac{[s][m]}{[n+s][n+s+1]}P_n(x) = \frac{x}{\rho}P_n(x)
\end{aligned} \tag{14}$$

(here $n = l-k$) and the initial conditions $P_0(x) = 1$, $P_{-1}(x) = 0$.

Making in (14) the substitution

$$P_n(x) = q^{-n(n+2s+1)/4}\left(\frac{[n+2s]![n+s+m]![2n+2s+1]}{[n]![n+s-m]![s+m]![2s+1]!}\right)^{1/2} P'_n(x),$$

where $[n]! = [n][n-1]\ldots[1]$, we reduce (22) to the recurrence relation

$$\begin{aligned}
&\frac{(1-q^{n+2s+1})(1-q^{n+s+m+1})(1+q^{n+s+1})}{(1-q^{2n+2s+1})(1-q^{2n+2s+2})}P'_{n+1}(x)\\
&\quad+\frac{q^{n+2s+m+1}(1-q^n)(1+q^{n+s})(1-q^{n+s-m})}{(1-q^{2n+2s+1})(1-q^{2n+2s})}P'_{n-1}(x)\\
&\quad-\frac{q^{n+s+1}(1-q^m)(1-q^s)}{(1-q^{n+s})(1-q^{n+s+1})}P'_n(x) = q^{(s+m+1)/2}\frac{x}{\rho}P'_n(x).
\end{aligned} \tag{15}$$

To solve this recurrence relation, we use the following recurrence relation

$$A_n p_{n+1}(y) - C_n p_{n-1}(y) - (A_n - C_n - 1)p_n(y) = y p_n(y) \tag{16}$$

for big q-Jacobi polynomials [6]

$$p_n(y) \equiv p_n(y; a, b, c|q) = {}_3\varphi_2 \left(\begin{matrix} q^{-n}, \ abq^{n+1}, \ y \\ aq, \ \ cq \end{matrix} ; q, \ q \right),$$

where ${}_3\varphi_2$ is the q-hypergeometric function and

$$A_n = \frac{(1-aq^{n+1})(1-cq^{n+1})(1-acq^{n+1})}{(1-abq^{2n+1})(1-abq^{2n+2})}, \quad C_n = \frac{(1-q^n)(1-bq^n)(1-abc^{-1}q^n)acq^{n+1}}{(1-abq^{2n})(1-abq^{2n+1})}.$$

Setting into (16)

$$a = q^{s+m}, \quad b = q^{s-m}, \quad c = -q^s, \quad y = \frac{x}{\rho} q^{(m+s+1)/2}, \tag{17}$$

after some calculation, we reduce (16) to (15). This means that the solution of the recurrence relations (14), normed by the condition $P_0(x) = 1$, is the polynomial

$$P_n(x) = N_n^{1/2} p_n(y; q^{s+m}, \ q^{s-m}, \ -q^s \mid q), \tag{18}$$

where

$$N_n = q^{-n(n+2s+1)/2} \frac{[n+2s]![n+s+m]![s-m]![2n+2s+1]}{[n]![n+s-m]![s+m]![2s+1]!}. \tag{19}$$

Note that the same result is obtained by setting $a = b = -q^s$, $c = q^{s+m}$ in (16) and retaining y from (17).

The big q-Jacobi polynomials in a general case satisfy the orthogonality relation, which can be given by formulas (7.3.12-14) from [6] (the formula (7.3.13) is corrected):

$$\int_{cq}^{aq} p_n(y; a, b, c|q) p_m(y; a, b, c|q) \mu(y) d_q y = \delta_{nm}/h_\infty \cdot h_n, \tag{20}$$

where

$$\mu(y) = (y/a; q)_\infty (y/c; q)_\infty / (y; q)_\infty (by/c; q)_\infty,$$

$$h_\infty = \frac{(aq;q)_\infty (bq;q)_\infty (cq;q)_\infty (abq/c;q)_\infty}{aq(1-q)(c/a;q)_\infty (aq/c;q)_\infty (q;q)_\infty (abq^2;q)_\infty}$$

and

$$h_n = \frac{(1-abq^{2n+1})(abq;q)_n (aq;q)_n (cq;q)_n}{(1-abq)(q;q)_n (abq/c;q)_n (bq;q)_n} (-ac)^{-n} q^{-n(n+3)/2}.$$

Here $(a;q)_\infty = \prod_{i=0}^{\infty}(1-aq^i)$, $(a;q)_n = (a;q)_\infty/(aq^n;q)_\infty$. The integral on the left-hand side of (19) is understood as a q-integral, see [6]:

$$\int_a^b f(t) d_q t = \int_0^b f(t) d_q t - \int_0^a f(t) d_q t, \quad \int_0^a f(t) d_q t = a(1-q) \sum_{n=0}^{\infty} f(aq^n) q^n.$$

Let us express $\mu(y)$, h_∞, and h_n through the parameters a, b, c, and y from (17). Taking into account that $(aq^{-n};q)_n = (-a)^n q^{-n(n+1)/2}(qa^{-1};q)_n$, we have

$$\mu(y) = (yq^{-s-m};q)_{s+m}(-yq^{-s};q)_s/(-yq^{-m};q)_m.$$

With formulas

$$(-q^{n+1};q)_\infty/(-q^{m+1};q)_\infty = q^{-\alpha}[n]![2m]!!/[m]![2n]!!,$$

$$(-q^{n+1};q)_\infty/(-q^{-m};q)_\infty = q^{-\alpha}[n]![m]!/2[2m]!![2n]!!$$

and

$$(-q;q)_n = q^{n/2}[2n](-1;q)_n/2[n] = q^{n(n+1)/4}[2n]!!/[n]!,$$

where $\alpha = (n-m)(n+m+1)/4$, $(q;q)_n = q^{n(n-1)/4}(1-q)^n[n]!$, $(q^2;q^2)_n = q^{n^2/2}(1-q)^n[2n]!!$, $[n]!! = [n][n-2]\cdots[1]$ or $[2]$, the expression for h_∞ can be transformed into the following:

$$h_\infty = \frac{1}{2}q^{-(s+m+2)/2}\left(\frac{[s]!}{[2s]!!}\right)^2 \frac{[2s+1]}{[s-m]![s+m]!}.$$

For $a = b = q^{-s}$, $c = q^{s+m}$, the quantity h_∞ is negative.

The expression for h_n coincides with that for N_n in (18), as it should be. This can be easily verified with the help of

$$(q^{m+1};q)_n = q^{n(n+2m-1)/4}(1-q)^n[m+n]!/[m]!. \tag{21}$$

The facts mentioned above imply that the polynomials $P_n(x)$ in (18) satisfy the orthogonality relation

$$\int_{-q^{s+1}}^{q^{s+m+1}} P_n(x)P_m(x)w(y)d_q(y) = \delta_{mn}, \tag{22}$$

where

$$w(y) = \frac{1}{2}q^{-(s+m+2)/2}\left(\frac{[s]!}{[2s]!!}\right)^2 \frac{[2s+1]!}{[s-m]![s+m]!}\frac{(-yq^{-s};q)_s(yq^{-s-m};q)_{s+m}}{(-yq^{-m};q)_m},$$

$y = xq^{(s+m+1)/2}/\rho$. In more explicit form,

$$\sum_{k=0}^{\infty} P_n(z_k)P_m(z_k)W(r_k) + \sum_{k=0}^{\infty} P_n(z'_k)P_m(z'_k)W'(r_k) = \delta_{mn}, \tag{23}$$

where $z_k = \rho q^{k+(s+m+1)/2}$, $z'_k = -q^{-m}z_k$, $r_k = q^{k+s+m+1}$ and $W(r_k) = (1-q)r_k w(r_k)$, $W'(r_k) = -W(-q^{-m}r_k)$. Straightforward calculation with (20) and

$$(-q^{m+1};q)_n = q^{n(n+2m+1)/4}[m]![2m+2n]!!/[m+n]![2m]!!$$

leads to

$$W(r_k) = \frac{1}{2}q^{s(s+m+1)/2+k(s+1)}(1-q)^{s+m+1}\left(\frac{[s]!}{[2s]!!}\right)^2 \frac{[2s+1]!}{[s-m]![s+m]!} \times \frac{[k+s+m]![k+m]![2k+2s]!!}{[k]![k+s]![2k+2m]!!},$$

and $W'(r_k) = W(r_k)_{m\to -m}$.

Formula (22) demonstrates that the spectrum of the operator L_ρ on the subspace V_{sm} is a discrete set of points $-\rho q^{k+(s-m+1)/2}$ and $\rho q^{k+(s+m+1)/2}$, $k = 0, 1, 2, \cdots$. Since $0 < q < 1$, the accumulation point of the spectrum is zero. Joining spectra for all subspaces V_{sm}, we obtain the spectrum of the operator $T_{\rho s}(I_3)$ on V_s.

6. Case $q = 1$

Let $|x, m\rangle$ at $q = 1$ be an eigenvector of the operator L_ρ with an eigenvalue x, $L_\rho|x, m\rangle = x|x, m\rangle$ and

$$|x, m\rangle = \sum_{l=s}^{\infty} \tilde{P}_{l-s}(x)|x, m\rangle. \tag{24}$$

The formula of action of the operator L_ρ upon the basis vectors $(-\mathrm{i})^{-l}|l, m\rangle$ at $q = 1$ is obtained from (12) by the substitution $[r] \to r$ for any c-number r. Then, repeating the procedure of the preceding section, we find that the functions $\tilde{P}_n(x)$ in (23) take the form

$$\tilde{P}_n(x) = \left(\frac{(s-m)!(s+m)!n!(n+2s)!(2n+2s+1)}{(2s+1)!(n+s-m)!(n+s+m)!}\right)^{1/2} P_n^{s+m,s-m}(x/\rho), \tag{25}$$

where $P_n^{(\alpha,\beta)}(x)$ is an ordinary Jacobi polynomial. With use of the orthogonality relation for these polynomials (see, e.g., [12]), we obtain

$$\int_{-1}^{1} \tilde{P}_n(x)\tilde{P}_m(x)\tilde{w}(y)dy = \delta_{mn}. \tag{26}$$

Here $y = x/\rho$ and

$$\tilde{w}(y) = 2^{-(2s+1)}\frac{(2s+1)!}{(s-m)!(s+m)!}(1-y)^{s+m}(1+y)^{s-m}.$$

We see that the spectrum of the operator L_ρ on the subspace V_{sm} is continuous at $q = 1$ and consists of points in the interval $[-\rho, \rho]$.

Remark that big q-Jacobi polynomials have the property

$$\lim_{q\to 1} p_n(x; q^\alpha, q^\beta, -q^\lambda|q) = P_n^{(\alpha,\beta)}(x)/P_n^{(\alpha,\beta)}(1).$$

In this limit, $P_n(x) \to \tilde{P}_n(x)$ (see formulas (18) and (24)) and $w(y) \to \tilde{w}(y)$ (see (21) and (25)). If, in additions, to use the property

$$\lim_{q\to 1} \int_0^a f(t)d_q(t) = \int_0^a f(t)d(t),$$

where f is a continuous function on $[0, a]$, one easily verifies that the orthogonality relation (21) for the polynomials $P_n(x)$ at $q \to 1$ goes to the orthogonality relation (25) for the polynomials $\tilde{P}_n(x)$.

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