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A Dynamical Mapping Method in Nonrelativistic Models of Quantum Field Theory *

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Abstract

Exact solutions of Heisenberg equations and two-particle eigenvalue problems for the nonrelativistic four-fermion interaction and N,Θ model are obtained in the framework of a dynamical mapping method. Equivalence of different dynamical mappings is shown.

1. General consideration

The main problem of QFT follows from the fact that any solutions of Heisenberg equations (HE) are operator distributions which products, always appearing in those equations, are ill-defined [1].

$$(i\partial_t - \mathcal{E}(\mathbf{P}))\,\Psi_\alpha(\vec{\mathrm{x}},t) = [\Psi_\alpha(\vec{\mathrm{x}},t)\,,\mathrm{H}_I\{\Psi\}] =? \;\text{ for }\quad \mathrm{H}\{\Psi\} = \mathrm{H}_0\{\Psi\} + \mathrm{H}_I\{\Psi\}. \qquad (1)$$

So, the correct definition of field equations (and Hamiltonian itself) implies some knowledge about qualitative properties of their solutions which, in turn, depend on the form of these equations in a very singular manner. The usual way to go out from this closed circle is connected with perturbation theory. It is based on the assumption that a product of Heisenberg fields (HF) may be defined like for the free ones and a solution of HE may be obtained by perturbation in the Fock space of renormalized *free* fields. However, it is impossible in such a way to treat a nonrenormalizable theory and to understand the origin of bound states. We consider another possibility which is based on the idea of dynamical mapping and reduce the product of HF to the normal ordering of the product of *physical fields*. The idea originates from the works of R. Haag [2], O. Greenberg [3], H. Umezawa [4], and L.D. Faddeev, M.I. Shirokov [5] (see also [6]). In this approach, the problem to make formal expression of HF

$$\Psi(\vec{\mathrm{x}},t) = e^{i\mathrm{H}(t-t_0)}\,\Psi(\vec{\mathrm{x}},t_0)\,e^{-i\mathrm{H}(t-t_0)} \Longrightarrow \mathcal{F}^t\,[\Psi(\vec{\mathrm{x}},t_0)]\,, \qquad (2)$$

for the Hamiltonian given as a functional $\mathrm{H} = \mathrm{H}\,[\Psi(\vec{\mathrm{x}},t)]$, be meaningful, is divided into two parts. The first one is to construct such an operator realization of the initial fields

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$\Psi(\vec{x}, t_0) = \Psi[\psi]$ via *physical fields* $\psi(\vec{x}, t) \equiv \psi\left\{A_\alpha(\vec{k})\right\}$, which, on the one hand, must be consistent with CCR (CAR) $(\alpha = 1, 2)$

$$\begin{aligned} &\{\Psi_\alpha(\vec{x}, t)\,,\, \Psi_\beta(\vec{y}, t)\} = 0 = \{\psi_\alpha(\vec{x}, t)\,,\, \psi_\beta(\vec{y}, t)\}, \\ &\{\Psi_\alpha(\vec{x}, t)\,,\, \Psi^\dagger_\beta(\vec{y}, t)\} = \delta_3(\vec{x} - \vec{y})\, \delta_{\alpha\beta} = \{\psi_\alpha(\vec{x}, t)\,,\, \psi^\dagger_\beta(\vec{y}, t)\}, \\ &\{A_\alpha(\vec{k})\,,\, A_\beta(\vec{q})\} = 0; \quad \{A_\alpha(\vec{k})\,,\, A^\dagger_\beta(\vec{q})\} = \delta_3(\vec{k} - \vec{q})\, \delta_{\alpha\beta}, \end{aligned} \tag{3}$$

and, on the other hand, leads to the unique stable vacuum $\mid 0\rangle$ and one-particle state $\mid 1\vec{k}, \alpha\rangle$ with a definite spectrum $E(\vec{k})$:

$$H \mid 0\rangle = V\, w_0 \mid 0\rangle; \qquad A_\alpha(\vec{k}) \mid 0\rangle = 0; \qquad V \text{ is space volume;} \tag{4}$$

$$[H, A^\dagger_\alpha(\vec{k})] \mid 0\rangle = E(k) A^\dagger_\alpha(\vec{k}) \mid 0\rangle = E(k) \mid 1\vec{k}, \alpha\rangle = (H - V\, w_0) \mid 1\vec{k}, \alpha\rangle. \tag{5}$$

Moreover, let us suppose that for this operator realization the reduced (time-independent) Hamiltonian does not contain "fluctuation" terms [7] at a definite moment $t = t_0$ up to the fourth order and looks like :

$$\begin{aligned} &H \equiv H\{A\} = V\, w_0 + \hat{H}\{A\} =: \mathrm{H}\{\Psi\,[\psi[A]]\} := V\, w_0 + \hat{H}_0\{A\} + \hat{H}_I\{A\}; \\ &\hat{H} \mid 0\rangle = \hat{H}_0 \mid 0\rangle = \hat{H}_I \mid 0\rangle = 0; \\ &\hat{H}_0\{A\} \stackrel{def}{\equiv} \int d^3k E(k) A^\dagger_\alpha(\vec{k}) A_\alpha(\vec{k}); \quad [\hat{H}_0, A^\dagger_\alpha(\vec{k})] \equiv E(k) A^\dagger_\alpha(\vec{k}); \\ &\hat{H}_I\{A\} = \int d^3q \int d^3p \int d^3\kappa \int d^3l\; \delta_3(\vec{q} + \vec{p} - \vec{\kappa} - \vec{l})\; K^{q+p}_{22}\left(\frac{\vec{q} - \vec{p}}{2}, \frac{\vec{\kappa} - \vec{l}}{2}\right) \cdot \\ &\cdot A^\dagger_\alpha(\vec{\kappa}) A^\dagger_\beta(\vec{l}) A_\beta(\vec{p})\, A_\alpha(\vec{q}) + \sim \sum (A^\dagger)^m (A)^n; \qquad m, n \geq 3. \\ &K^{\mathcal{P}}_{22}(\vec{r}, \vec{s}) = K^{\mathcal{P}}_{22}(-\vec{r}, -\vec{s}); \qquad \overset{*}{K}{}^{\mathcal{P}}_{22}(\vec{s}, \vec{r}) = K^{\mathcal{P}}_{22}(\vec{r}, \vec{s}); \end{aligned} \tag{6}$$

General conditions of the existence of such operator realizations lie beyond of issues of our work. They always exist for the Lee models considered below.

This work is devoted to the second part of the problem, that is to construct the corresponding dynamical mapping (Haag expansion) $\mathcal{F}^t\,[\Psi(\vec{x}, t_0)]$ as a series of normal ordered products of *physical fields* $\psi(\vec{x}),\ \ A(\vec{k})$:

$$e^{iH(t-t_0)} A_\alpha(\vec{k}) e^{-iH(t-t_0)} \equiv e^{-iE(k)(t-t_0)} a_\alpha(\vec{k}, t) = e^{-iE(k)(t-t_0)}\; \mathcal{A}^t_{t_0}\left[A_\alpha(\vec{k})\right], \tag{7}$$

$$a_\alpha(\vec{k}, t) = A_\alpha(\vec{k}) + \int d^3l \int d^3p\, A^\dagger_\beta(\vec{l} + \vec{p}) A_\beta(\vec{k} + \vec{p}) A_\alpha(\vec{l})\, F^{(1)}_A(t; \vec{p} \mid \vec{l}, \vec{k}) + \ldots \tag{8}$$

$$\begin{aligned} &(\text{for } m = n) \;\; \Longrightarrow \mathcal{A}^t_{t_0}\left[A_\alpha(\vec{k})\right] = A_\alpha(\vec{k}) + \sum_{n=1}^{\infty} \int d^3l \prod_{j=1}^{n} \left\{\int d^3q_j \int d^3p_j\right\} \cdot \\ &\qquad \cdot \prod_{j=1}^{n} \left\{A^\dagger_{\beta_j}(\vec{q}_j)\right\} \prod_{j=n}^{1} \left\{A_{\beta_j}(\vec{p}_j)\right\} A_\alpha(\vec{l})\, Y^{(n)}_A(t; \vec{k}; \{\vec{q}_j\}^n_1 \,|\{\vec{p}_j\}^1_n; \vec{l}), \end{aligned} \tag{9}$$

and to exploit these coefficient functions F, Y for the eigenvalue problem. The condition $m = n$ in eq.(9) means the absence of "fluctuation terms" (with $m \neq n$) when the reduced Hamiltonian (6) commutes with the particle number operator.

It follows from eqs. (7), (8) that vacuum and one-particle states remain stable for all t,

$$\begin{aligned}
&a_\alpha(\vec{\mathrm{k}},t)\mid 0\rangle \Longrightarrow A_\alpha(\vec{\mathrm{k}})\mid 0\rangle \equiv 0; \quad \mid 1\vec{\mathrm{k}},\alpha\rangle = a^\dagger_\alpha(\vec{\mathrm{k}},t)\mid 0\rangle \Longrightarrow A^\dagger_\alpha(\vec{\mathrm{k}})\mid 0\rangle;\\
&\Big[H,\, a^\dagger_\alpha(\vec{\mathrm{k}},t)\Big]\mid 0\rangle = E(k)\, A^\dagger_\alpha(\vec{\mathrm{k}})\mid 0\rangle,
\end{aligned} \tag{10}$$

that allows one to *define* the normal ordering directly for HF and the normal ordered Hamiltonian (6) now correctly determines the nonlinear terms in reduced HE:

$$(i\partial_t - E(\mathbf{P}))\,\Psi_\alpha(\vec{\mathrm{x}},t) = \Big[\Psi_\alpha(\vec{\mathrm{x}},t)\,,\hat{H}_I\{\Psi\}\Big]\,. \tag{11}$$

$$i\partial_t a_\alpha(\vec{\mathrm{k}},t) = \Big[a_\alpha(\vec{\mathrm{k}},t),\, \hat{H}_I\{a\}\Big] \Longrightarrow \int d^3l\, Q_{(a)}(\vec{\mathrm{k}},\vec{\mathrm{l}};t)\, a_\alpha(\vec{\mathrm{l}},t); \tag{12}$$

$$\begin{aligned}
Q_{(a)}(\vec{\mathrm{k}},\vec{\mathrm{l}};t) = &\int d^3q \int d^3p\, a^\dagger_\beta(\vec{\mathrm{q}},t)\, a_\beta(\vec{\mathrm{p}},t)\, e^{it[E(k)+E(q)-E(p)-E(l)]}\cdot\\
&\cdot\delta_3(\vec{\mathrm{k}}+\vec{\mathrm{q}}-\vec{\mathrm{p}}-\vec{\mathrm{l}})\,\frac{2}{i}\,K_{22}^{\vec{\mathrm{l}}+\vec{\mathrm{p}}}\left(\frac{\vec{\mathrm{l}}-\vec{\mathrm{p}}}{2},\frac{\vec{\mathrm{k}}-\vec{\mathrm{q}}}{2}\right).
\end{aligned} \tag{13}$$

Case (9) means, moreover, stability not only for vacuum and one-particle states but also for arbitrary N-particle ones. So, for arbitrary N, one can reduce the product:

$$\langle 0\mid \prod_{i=1}^{N} a_\alpha(\vec{\mathrm{k}}_i,t) \xrightarrow{to}\sim \langle 0\mid \sum \prod_{i=1}^{N}\int d^3s_i\, A_\alpha(\vec{\mathrm{s}}_i). \tag{14}$$

However, if fluctuation terms appear with min $(m,n)\ge N_0$, then reduction (14) is possible only for $N<N_0$.

There exist two essentially different choices for the initial moment t_0 leading to different choices of physical fields, correspondingly:

$t_0\to-\infty$, (Greenberg, Umezawa)	$t_0=0$, (Faddeev, Shirokov)
nonoperator initial condition	operator initial condition
$\mathrm{w}\lim\limits_{t\to-\infty}\langle f_{in}\mid\Psi(\vec{\mathrm{x}},t)-\psi_{in}(\vec{\mathrm{x}},t)\mid i_{in}\rangle=0$	$\mathrm{s}\lim\limits_{t\to0}\Psi(\vec{\mathrm{x}},t)=\Psi\left[\psi(\vec{x}.0)\right]$
$\{\psi_{in}[A_{in}]\}\to$ incomplete Fock space	$\{\psi[A]\}\to$ complete Fock space
new fields V_{in} for every bound state	no any new fields for bound states
$\hat{H}\overset{weak}{=}\hat{H}_0\{A_{in}\}+\hat{H}_0\{V_{in}\}+\dots$	$\hat{H}=\hat{H}_0\{A\}+\hat{H}_I\{A\}$

The second choice $t_0=0$ is used here. It seems more economical, and both bound and same scattering eigenstates are treated on the basis in this case.

One can check by direct substitution that solutions of both scattering and bound state two-particle eigenvalue problems

$$\begin{aligned}
&\hat{H}\mid R^{\pm}_{\alpha\beta}(\mathcal{P},\vec{\mathrm{q}})\rangle = E_2(\mathcal{P},q)\mid R^{\pm}_{\alpha\beta}(\mathcal{P},\vec{\mathrm{q}})\rangle;\quad E_2(\mathcal{P},q)\equiv E\left(\frac{\mathcal{P}}{2}+\vec{\mathrm{q}}\right)+E\left(\frac{\mathcal{P}}{2}-\vec{\mathrm{q}}\right);\\
&\hat{H}\mid \mathrm{B}^{\mathcal{P}}_{\alpha\beta}\rangle = M_2(\mathcal{P})\mid \mathrm{B}^{\mathcal{P}}_{\alpha\beta}\rangle;
\end{aligned} \tag{15}$$

exist in the Fock eigenspace of the kinetic part of the reduced Hamiltonian (6)

$$\hat{H}_0\{A\}\, A^\dagger_{\alpha_1}(\vec{\mathrm{k}}_1)\cdots A^\dagger_{\alpha_n}(\vec{\mathrm{k}}_n)\mid 0\rangle = \left(\sum_{j=1}^{n}E(k_j)\right) A^\dagger_{\alpha_1}(\vec{\mathrm{k}}_1)\cdots A^\dagger_{\alpha_n}(\vec{\mathrm{k}}_n)\mid 0\rangle, \tag{16}$$

and can be written in the following form:

$$\begin{aligned} &\mid R^{\pm}_{\alpha\beta}(\mathcal{P},\vec{\mathrm{q}})\rangle = \int d^3\vec{\kappa} \mid R^0_{\alpha\beta}(\mathcal{P},\vec{\kappa})\rangle \Phi^{\pm}_{\mathcal{P}q}(\vec{\kappa}); \quad \mid \mathrm{B}^{\mathcal{P}}_{\alpha\beta}\rangle = \int d^3\vec{\kappa} \mid R^0_{\alpha\beta}(\mathcal{P},\vec{\kappa})\rangle B^{\mathcal{P}}(\vec{\kappa}); \\ &\mid R^0_{\alpha\beta}(\mathcal{P},\vec{\mathrm{q}})\rangle \equiv A^{\dagger}_{\alpha}\left(\frac{\mathcal{P}}{2}+\vec{\mathrm{q}}\right) A^{\dagger}_{\beta}\left(\frac{\mathcal{P}}{2}-\vec{\mathrm{q}}\right) \mid 0\rangle. \end{aligned} \tag{17}$$

Corresponding wave functions satisfy the usual Lippman-Schwinger equations:

$$\begin{aligned} &\Phi^{\pm}_{\mathcal{P}q}(\vec{\kappa}) = \delta_3(\vec{\kappa}-\vec{\mathrm{q}}) + \frac{1}{E_2(\mathcal{P},q)-E_2(\mathcal{P},\kappa)\pm i\delta}\cdot\int d^3r\; \Phi^{\pm}_{\mathcal{P}q}(\vec{\mathrm{r}})\, 2\,K^{\mathcal{P}}_{22}(\vec{\mathrm{r}},\vec{\kappa}), \\ &B^{\mathcal{P}}(\vec{\kappa}) = \frac{1}{M_2(\mathcal{P})-E_2(\mathcal{P},\kappa)}\cdot\int d^3r\; B^{\mathcal{P}}(\vec{\mathrm{r}})\, 2\,K^{\mathcal{P}}_{22}(\vec{\mathrm{r}},\vec{\kappa}). \end{aligned} \tag{18}$$

In turn, at $m,n \geq 3$ for the first coefficient function of (8)

$$Y^{(1)}_A(t;\vec{\mathrm{k}};\vec{\mathrm{q}}\mid\vec{\mathrm{p}};\vec{\mathrm{l}}) \equiv \delta_3(\vec{\mathrm{k}}+\vec{\mathrm{q}}-\vec{\mathrm{p}}-\vec{\mathrm{l}})\; F^{(1)}_A(t;\vec{\mathrm{p}}-\vec{\mathrm{k}}\mid\vec{\mathrm{l}},\vec{\mathrm{k}}) \tag{19}$$

an integral equation follows from the reduced HE (12) with the same kernel:

$$\begin{aligned} &F^{(1)}_A\left(t;-\vec{\kappa}-\vec{\mathrm{q}}\mid\frac{\mathcal{P}}{2}+\vec{\kappa},\frac{\mathcal{P}}{2}+\vec{\mathrm{q}}\right) \equiv F^{(1)}_A\left(t;\vec{\kappa}+\vec{\mathrm{q}}\mid\frac{\mathcal{P}}{2}-\vec{\kappa},\frac{\mathcal{P}}{2}-\vec{\mathrm{q}}\right) = \\ &\quad = \int_0^t d\eta\, e^{i\eta[E_2(\mathcal{P},q)-E_2(\mathcal{P},\kappa)]}\left[\frac{2}{i}K^{\mathcal{P}}_{22}(\vec{\kappa},\vec{\mathrm{q}}) + \int d^3r\, e^{i\eta[E_2(\mathcal{P},\kappa)-E_2(\mathcal{P},r)]}\cdot\right. \\ &\quad \left.\cdot\frac{2}{i}K^{\mathcal{P}}_{22}(\vec{\mathrm{r}},\vec{\mathrm{q}})\, F^{(1)}_A\left(\eta;\vec{\kappa}+\vec{\mathrm{r}}\mid\frac{\mathcal{P}}{2}-\vec{\kappa},\frac{\mathcal{P}}{2}-\vec{\mathrm{r}}\right)\right]. \end{aligned} \tag{20}$$

It contains all information about the two-particle sector directly determining the scattering wave function for $E_2(\mathcal{P},q) \to E_2(\mathcal{P},q) \pm i\delta,\ \delta \to 0+$:

$$\Phi^{\pm}_{\mathcal{P}q}(\vec{\kappa}) = \delta_3(\vec{\kappa}-\vec{\mathrm{q}}) + F^{(1)*}_A\left(t=\mp\infty;-\vec{\kappa}-\vec{\mathrm{q}}\mid\frac{\mathcal{P}}{2}+\vec{\kappa},\frac{\mathcal{P}}{2}+\vec{\mathrm{q}}\right), \tag{21}$$

In (21) the simply derived expression for the scattering state was used:

$$\begin{aligned} &\mid R^{\pm}_{\alpha\beta}(\mathcal{P},\vec{\mathrm{q}})\rangle = \mid R^0_{\alpha\beta}(\mathcal{P},\vec{\mathrm{q}})\rangle + \int d^3\kappa\, 2i\, K^{\mathcal{P}}_{22}(\vec{\mathrm{q}},\vec{\kappa})\cdot \\ &\quad\cdot\int_0^{\mp\infty} dt\; e^{-it(E_2(\mathcal{P},q)-E_2(\mathcal{P},\kappa)\pm i\delta)}\, a^{\dagger}_{\alpha}\left(\frac{\mathcal{P}}{2}+\vec{\kappa},t\right) a^{\dagger}_{\beta}\left(\frac{\mathcal{P}}{2}-\vec{\kappa},t\right) \mid 0\rangle, \end{aligned} \tag{22}$$

that follows directly from (7) and the definition of scattering state. It is a simple matter to see that all integral equations above are exactly solvable for degenerate kernels: $K^{\mathcal{P}}_{22}(\vec{\mathrm{r}},\vec{\mathrm{s}}) = \sum_n V_n(\vec{\mathrm{r}})\, U_n(\vec{\mathrm{s}})$.

The dynamical mapping formulae (7), (8) give two forms of the instantaneous Bethe-Salpeter matrix element $\langle 0 \mid a_\alpha(\frac{\mathcal{P}}{2}+\vec{\kappa},t)\, a_\beta(\frac{\mathcal{P}}{2}-\vec{\kappa},t) \mid 2,\mathcal{P}\rangle$, leading to the following identities when $\mid 2,\mathcal{P}\rangle$ stands for $\mid R^{\pm}_{\alpha\beta}(\mathcal{P},\vec{\mathrm{q}})\rangle$ or $\mid \mathrm{B}^{\mathcal{P}}_{\alpha\beta}\rangle$:

$$\left.\begin{aligned} \Phi^{\pm}_{\mathcal{P}q}(\vec{\kappa})\left[e^{it(E_2(\mathcal{P},\kappa)-E_2(\mathcal{P},q)\mp i\delta)}-1\right] \\ B^{\mathcal{P}}(\vec{\kappa})\left[e^{it(E_2(\mathcal{P},\kappa)-M_2(\mathcal{P}))}-1\right] \end{aligned}\right\} = \int d^3r \begin{Bmatrix} \Phi^{\pm}_{\mathcal{P}q}(\vec{\mathrm{r}}) \\ B^{\mathcal{P}}(\vec{\mathrm{r}}) \end{Bmatrix} F^{(1)}_A\left(t;\vec{\kappa}+\vec{\mathrm{r}}\mid\frac{\mathcal{P}}{2}-\vec{\kappa},\frac{\mathcal{P}}{2}-\vec{\mathrm{r}}\right) \tag{23}$$

which have a sense of the off-shell extension of a unitarity relation .

2. Four-fermion models

As a first example, let us consider the contact four-fermion model by defining

$$
\begin{aligned}
&x = (\vec{\mathrm{x}}, t);\;\; t = x_0;\quad \mathbf{P}_{\mathrm{x}} = -i\vec{\nabla}_{\mathrm{x}};\;\; \epsilon_{\alpha\beta} = -\epsilon_{\beta\alpha};\quad \chi_{(\Psi)}(x) = \epsilon_{\alpha\beta}\,\Psi_\alpha(x)\,\Psi_\beta(x);\\
&S_{(\Psi)}(x) = \Psi^\dagger_\alpha(x)\,\Psi_\alpha(x),\qquad \vec{J}_{(\Psi)}(x) = \Psi^\dagger_\alpha(x)\,\overleftrightarrow{\mathbf{P}}\,\Psi_\alpha(x);\\
&\{\Psi_\alpha(x)\,,\Psi_\beta(y)\}\Big|_{x_0=y_0} = 0;\quad \textbf{with the convention:}\\
&\left\{\Psi_\alpha(x)\,,\Psi^\dagger_\beta(y)\right\}\Big|_{x_0=y_0} = \delta_{\alpha\beta}\,\delta_3(\vec{\mathrm{x}}-\vec{\mathrm{y}}) \Rightarrow\Big|_{\vec{\mathrm{x}}=\vec{\mathrm{y}}}\delta_{\alpha\beta}\,\tfrac{1}{V^*},
\end{aligned}
\tag{24}
$$

and considering the following local densities:

$$
\mathcal{H}_1(x) = \Psi^\dagger_\alpha(x)\,E(\mathbf{P})\,\Psi_\alpha(x) - \frac{\lambda}{8}\,\chi^\dagger_{(\Psi)}(x)\;\chi_{(\Psi)}(x);
\tag{25}
$$

$$
\mathcal{H}_2(x) = \Psi^\dagger_\alpha(x)\,\mathcal{E}(\mathbf{P})\,\Psi_\alpha(x) - \frac{\lambda}{4}\,S^2_{(\Psi)}(x) + \frac{\mu}{4}\,\vec{J}^2_{(\Psi)}(x) \Longrightarrow
\tag{26}
$$

$$
\begin{aligned}
&:\mathcal{H}_2(x) := w_0 + \Psi^\dagger_\alpha(x)\,E(\mathbf{P})\,\Psi_\alpha(x) - \frac{\lambda}{8}\,\chi^\dagger_{(\Psi)}(x)\;\chi_{(\Psi)}(x)+\\
&\quad+\frac{\mu}{4}\,\Psi^\dagger_\alpha(x)\,\left(\vec{J}_{(\Psi)}(y)\cdot\overleftrightarrow{\mathbf{P}}_{\mathrm{x}}\right)\Psi_\alpha(x)\Big|_{x=y},
\end{aligned}
\tag{27}
$$

They provide the reduced HE (11), (12) for HF:

$$
\Psi_\alpha(x) = \int \frac{d^3k}{(2\pi)^{3/2}}\,e^{i(\vec{\mathrm{k}}\cdot\vec{\mathrm{x}})-iE(k)t}\,a_\alpha(\vec{\mathrm{k}},t),
\tag{28}
$$

$$
(i\partial_t - E(\mathbf{P}))\;\;\Psi_\alpha(x) = \hat{\mathrm{V}}_{(\Psi)}(t;\vec{\mathrm{x}},\mathbf{P})\,\Psi_\alpha(x) \equiv \mathcal{J}_\alpha(x),
\tag{29}
$$

$$
\begin{aligned}
&\hat{\mathrm{V}}_{(\Psi)}(t;\vec{\mathrm{x}},\mathbf{P}) = -\frac{\lambda}{2}S_{(\Psi)}(x) + \mu\left[\left(\vec{J}_{(\Psi)}(x)\cdot\mathbf{P}_{\mathrm{x}}\right) - \frac{i}{2}\left(\vec{\nabla}_{\mathrm{x}}\cdot\vec{J}_{(\Psi)}(x)\right)\right];\\
&Q_{(a)}(\vec{\mathrm{k}},\vec{\mathrm{l}};t) = \int d^3\mathcal{P}\,a^\dagger_\beta(\mathcal{P}-\vec{\mathrm{k}},t)\,a_\beta(\mathcal{P}-\vec{\mathrm{l}},t)\,e^{it[E_2(\mathcal{P},\frac{\mathcal{P}}{2}-\vec{\mathrm{k}})-E_2(\mathcal{P},\frac{\mathcal{P}}{2}-\vec{\mathrm{l}})]}\,.\\
&\cdot\frac{2}{i}K^{\mathcal{P}}_{22}\left(\vec{\mathrm{l}}-\frac{\mathcal{P}}{2},\vec{\mathrm{k}}-\frac{\mathcal{P}}{2}\right)\;\text{ is the Fourier-image of }\; e^{itE(\mathbf{P})}\hat{\mathrm{V}}_{(\Psi)}(t;\vec{\mathrm{x}},\mathbf{P})e^{-itE(\mathbf{P})},
\end{aligned}
$$

and, for the following simple operator realizations via physical fields

$$
\begin{aligned}
&\Psi_\alpha(\vec{\mathrm{x}},0) = \Psi_\alpha\left[\psi_\alpha\left\{A(\vec{\mathrm{k}})\right\}\right] = \begin{cases} -\epsilon_{\alpha\beta}\,\psi^\dagger_\beta(\vec{\mathrm{x}},0);\quad (+),\;\; E(k)\to E^+(k)\\ \quad\;\psi_\alpha(\vec{\mathrm{x}},0);\quad (-),\;\; E(k)\to E^-(k)\end{cases}\\
&\psi_\alpha(x) = \int \frac{d^3k}{(2\pi)^{3/2}}\,e^{i(\vec{\mathrm{k}}\cdot\vec{\mathrm{x}})-iE(k)t}A_\alpha(\vec{\mathrm{k}}) = e^{-itE(\mathbf{P})}\psi_\alpha(\vec{\mathrm{x}},0),
\end{aligned}
\tag{30}
$$

lead to the reduced Hamiltonian (6) $H\{A\} = :H_2\{A\}:$ (note, that $\mathcal{H}_1(x)$ looks like a normal form of $\mathcal{H}_2(x)$ for the case $\mu = 0$), with the following degenerate kernel and parameters:

$$
\frac{2}{i}K^{\mathcal{P}}_{22}(\vec{\mathrm{r}},\vec{\mathrm{s}}) = \frac{i}{2(2\pi)^3}\left\{L + \mu\,(\vec{\mathrm{r}}+\vec{\mathrm{s}})^2\right\};\quad L \equiv \lambda - \mu\mathcal{P}^2;
\tag{31}
$$

$$E(k) \to E^{\pm}(k) = \frac{\mu}{4V^*}\left(k^2 + < k^2 >\right) + \frac{\lambda}{4V^*} \mp \left(\mathcal{E}(k) - \frac{\lambda}{2V^*}\right); \tag{32}$$

$$w_0 \to w_0^{\pm} = \frac{1 \pm 1}{V^*}\left(< \mathcal{E}(k) > - \frac{\lambda}{2V^*}\right); \;\; < \mathcal{E}(k) > \stackrel{def}{=} V^* \int \frac{d^3k}{(2\pi)^3}\,\mathcal{E}(k).$$

Here $\mathcal{E}(k)$ is an arbitrary "bare" one-particle spectrum and V^* has a meaning of the volume of the excitation. Now the solution of (20) for the first coefficient function reads :

$$F^{(1)}\left(t; \vec{\kappa} + \vec{\mathrm{q}} \mid \frac{\mathcal{P}}{2} - \vec{\kappa}, \frac{\mathcal{P}}{2} - \vec{\mathrm{q}}\right) = \int_0^t d\eta\, e^{i\eta E_2(\mathcal{P},q)} \frac{\varepsilon(t)}{2\pi i} \int_{-i\infty+\Delta\cdot\varepsilon(t)}^{i\infty+\Delta\cdot\varepsilon(t)} \frac{d\sigma\, e^{\sigma\eta}}{\sigma + iE_2(\mathcal{P},\kappa)} \cdot$$
$$\cdot \left\{ \frac{D^{\mathcal{P}}_{\kappa\,1}(\sigma) + q^2\, D^{\mathcal{P}}_{\kappa\,2}(\sigma)}{D^{\mathcal{P}}_0(\sigma)} + 2\mu\, q^j \left[\frac{\prod^{jl}_{\perp}(\mathcal{P})}{1 - 2\mu \mathrm{J}^{\mathcal{P}}_{\delta}(\sigma)} + \frac{\prod^{jl}_{\parallel}(\mathcal{P})}{1 - 2\mu \mathrm{J}^{\mathcal{P}}_{D}(\sigma)} \right] \kappa^l \right\}, \tag{33}$$

where $\varepsilon(\eta) = \varepsilon(t) \equiv \mathrm{sign}(t)$, $\Delta > 0$, $\Delta \to 0+$ and

$$\left\{ \begin{array}{c} \mathrm{J}^{\mathcal{P}}_n(\sigma) \\ \mathrm{J}^{\mathcal{P}}_D(\sigma) \\ \mathrm{J}^{\mathcal{P}}_{jl}(\sigma) \end{array} \right\} = \frac{1}{2} \int \frac{d^3r}{(2\pi)^3} \cdot \frac{1}{E_2(\mathcal{P}, r) - i\sigma} \cdot \left\{ \begin{array}{c} \left(r^2\right)^n \\ \frac{(\vec{\mathrm{r}}\cdot\mathcal{P})^2}{\mathcal{P}^2} \\ r^j r^l \end{array} \right. ; \tag{34}$$

$$\mathrm{J}^{\mathcal{P}}_{jl}(\sigma) = \delta_{jl} \mathrm{J}^{\mathcal{P}}_{\delta}(\sigma) + \frac{\mathcal{P}^j \mathcal{P}^l}{\mathcal{P}^2} \mathrm{J}^{\mathcal{P}}_R(\sigma), \quad \mathrm{J}^{\mathcal{P}}_{\delta}(\sigma) = \frac{1}{2}\left[\mathrm{J}^{\mathcal{P}}_1(\sigma) - \mathrm{J}^{\mathcal{P}}_D(\sigma)\right];$$

$$\left. \begin{array}{c} D^{\mathcal{P}}_0(\sigma) \\ D^{\mathcal{P}}_{\kappa\,1}(\sigma) \\ D^{\mathcal{P}}_{\kappa\,2}(\sigma) \end{array} \right\} = \left\{ \begin{array}{c} \left[\mu\, \mathrm{J}^{\mathcal{P}}_1(\sigma) - 1\right]^2 - \mu^2\, \mathrm{J}^{\mathcal{P}}_0(\sigma) \mathrm{J}^{\mathcal{P}}_2(\sigma) - L\, \mathrm{J}^{\mathcal{P}}_0(\sigma). \\ L + \mu\kappa^2 + \mu^2 \left[\mathrm{J}^{\mathcal{P}}_2(\sigma) - \kappa^2 \mathrm{J}^{\mathcal{P}}_1(\sigma)\right]. \\ \mu + \mu^2 \left[\kappa^2 \mathrm{J}^{\mathcal{P}}_0(\sigma) - \mathrm{J}^{\mathcal{P}}_1(\sigma)\right]. \end{array} \right. . \tag{35}$$

Setting $I^{\mathcal{P}(\pm)}_{\{\ldots\}}(q) = \mathrm{J}^{\mathcal{P}}_{\{\ldots\}}(\pm\delta - iE_2(\mathcal{P}, q))$, $\mathcal{D}^{\mathcal{P}(\pm)}_{\{\ldots\}}(q) = D^{\mathcal{P}}_{\{\ldots\}}(\pm\delta - iE_2(\mathcal{P}, q))$, from eq.(18) or from (21), (33) with $\delta > \Delta$, $\delta \to 0+$, for scattering eigenfunctions with the fixed parity (angular momentum) $l = 0, 1$ and spin J=0,1, defined in a symmetric basis (σ_j, $j = 1, 2, 3$, are usual Pauli matrices):

$$(\delta_{\alpha\beta}, (\sigma_j)_{\alpha\beta}) \longrightarrow (\epsilon_{\alpha\beta}, \tau^j_{\alpha\beta}), \quad \epsilon_{\alpha\beta} = i(\sigma_2)_{\alpha\beta}, \quad \tau^j_{\alpha\beta} = \tau^j_{\beta\alpha} = i(\sigma_2\sigma_j)_{\alpha\beta}, \tag{36}$$

$$\text{as:} \qquad \phi^{\pm(0,0)}_{\mathcal{P}q}(\vec{\kappa})_{\alpha\beta} = \epsilon_{\alpha\beta}\, \Phi^{\pm(0)}_{\mathcal{P}q}(\vec{\kappa}); \qquad \phi^{\pm(1,1,m)}_{\mathcal{P}q}(\vec{\kappa})_{\alpha\beta} = \tau^{(m)}_{\alpha\beta}\, \Phi^{\pm(1)}_{\mathcal{P}q}(\vec{\kappa}); \tag{37}$$

$$\tau^{(0)}_{\alpha\beta} = i\tau^3_{\alpha\beta}; \quad \tau^{(\pm1)}_{\alpha\beta} = \frac{\mp i}{\sqrt{2}}\left(\tau^1_{\alpha\beta} \pm i\tau^2_{\alpha\beta}\right); \qquad \phi^{\pm(l,\mathrm{J},\mathrm{m})}_{\mathcal{P}q}(\vec{\kappa})_{\alpha\beta} = -\phi^{\pm(l,\mathrm{J},\mathrm{m})}_{\mathcal{P}q}(-\vec{\kappa})_{\beta\alpha};$$

$$\mid l, J, m; \mathcal{P}, q\rangle = \int d^3\kappa\, \phi^{\pm(l,\mathrm{J},\mathrm{m})}_{\mathcal{P}q}(\vec{\kappa})_{\alpha\beta} \mid R^0_{\alpha\beta}(\mathcal{P}, \vec{\kappa})\rangle;$$

one has the following expressions:

$$\Phi^{\pm(l)}_{\mathcal{P}q}(\vec{\kappa}) = \frac{1}{2}\left[\Phi^{\pm}_{\mathcal{P}q}(\vec{\kappa}) + (-1)^l \Phi^{\pm}_{\mathcal{P}q}(-\vec{\kappa})\right]; \tag{38}$$

$$\Phi^{\pm(l)}_{\mathcal{P}q}(\vec{\kappa}) = \frac{1}{2}\left[\delta_3(\vec{\kappa} - \vec{\mathrm{q}}) + (-1)^l \delta_3(\vec{\kappa} + \vec{\mathrm{q}}) + \frac{\mathcal{T}^{(l)(\pm)}_{\mathcal{P}}(q; \kappa)}{E_2(\mathcal{P}, \kappa) - E_2(\mathcal{P}, q) \mp i0}\right];$$

$$\mathcal{T}^{(0)(\pm)}_{\mathcal{P}}(q;k) = \mathcal{C}^{\mathcal{P}(\pm)}_1(q) + k^2\,\mathcal{C}^{\mathcal{P}(\pm)}_2(q) = \frac{\mathcal{D}^{\mathcal{P}(\pm)}_{q1}(q) + k^2\,\mathcal{D}^{\mathcal{P}(\pm)}_{q2}(q)}{(2\pi)^3\mathcal{D}^{\mathcal{P}(\pm)}_0(q)}, \tag{39}$$

$$\mathcal{T}^{(1)(\pm)}_{\mathcal{P}}(q;k) = \left(\vec{\mathrm{k}}\cdot\mathcal{C}^{\mathcal{P}(\pm)}_3(q)\right) = \frac{2\mu}{(2\pi)^3}k^j\left\{\frac{\prod^{jl}_{\perp}(\mathcal{P})}{1-2\mu I^{\mathcal{P}(\pm)}_{\delta}(q)} + \frac{\prod^{jl}_{\parallel}(\mathcal{P})}{1-2\mu I^{\mathcal{P}(\pm)}_{D}(q)}\right\}q^l;$$

$$\text{where projectors are } \prod\nolimits^{jl}_{\perp}(\mathcal{P}) = \left(\delta_{jl} - \frac{\mathcal{P}^j\mathcal{P}^l}{\mathcal{P}^2}\right); \quad \prod\nolimits^{jl}_{\parallel}(\mathcal{P}) = \frac{\mathcal{P}^j\mathcal{P}^l}{\mathcal{P}^2}. \tag{40}$$

The bound-state wave functions look like simple residues of the scattering one $\Phi^{\pm(l)}_{\mathcal{P}q}(\vec{\mathrm{k}})$ at the corresponding poles for $E_2(\mathcal{P},q) \pm i0 \to i\sigma \to M^{(l)}_2(\mathcal{P})$, $q \to ib(\mathcal{P})$:

$$\begin{aligned}
&D^{\mathcal{P}}_0(\sigma) = 0; \quad 1 - 2\mu \mathrm{J}^{\mathcal{P}}_{\delta}(\sigma) = 0; \quad 1 - 2\mu \mathrm{J}^{\mathcal{P}}_{D}(\sigma) = 0; \quad C_n(\mathcal{P}) \sim \mathcal{C}^{\mathcal{P}(\pm)}_n(q);\\
&\Phi^{\pm(l)}_{\mathcal{P}q}(\vec{\mathrm{k}}) \to B^{\mathcal{P}(l)}(\vec{\mathrm{k}}) = Z^{(l)}(\mathcal{P},\vec{\mathrm{k}})\left[E_2(\mathcal{P},k) - M^{(l)}_2(\mathcal{P})\right]^{-1};\\
&Z^{(0)}(\mathcal{P},\vec{\mathrm{k}}) = C_1(\mathcal{P}) + k^2\,C_2(\mathcal{P}); \quad Z^{(1)}_{\perp,\parallel}(\mathcal{P},\vec{\mathrm{k}}) = \left(\vec{\mathrm{k}}\cdot\vec{\mathrm{C}}^{\perp,\parallel}_3(\mathcal{P})\right);
\end{aligned} \tag{41}$$

$$\text{For the case} \quad \mu \to 0: \quad \mathcal{C}^{\mathcal{P}(\pm)}_1(q) \to \frac{\lambda(2\pi)^{-3}}{\lambda I^{\mathcal{P}(\pm)}_0(q) - 1}; \qquad \mathcal{C}^{\mathcal{P}(\pm)}_{2,3}(q) \to 0. \tag{42}$$

The obtained solutions (33-41) directly satisfy the extended unitarity relation (23).

3. Case $\mu = 0$ and linearization of HE

Returning to HE, let us consider the conserved charge densities $S_{(\Psi)}(x)$:

$$\begin{aligned}
&i\partial_t S_{(\Psi)}(x) = \Psi^{\dagger}_{\alpha}(x)\left[E(\overrightarrow{\mathbf{P}}) - E(\overleftarrow{\mathbf{P}})\right]\Psi_{\alpha}(x) - i\mu\,\vec{\nabla}_{\mathrm{x}}\cdot\left(\Psi^{\dagger}_{\alpha}(x)\,\vec{J}_{(\Psi)}(x)\Psi_{\alpha}(x)\right);\\
&Q(t) = \int d^3\mathrm{x}\,S_{(\Psi)}(\vec{\mathrm{x}},t); \quad \partial_t\,Q(t) = -\mu\oint_{\Sigma_R} d\vec{\sigma}\cdot\left(\Psi^{\dagger}_{\alpha}(x)\,\vec{J}_{(\Psi)}(x)\Psi_{\alpha}(x)\right) \overset{R\to\infty}{\longrightarrow} 0.
\end{aligned} \tag{43}$$

It is clear that, for $\mu = 0$, the HE for $S_{(\Psi)}(x)$ contains only a kinetic term. Then the initial conditions lead to the simple form of dynamical mapping for this operator:

$$\begin{aligned}
&S_{(\Psi)}(\vec{\mathrm{x}},t) \equiv e^{iHt}\,S_{(\Psi)}(\vec{\mathrm{x}},0)\,e^{-iHt} \Longrightarrow\\
&\qquad \Longrightarrow e^{iH_0t}\,S_{(\Psi)}(\vec{\mathrm{x}},0)\,e^{-iH_0t} = e^{iH_0t}\,S_{(\psi)}(\vec{\mathrm{x}},0)\,e^{-iH_0t} \equiv S_{(\psi)}(\vec{\mathrm{x}},t).
\end{aligned} \tag{44}$$

So, HE (29) becomes a *linear* equation with respect to HF $\Psi_{\alpha}(\vec{x},t)$:

$$\left(i\partial_t - E(\mathbf{P})\right)\;\Psi_{\alpha}(\vec{\mathrm{x}},t) = -\frac{\lambda}{2}S_{(\Psi)}(\vec{\mathrm{x}},t)\,\Psi_{\alpha}(\vec{\mathrm{x}},t) \Longrightarrow -\frac{\lambda}{2}S_{(\psi)}(\vec{\mathrm{x}},t)\,\Psi_{\alpha}(\vec{\mathrm{x}},t), \tag{45}$$

and its solution gives a closed expression of HF in terms of the physical one:

$$\begin{aligned}
\Psi_{\alpha}(\vec{\mathrm{x}},t) &= \mathbf{T}\,\exp\left\{i\int_0^t d\eta\left[\frac{\lambda}{2}S_{(\psi)}(\vec{\mathrm{x}},\eta) - E(\mathbf{P})\right]\right\}\psi_{\alpha}(\vec{\mathrm{x}},0) =\\
&= e^{-itE(\mathbf{P})}\mathbf{T}\,\exp\left\{i\,\frac{\lambda}{2}\int_0^t d\eta\,S_{(\psi)}\left(\vec{\mathrm{x}} + \eta\vec{\mathrm{v}}(\mathbf{P}),\eta\right)\right\}\psi_{\alpha}(\vec{\mathrm{x}},0),
\end{aligned} \tag{46}$$

where $\vec{v}(\mathbf{P}) = \vec{\nabla}_{\mathbf{P}} E(\mathbf{P})$ is the corresponding group velocity. Dynamical mapping is given by a normal ordering of this formal solution. It seems difficult to obtain such a solution in terms of in-fields.

4. N, Θ model

This model is determined by the following Hamiltonian, CCR, and HE (see e.g. [4]):

$$\begin{aligned} H = \int d^3\mathrm{x} \Big\{ & N^\dagger(x)\mu(\nabla^2)N(x) + \Theta^\dagger(x)w(\nabla^2)\Theta(x) + \\ & +\lambda \int d^4y \int d^4z \bar{\alpha}(x-y)\bar{\alpha}(x-z)N^\dagger(x)N(x)\Theta^\dagger(y)\Theta(z) \Big\}, \end{aligned} \tag{47}$$

$$\bar{\alpha}(x-y) = \alpha(\mathrm{x}-\mathrm{y})\delta(t_x - t_y); \;\; \mu(\nabla^2)e^{ik\mathrm{x}} = m(k)e^{ik\mathrm{x}}; \;\; w(\nabla^2)e^{ik\mathrm{x}} = \omega(k)e^{ik\mathrm{x}}.$$

$$\{N(x), N^\dagger(y)\}\delta(t_x - t_y) = \delta_4(x-y), \quad [\Theta(x), \Theta^\dagger(y)]\delta(t_x - t_y) = \delta_4(x-y),$$

$$\begin{aligned} (i\partial_t - \mu(\nabla^2))N(x) &= \lambda \int d^4y \int d^4z \bar{\alpha}(x-y)\bar{\alpha}(x-z)\Theta^\dagger(y)N(x)\Theta(z), \\ (i\partial_t - w(\nabla^2))\Theta(x) &= \lambda \int d^4y \int d^4z \bar{\alpha}(y-x)\bar{\alpha}(y-z)N^\dagger(y)N(y)\Theta(z). \end{aligned} \tag{48}$$

All other (anti) commutators vanish. From this HE It is easy to see from this HE that the HE for the operator $N^\dagger(x)N(x)$, as in the previous case (43), contains only a kinetic term:

$$i\partial_t(N^\dagger(x)N(x)) = N^\dagger(x)\left[\mu(\overrightarrow{\nabla}^2) - \mu(\overleftarrow{\nabla}^2)\right]N(x). \tag{49}$$

That means $N^\dagger(x)N(x) = N_0^\dagger(x)N_0(x)$. Defining HF $N(x)$, the physical field $N_0(x)$ and HE in the momentum representation

$$\left.\begin{matrix} N(x) \\ \Theta(x) \end{matrix}\right\} = \int \frac{d^3k}{(2\pi)^{3/2}} \left\{\begin{matrix} n(k,t)\, e^{ik\mathrm{x} - im(k)t} \\ o(k,t)\, e^{ik\mathrm{x} - i\omega(k)t} \end{matrix}\right\}; \quad \tilde{\alpha}(p) = \int \frac{d^3\mathrm{x}}{(2\pi)^{3/2}} e^{ip\mathrm{x}}\alpha(\mathrm{x}); \tag{50}$$

$$\begin{aligned} i\partial_t n(l,t) &= \lambda \int d^3p \int d^3q\, \tilde{\alpha}(q)\tilde{\alpha}(-p) e^{i(E_q^{q+l} - E_p^{q+l})t} o^\dagger(q,t)o(p,t)n(l+q-p,t), \\ i\partial_t o(l,t) &= \lambda\tilde{\alpha}(l) \int d^3p \int d^3q\, \tilde{\alpha}(p-q-l) e^{i(E_l^{q+l} - E_{l+q-p}^{q+l})t} n^\dagger(q,t)n(p,t)o(l+q-p,t), \end{aligned} \tag{51}$$

where $E_q^{p+q} = \omega(q) + m(p)$; and $N_0(x), \Theta_0(x)$ are introduced by the same identities (50) with $n(k,t) \to N(k)$, $o(k,t) \to \Theta(k)$, one has, as above, the linear HE for the operator $o(k,t)$

$$\begin{aligned} o(k,t) &= \Theta(k) + \int_0^t d\eta \int d^3l\, \mathcal{K}_N(k,l;\eta)o(l,\eta), \\ \mathcal{K}_N(k,l;t) &= -i\lambda\tilde{\alpha}(k)\tilde{\alpha}(-l) \int d^3q\, e^{i(E_k^{q+k} - E_l^{q+k})t} N^\dagger(q)N(q+k-l). \end{aligned} \tag{52}$$

With the initial conditions $o(k,0) = \Theta(k)$, $n(k,0) = N(k)$, it has the similar formal solution

$$o(k,t) = \Theta(k) + \int d^3l\, \mathcal{R}_N(k,l;t)\Theta(l) = \mathrm{T}\left\{\exp\left[\int_0^t d\eta \hat{K}_N(\eta)\right]\Theta\right\}(k), \tag{53}$$

where $\hat{K}_N(\eta)$ is an integral operator with the kernel $\mathcal{K}_N(k,l;\eta)$. Note that, for a given $o(k,t)$, Eq. (51) for $n(k,t)$ is also linear.

For the reduced Hamiltonian

$$\begin{aligned} H_I &= \lambda \int d^3p \int d^3k \int d^3q \tilde{\alpha}(-q)\tilde{\alpha}(k+q-p)\, N^\dagger(p)N(k)\Theta^\dagger(k+q-p)\Theta(q), \\ H_0 &= \int d^3p \left[m(p)\, N^\dagger(p)N(p) + \omega(p)\,\Theta^\dagger(p)\Theta(p)\right], \qquad H = H_0 + H_I, \end{aligned} \tag{54}$$

it's also possible to find coefficient functions of dynamical mapping and two-particle bound and scattering eigenstates. The dynamical mapping up to the third order now reads :

$$\begin{aligned} o(p,t) &= \mathcal{O}_0^t[\Theta(k), N(k)] = \Theta(p) + \int d^3q \int d^3k N^\dagger(q)N(p+q-k)\Theta(k)\cdot \\ &\quad \cdot F(t;q,p;p+q-k,k) + \ldots \\ n(p,t) &= \mathcal{N}_0^t[N(k),\Theta(k)] = N(p) + \int d^3q \int d^3k \Theta^\dagger(q)N(p+q-k)\Theta(k)\cdot \\ &\quad \cdot F(t;p,q;p+q-k,k) + \ldots, \end{aligned} \tag{55}$$

where the first coefficient function may be found as

$$\begin{aligned} F(t;l,q;l+q-p,p) &= -i\lambda\tilde{\alpha}(q)\tilde{\alpha}(-p)\int_0^t d\xi e^{i\xi E_q^{l+q}} \frac{\varepsilon(t)}{2\pi i}\cdot \\ &\cdot \int_{-i\infty+\Delta\cdot\varepsilon(t)}^{i\infty+\Delta\cdot\varepsilon(t)} d\sigma \frac{e^{\sigma\xi}}{(\sigma + iE_p^{l+q})\left[1 + I^{l+q}(\sigma)\right]}; \quad I^{l+q}(\sigma) = \lambda \int d^3k \frac{\mid\tilde{\alpha}(k)\mid^2}{E_k^{l+q} - i\sigma}. \end{aligned} \tag{56}$$

The well-known solutions of Eqs. (15), (18) for bound and scattering eigenstates of Hamiltonian (54) with two-particle energies $M(\mathcal{P})$, E_q^p, [4], [7]:

$$\mid R^\pm\{N(p-k)\Theta(k)\}\rangle = \int d^3q R_\pm^{p,k}(q)\, N^\dagger(p-q)\Theta^\dagger(q)\mid 0\rangle, \tag{57}$$

$$R_\pm^{p,k}(q) = \delta_3(k-q) + \frac{\lambda\tilde{\alpha}(q)\tilde{\alpha}(-k)}{E_k^p - E_q^p \pm i\delta} Q_k^{p(\pm)}; \quad Q_k^{p(\pm)} = \left[1 + I^p\left(\pm\delta - iE_k^p\right)\right]^{-1};$$

$$\mid \mathrm{B}^p\rangle = \int d^3q B^p(q)\; N^\dagger(p-q)\Theta^\dagger(q)\mid 0\rangle; \quad 1 + I^p\left(-iM(p)\right) = 0. \tag{58}$$

$$B^p(q) = Z^p \frac{\tilde{\alpha}(q)}{M(p) - E_q^p}; \quad J^p(\sigma) = \int d^3k \frac{\mid\tilde{\alpha}(k)\mid^2}{(E_k^p - i\sigma)(E_k^p - M(p))};$$

$$\int d^3k\, \overset{*}{B^p}(k) B^p(k) = \mid Z^p\mid^2 J^p(-iM(p)) = 1, \tag{59}$$

satisfy the orthogonality conditions:

$$\begin{aligned} &\langle R^\pm\{N(p-k)\Theta(k)\}\mid R^\pm\{N(p_1-k_1)\Theta(k_1)\}\rangle = \\ &\quad = \delta_3(p-p_1)\int d^3q\, \overset{*}{R_\pm^{p,k}}(q)\, R_\pm^{p,k_1}(q) = \delta_3(p-p_1)\,\delta_3(k-k_1); \end{aligned} \tag{60}$$

$$\langle \mathrm{B}^{p_1} \mid R^{\pm}\{N(p-k)\Theta(k)\}\rangle = \delta_3(p_1-p)\int d^3q B^{p*}(q)R^{p,k}_{\pm}(q) = 0. \tag{61}$$

By definition, the S-matrix reads:

$$\begin{aligned}
&\langle N_{in}(p-k)\Theta_{in}(k) \mid \hat{\mathrm{S}}^{\pm 1} \mid N_{in}(p_1-k_1)\Theta_{in}(k_1)\rangle \stackrel{def}{=} \\
&\quad \stackrel{def}{=} \langle R^{\pm}\{N(p-k)\Theta(k)\} \mid R^{\mp}\{N(p_1-k_1)\Theta(k_1)\}\rangle = \\
&\quad = \delta_3(p-p_1)\int d^3q \, \overset{*}{R^{p,k}_{\pm}}(q)\, R^{p,k_1}_{\mp}(q) = \delta_3(p-p_1)\, S^p_{\pm}(k,k_1); \\
&S^p_{\pm}(k,k_1) = \left\{\delta_3(k-k_1) \mp 2\pi i\, \delta(E^p_{k_1}-E^p_k)\, \lambda\, \tilde{\alpha}(-k_1)\tilde{\alpha}(k)\, Q^{p(\pm)}_k\right\}.
\end{aligned} \tag{62}$$

This model was considered by Umezawa, Matsumoto, Tachiki [4] in the framework of the dynamical mapping method using "in" -fields. The dynamical mapping for this case has the form:

$$\begin{aligned}
&n(l,t) \equiv \mathcal{N}^t_{-\infty}[N_{in}(k),\Theta_{in}(k)] = N_{in}(l) + \int d^3q \int d^3k \left[R^{l+q,k}_{\pm}(q) - \delta_3(k-q)\right]\cdot \\
&\quad \cdot e^{i(E^{l+q}_q - E^{l+q}_k)t}\Theta^{\dagger}_{in}(q)N_{in}(l+q-k)\Theta_{in}(k) + \\
&\quad + \int d^3k B^{l+k}(k)\, e^{i(E^{l+k}_k - M(l+k))t}\Theta^{\dagger}_{in}(k)V_{in}(l+k) + \ldots; \\
&e^{-itE^p_q}o(q,t)n(p-q,t) = e^{-itM(p)}B^p(q)V_{in}(p) + \\
&\quad + \int d^3k\, e^{-itE^p_k}R^{p,k}_{+}(q)\Theta_{in}(k)N_{in}(p-k) + \ldots.
\end{aligned} \tag{63}$$

Unfortunately, the second term in the last equation was absent in [4]. With this correction, one can compare their results with our approach using the uniqueness of HF and making the dynamical mapping onto the "in" -field in two ways:

$$n(k,t) = \mathcal{N}^t_{-\infty}[N_{in}(k)] = \mathcal{N}^t_0[N(k)] = \mathcal{N}^t_0\left\{\mathcal{N}^0_{-\infty}[N_{in}(k)]\right\}. \tag{64}$$

One can see that the obtained consistency conditions have the same form as the above-mentioned off-shell extended unitarity relations (23):

$$\begin{aligned}
&B^{l+q}(q)\left\{e^{i(E^{l+q}_q - M(l+q))t} - 1\right\} = \int d^3p\, B^{l+q}(p)F(t;l,q;l+q-p,p), \\
&R^{l+q,k}_{\pm}(q)\left\{e^{i(E^{l+q}_q - E^{l+q}_k)t} - 1\right\} = \int d^3p\, R^{l+q,k}_{\pm}(p)F(t;l,q;l+q-p,p),
\end{aligned} \tag{65}$$

and are held identically for solutions (56-59). Moreover, substituting (63) into (57), (58), and using (60), (61), (62), it is a simple matter to see that scattering and bound states for these two different approaches are connected correspondingly as:

$$\langle \mathrm{B}^p \mid = \langle 0 \mid \int d^3q B^{p*}(q)\left\{B^p(q)\, V_{in}(p) + \int d^3k R^{p,k}_{+}(q)\Theta_{in}(k)N_{in}(p-k) + \ldots\right\} = \langle 0 \mid V_{in}(p);$$

$$\mid R^{\pm}\{N(p-k)\Theta(k)\}\rangle = \int d^3l\, N^{\dagger}_{in}(p-l)\Theta^{\dagger}_{in}(l)\ \mid 0\rangle \left\{\begin{array}{ll} \delta_3(l-k), & (+) \\ S^p_{+}(l,k) & (-). \end{array}\right.$$

So, as expected, the bound state $\mid \mathrm{B}^p\rangle$, obtained by the self-consistency method [4] with the help of a new bound state operator $V_{in}(p)$, coincides with one derived from a direct solution of the eigenvalue problem in terms of constituent fields, and, in turn, the scattering eigenstates are nothing but two-particle in- and out- states from [4].

5. Conclusions

In this work, we demonstrate the convenience to use Schrödinger fields (SF) $\Psi_\alpha(\vec{x}, 0)$ as physical ones instead of the asymptotical fields (AF) $\psi_{in}(x)$ on exactly solvable four-fermion and N, Θ models. The point is that SF and HF $\Psi_\alpha(\vec{x}, t)$, contrary to AF, form a complete irreducible representation of CCR (CAR) also in the presence of bound states, whereas a complete set of AF must incorporate a new field for every bound state. Thus, the dynamical mapping of HF onto SF is simpler than on AF and, for several cases, may be found in closed form (46), (53).

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