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# On Radial Schrödinger Equations in Curved Spaces and Their Spectra Through Nonlinear Constraints

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*This paper is dedicated to our friend Wilhelm Fushchych in honour of his sixtieth birthday*

**Abstract**

First, we determine the radial Schrödinger equation in $D$-dimensional curved spaces when central problems are considered. Second, we develop the so-called factorization method on the basis of supersymmetric arguments for solving such radial equations when $D = 1, 2, 3$-harmonic oscillator and $D = 3$-hydrogen atom physical interactions are studied in such spherical geometries. Comparisons with conventional methods are pointed out as well as with the corresponding results in flat spaces.

## 1 Introduction

Up to its proper results as collected in recent reviews [1, 2], supersymmetric quantum mechanics (SSQM) – developed since the eighties by Witten [3] from particle physics characteristics – can be very useful also for solving in a simpler way nonsupersymmetric Schrödingerlike equations when specific interactions are introduced. In that direction, let us mention the famous *method of factorization* essentially due to Sukumar [4] by exploiting the original Schrödinger method [5] already reviewed by Infeld and Hull [6]. Due to the fact that the Hamiltonian of a physical quantum problem can be factorized into a product of a pair of linear differential operators, the language of SSQM is indeed well adapted for formulating a generalized operator method yielding all energy eigenvalues and eigenfunctions of the system described by time-independent equations in ordinary spaces.

As our knowledge, such a factorization method has never been applied to physical problems formulated in $D$-dimensional curved spaces ($D = 1, 2, 3, ...$) like those tackled by Higgs [7] and his collaborator Leemon [8] on interesting subjects dealing with dynamical symmetries in a spherical geometry. Their explicit physical applications were concerned with the quantum mechanical Kepler and isotropic oscillator systems on specific $D$-spheres.

Through SSQM-properties, we plan to come back here on these problems and to exploit nonlinear (Riccati) equations appearing as typical constraints of the corresponding SSQM-method. The contents are then distributed as follows. In Section 2, we take care of the central character of our problems by putting in evidence the radial (Schrödinger) equation in $D$-dimensional curved spaces associated with the Higgs study. Three specific cases $D = 1, 2, 3$ are pointed out for evident reasons in connection with the parallel approach in flat spaces. Section 3 is devoted to the factorization method applied to such radial equations and to the corresponding SSQM-constraints. Finally, Section 4 contains the applications of these results to the two above-mentioned quantum mechanical systems but in dimensions $D = 1, 2, 3$ for the harmonic oscillator and only in dimension $D = 3$ for the Hydrogen atom. Such results can then be compared with those obtained in a very heavy way by Leemon [8] when energy eigenvalues and eigenfunctions have to be determined. Typical results for flat space considerations are also recovered.

## 2 Radial Schrödinger Equations in D-Dimensional Curved Spaces

The quantum mechanical Hamiltonian for a system submitted to central forces (derived from potentials $V(r)$) on a $D$-dimensional sphere can be written as

$$H = H_0 + V(r), \qquad r = |\vec{r}|. \tag{1}$$

The free Hamiltonian $H_0$ [7] is given by

$$H_0 = \frac{1}{2}(\vec{\pi}^{\,2} + \lambda \vec{L}^{\,2}) \tag{2}$$

in terms of the Hermitian linear momentum

$$\vec{\pi} = \vec{p} + \frac{1}{2}\lambda \vec{x}(\vec{x}\cdot\vec{p}) + \frac{1}{2}\lambda(\vec{p}\cdot\vec{x})\vec{x} \tag{3}$$

expressed in terms of $D$-dimensional vectors with cartesian coordinates $(x_1, x_2, ..., x_D)$ while $\vec{L}$ refers as usual to the angular momentum operators such that

$$\vec{L}^2 = \frac{1}{2}L_{ij}L_{ij}, \qquad L_{ij} = x_i p_j - x_j p_i, \qquad i, j = 1, 2, ..., D. \tag{4}$$

Here $\lambda$ is the deformation parameter which characterizes the curvature of a $D$-sphere (giving back the flat space when $\lambda \to 0$). For specific interactions such as those of the (isotropic) harmonic oscillator (of angular frequency $\omega = 1$) or of the Coulomb problem (for the hydrogen atom with $|e| = 1$), we are thus led to stationary Schrödinger equations of the form

$$H\Psi(\vec{x}) = E\Psi(\vec{x}) \tag{5}$$

which have to be solved through usual methods, essentially based on the separation of (radial and angular) variables.

Let us develop this method here due to technical difficulties typical of the curved context. By factorizing the wavefunction in terms of radial $R(r)$ and angular $Y(\theta, ...)$

parts, we can show by taking care of specific already known results [9] that the angular equation does not depend on $\lambda$ and leads to the eigenvalue relation

$$\Delta_\theta Y(\theta, ...) = -l(l + D - 2)Y(\theta, ...) \tag{6}$$

if the $D$-dimensional Laplacian operator is written as

$$\Delta \equiv \frac{\partial^2}{\partial r^2} + \frac{D-1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta_\theta. \tag{7}$$

We thus get at this stage a radial equation of the form:

$$\begin{aligned} &\left\{(1+\lambda r^2)^2 \frac{d^2}{dr^2} + \frac{1}{r}\left[(D-1+3\lambda r^2+\lambda D r^2)(1+\lambda r^2)\right]\frac{d}{dr} - \frac{\ell(\ell+D-2)}{r^2}(1+\lambda r^2)\right. \\ &\qquad \left. + \frac{\lambda}{2}D(D+1) + \frac{\lambda^2}{4}(D+1)(D+3)r^2 + 2(E - V(r))\right\} R(r) = 0. \end{aligned} \tag{8}$$

As a last step, we successively realize a factorization

$$R(r) = f(r)\Lambda(r) \tag{9}$$

with the constraint

$$\frac{d}{dr}f(r) = -\frac{D-1+3\lambda r^2+\lambda D r^2}{2r(1+\lambda r^2)}f(r), \tag{10}$$

a new change of variables

$$r = \frac{1}{\sqrt{\lambda}}\tan(\sqrt{\lambda}\, y) \tag{11}$$

and another factorization

$$\Lambda(y) = g(y)\chi(y) \tag{12}$$

with the constraint

$$\frac{dg(y)}{dy} = \sqrt{\lambda}\ \tan(\sqrt{\lambda}\, y)\, g(y). \tag{13}$$

In that way, we obtain the final form of the "radial" Schrödinger equations given by

$$\left[-\frac{1}{2}\frac{d^2}{dy^2} + \mathcal{V}(y)\right]\chi(y) = E\chi(y), \tag{14}$$

where the "potential" $\mathcal{V}(y)$ takes specific expressions depending on the chosen dimension $D$ and on the typical (central) problem that we want to handle.

Let us consider the four cases announced in Introduction and solved in Section 4 through the method that we plan to discuss on SSQM-properties in the following section. For the (isotropic) harmonic oscillator, eq.(14) is characterized, when $D = 1$, by

$$\mathcal{V}(y) = \frac{1}{2\lambda}\tan^2(\sqrt{\lambda}\, y), \qquad \lambda > 0 \tag{15}$$

or, when $D = 2$, by

$$\mathcal{V}(y) = \frac{1}{2\lambda}\tan^2(\sqrt{\lambda}\,y) - \frac{\lambda}{8} - \frac{1}{2}\left(\frac{1}{4} - \ell^2\right)\frac{\lambda}{\sin^2(\sqrt{\lambda}\,y)} \tag{16}$$

or, when $D = 3$, by

$$\mathcal{V}(y) = \frac{1}{2\lambda}\tan^2(\sqrt{\lambda}\,y) - \frac{\lambda}{2} + \frac{\lambda\ell(\ell+1)}{2\sin^2(\sqrt{\lambda}\,y)}. \tag{17}$$

Finally, for the Hydrogen atom in $D$=3-dimensions, the "potential" in eq.(14) takes the form

$$\mathcal{V}(y) = -\frac{\sqrt{\lambda}}{\tan(\sqrt{\lambda}\,y)} - \frac{\lambda}{2} + \frac{\lambda\ell(\ell+1)}{2\sin^2(\sqrt{\lambda}\,y)}. \tag{18}$$

## 3 The Factorization Method and its Nonlinear SSQM-Constraints

One of the most interesting properties of SSQM is that its Hamiltonian may always be understood as a set of two ordinary (super)partners which, in the 1-dimensional context for example, are given by

$$H_{\mp} = \frac{1}{2}p^2 + \frac{1}{2}[W^2(x) \mp W'(x)] \tag{19}$$

where $W(x)$ is an arbitrary function (and $W'(x)$ its first derivative with respect to $x$) called the "superpotential" with a specific value for each physical application. These two Hamiltonians have an identical spectrum up, eventually, to the lowest level of one of them and, in this latter case, its energy is necessarily zero. That twofold degeneracy of all levels with energy $E > 0$ and the vanishing of the groundstate energy help us to find the whole spectrum as well as the corresponding eigenfunctions for a lot of problems [1, 2].

Such properties are also intimately connected with the fact that (as it has been mentioned a long time ago [5, 6]) the Schrödinger Hamiltonians can be expressed as products of two linear differential operators, the starting point of the factorization method recently introduced in SSQM [4].

Let us now take care of these observations for the study of Schrödinger equations of the type (14), the 1-dimensional context being an ideal one for our <u>radial</u> considerations of Section 2. The first step is to rewrite eq.(14) in the form

$$\left(-\frac{1}{2}\frac{d^2}{dy^2} + \frac{1}{2}{W_0}^2(y) - \frac{1}{2}W_0'(y) + E_0\right)\chi(y) = 0, \tag{20}$$

where $E_0$ is the fundamental (lowest) energy corresponding to the potential $\mathcal{V}(y)$. Identifying eqs.(14) and (20), we get a Riccatti equation of the type

$$\mathcal{V}(y) = \frac{1}{2}{W_0}^2(y) - \frac{1}{2}W_0'(y) + E_0. \tag{21}$$

Solving this equation, we obtain the corresponding superpotential $W_0$ and energy eigenvalue $E_0$ as well as, consequently, the fundamental eigenfunction $\chi_0$ which is given (up to a normalization factor) by

$$\chi_0 \sim \exp\Big[-\int W_0\, dy\Big], \tag{22}$$

as can be verified. All these results concern the superpartner $H_-$ defined in eq.(19) and suggest now the construction of the associated $H_+$ leading to a new potential function $\mathcal{V}_1(y)$ defined as

$$\mathcal{V}_1(y) = \frac{1}{2}{W_0}^2(y) + \frac{1}{2}W_0'(y) + E_0. \tag{23}$$

The other steps are then based on the algorithm which can now begin by applying once again the first step described above but on this new potential $\mathcal{V}_1(y)$. This means that we introduce another superpotential $W_1(y)$ and determine it (with $E_1$, the first energy associated with $\mathcal{V}_1$ but also – this is the supersymmetric degeneracy – the second energy level of $\mathcal{V}(y)$) through the new Riccatti constraint

$$\mathcal{V}_1(y) = \frac{1}{2}{W_1}^2(y) - \frac{1}{2}W_1'(y) + E_1. \tag{24}$$

The second eigenfunction corresponding to $\mathcal{V}(y)$ is then given by

$$\chi_1 \sim \Big[\frac{d}{dy} - W_0(y)\Big]\exp\Big[-\int W_1\, dy\Big], \tag{25}$$

as can be verified.

Then, we immediately deduce a recurrent procedure characterized by the nonlinear constraints

$${W_{n-1}}^2 + W_{n-1}' + 2E_{n-1} = {W_n}^2 - W_n' + 2E_n, \qquad \forall\, n = 0, 1, ... \tag{26}$$

leading to the knowledge of $W_n$ and $E_n$. On the level of eigenfunctions, we have

$$\chi_n \sim \Big(\frac{d}{dy} - W_0\Big)\Big(\frac{d}{dy} - W_1\Big)\cdots\Big(\frac{d}{dy} - W_{n-1}\Big)\exp\Big[-\int W_n\, dy\Big]. \tag{27}$$

We thus get the whole set of energy eigenvalues and eigenfunctions as already discussed in supersymmetric developments [1, 2] for chains of Hamiltonians whose consecutive pairs are supersymmetric partners [4].

Let us now apply these properties to our specific applications in a <u>curved</u> space with the corresponding Schrödinger equations (14) when the potentials (15)-(18) enter in.

## 4 Some results and conclusions

### a) <u>The $D$=1-Harmonic Oscillator</u>

The simplest application evidently concerns "potential" (15) which corresponds to the well-known example

$$V^{H.O.}(r) = \frac{1}{2}\,r^2 \tag{28}$$

when the change of variables (11) is introduced. An immediate conclusion is that there is no deformation in this case except that the parameter $\lambda$ enters into function (15): this means that the "curved" harmonic oscillator, when $\lambda > 0$, has the same spectrum as the interaction characterized by a ($\tan^2$)-potential in flat space and as another one characterized by a ($\tanh^2$)-potential when $\lambda < 0$, this latter context corresponding on an hyperboloid to the well-known modified Poschl-Teller [10] potential in flat space. Let us only mention that, for $\lambda > 0$, according to the nonlinear constraints (26), we get here the superpotentials

$$W_n = \left[\left(n+\frac{1}{2}\right)\sqrt{\lambda} + \frac{1}{\sqrt{\lambda}}\sqrt{1+\frac{1}{4}\lambda^2}\right]\tan(\sqrt{\lambda}\,y) \tag{29}$$

and the energy eigenvalues

$$E_n = \left[\left(n+\frac{1}{2}\right)\sqrt{1+\frac{1}{4}\lambda^2} + \frac{\lambda}{2}\left(n^2+n+\frac{1}{2}\right)\right]. \tag{30}$$

The corresponding eigenfunctions can also be constructed from

$$\chi_0 = [\cos(\sqrt{\lambda}\,y)]^{\frac{1}{2}+\frac{1}{\lambda}\sqrt{1+\frac{1}{4}\lambda^2}},$$

we easily get by exploiting relation (27)

$$\chi_n = [\cos(\sqrt{\lambda}\,y)]^{\frac{1}{2}+\frac{1}{\lambda}\sqrt{1+\frac{1}{4}\lambda^2}} F\left(\frac{1}{\lambda}\sqrt{1+\frac{1}{4}\lambda^2}+\frac{n+1}{2}, -\frac{n}{2}; \frac{1}{2}; \sin^2(\sqrt{\lambda}\,y)\right) \tag{31}$$

in terms of usual hypergeometric functions $F(a,b\,;c\,;z)$. Results (30) and (31) coincide with those established by Leemon [8].

**b) The $D$=2-Harmonic Oscillator**

Due to the extra $\lambda$-terms contained in "potential" (16) with respect to (15), we immediately point out here a deformation of the original Schrödinger equation and note that our SSQM-method is still working.

We have, in particular,

$$E_0 = (1+\ell)\sqrt{1+\frac{1}{4}\lambda^2} + \frac{\lambda}{2}(1+\ell)^2 \tag{32}$$

and

$$\chi_0 = [\cos(\sqrt{\lambda}\,y)]^{\frac{1}{2}+\frac{1}{\lambda}\sqrt{1+\frac{1}{4}\lambda^2}}\,[\sin(\sqrt{\lambda}\,y)]^{l+\frac{1}{2}} \tag{33}$$

but, in general, $\forall\, n = 0, 1, 2, ...$

$$W_n = \left[\left(n+\frac{1}{2}\right)\sqrt{\lambda} + \frac{1}{\sqrt{\lambda}}\sqrt{1+\frac{1}{4}\lambda^2}\right]\tan(\sqrt{\lambda}\,y) - \sqrt{\lambda}\left(\ell+n+\frac{1}{2}\right)\tan^{-1}(\sqrt{\lambda}\,y) \tag{34}$$

and

$$E_n = (2n+1+\ell)\sqrt{1+\frac{1}{4}\lambda^2} + \frac{\lambda}{2}(2n+1+\ell)^2. \tag{35}$$

The eigenfunctions have once again the same structure as those given in (31) but are now $\ell$-dependent. We get, for arbitrary $n$,

$$\begin{aligned}\chi_n &= [\cos(\sqrt{\lambda}\, y)]^{\frac{1}{2}+\frac{1}{\lambda}\sqrt{1+\frac{1}{4}\lambda^2}}\,[\sin(\sqrt{\lambda}\, y)]^{\ell+\frac{1}{2}} = \\ &= CAF\left(1+\ell+n+\frac{1}{\lambda}\sqrt{1+\frac{1}{4}\lambda^2}, -n\,;\ell+1\,;\sin^2(\sqrt{\lambda}\, y)\right).\end{aligned} \tag{36}$$

### c) The $D$=3-Harmonic Oscillator

The results of this context are readily obtained from those of the $D$=2-case due to the similarity of the extra $\lambda$-terms contained in "potentials" (17) and (18). We get now, for arbitrary $n$,

$$W_n = \left[\left(n+\frac{1}{2}\right)\sqrt{\lambda}+\frac{1}{\sqrt{\lambda}}\sqrt{1+\frac{1}{4}\lambda^2}\right]\tan(\sqrt{\lambda}\, y) - \sqrt{\lambda}\,(\ell+n+1)\tan^{-1}(\sqrt{\lambda}\, y) \tag{37}$$

and

$$E_n = \left(2n+\frac{3}{2}+\ell\right)\sqrt{1+\frac{1}{4}\lambda^2}+\frac{\lambda}{2}\left[(2n+\ell)^2+3(2n+\ell)+\frac{3}{2}\right]. \tag{38}$$

in correspondence with the eigenfunctions

$$\begin{aligned}\chi_n &= [\cos(\sqrt{\lambda}\, y)]^{\frac{1}{2}+\frac{1}{\lambda}\sqrt{1+\frac{1}{4}\lambda^2}}\,[\sin(\sqrt{\lambda}\, y)]^{\ell+1}\times \\ &\times F\left(\frac{3}{2}+\ell+n+\frac{1}{\lambda}\sqrt{1+\frac{1}{4}\lambda^2}, -n\,;\ell+\frac{3}{2}\,;\sin^2(\sqrt{\lambda}\, y)\right).\end{aligned} \tag{39}$$

### d) The $D$=3-Hydrogen Atom

The main difference of this context with respect to the preceding case is evidently contained in the first terms of the right sides of eqs.(17) and (18) but our SSQM-method is still straightforward. We get

$$W_0 = -\sqrt{\lambda}\,(\ell+1)\tan^{-1}(\sqrt{\lambda}\, y) + \frac{1}{\ell+1}, \tag{40}$$

$$E_0 = -\frac{1}{2(\ell+1)^2}+\frac{\lambda}{2}\,\ell(\ell+2) \tag{41}$$

and

$$\chi_0 = \exp\left(-\frac{y}{\ell+1}\right)\cdot[\sin(\sqrt{\lambda}\, y)]^{\ell+1}. \tag{42}$$

For arbitrary $n$, these expressions are generalized as

$$W_n = -\sqrt{\lambda}\,(\ell+n+1)\tan^{-1}(\sqrt{\lambda}\, y) + \frac{1}{\ell+n+1}, \tag{43}$$

$$E_n = -\frac{1}{2(\ell+n+1)^2}+\frac{\lambda}{2}\,(\ell+n)(\ell+n+2) \tag{44}$$

and

$$\chi_n = \exp\left(-\frac{y}{n+\ell+1}\right) \cdot [\sin(\sqrt{\lambda}\, y)]^{\ell+1} f_{n\,,\lambda}(y), \tag{45}$$

where the $f_{n\,,\lambda}(y)$ are nonidentified special functions. In terms of the new variable

$$q = \tan^{-1}(\sqrt{\lambda}\, y), \tag{46}$$

these special functions have to satisfy the following second-order equation

$$\left[(1+q^2)\frac{d^2}{dq^2} - 2(n+\ell)\, q\, \frac{d}{dq} + \frac{2}{\sqrt{\lambda}(n+\ell+1)}\frac{d}{dq} + n(n+2\ell+1)\right] f_{n\,,\lambda}(q) = 0 \tag{47}$$

leading to solutions given as polynomials of degree $n$ in the variable $q$:

$$\begin{aligned} &F_{0\,,\lambda} = N_0, \qquad F_{1\,,\lambda} = N_1\left(q - \frac{1}{\sqrt{\lambda}(\ell+1)(\ell+2)}\right), \\ &F_{2\,,\lambda} = N_2\left(q^2 - \frac{2q}{\sqrt{\lambda}(\ell+1)(\ell+3)} - \frac{1}{2\ell+3} + \frac{2}{\lambda(\ell+1)(\ell+3)^2(2\ell+3)}\right), \ldots . \end{aligned} \tag{48}$$

As a final comment on this specific application - the interesting $D$=3-Coulomb interaction –, let us mention that, besides the results obtained by Higgs [7] and Leemon [8], Schrödinger himself had already studied this context [11] in complete agreement with the above results.

As a conclusion, we have developed through SSQM-properties a method which is technically equivalent to that using raising and lowering operators of the symmetry algebras subtended by the Schrödinger equations. It is easy to see that our results coincide with those obtained by Leemon [8], the cases $D = 3$ being particularly instructive. Moreover, simple connections between these “curved” results with “flat” ones are readily analyzed in all the “physical” cases that we have considered when $\lambda \to 0$. These developments really enhance the interest of the nonlinear constraints (26) between the superpotentials associated with superpartners in Quantum Mechanics.

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