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Symmetry of Equations with Convection Terms

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Abstract

We study symmetry properties of the heat equation with convection term (the equation of convection diffusion) and the Schrödinger equation with convection term. We also investigate the symmetry of systems of these equations with additional conditions for potentials. The obtained results are applied to construction of exact solutions of the system of the Schrödinger equation with convection term and the Euler equations for potentials.

Study of symmetry properties of evolution equations is an important problem in mathematical physics. These equations are thoroughly investigated by a number of authors (see, e.g., [1, 2, 3]). The fundamental property of these equations is the fact that they are invariant under the Galilei transformations.

It is known [4] that the nonlinear heat equation

$$\frac{\partial u}{\partial t} - \lambda \triangle u = F(u) \tag{1}$$

is not invariant under the Galilei transformations if $F(u) \neq 0$. It is Galilei–invariant only in the case of linear equation, i.e., in the case where $F(u) = 0$ (up to equivalence transformations). Therefore, it is important to consider nonlinear evolution equations which admit the Galilei operator.

In the present paper, we study symmetry properties of equations with convection terms, namely, the heat equation with convection term (the equation of convection diffusion) and the Schrödinger equation with convection term. We also investigate the symmetry of systems of these equations with additional conditions for potentials V_k. The results of symmetry classification are applied to constructing exact solutions of the system of the Schrödinger equation with convection term and the Euler equations for potentials.

1 Symmetry of the Equation of Convection Diffusion

The equation of convection diffusion has the form

$$\frac{\partial u}{\partial t} - \lambda \triangle u = V_k \frac{\partial u}{\partial x_k}, \tag{2}$$

where $u = u(t, \vec{x})$ is a real function, λ is a real parameter, the index k varies from 1 to n.

To extend the symmetry of equation (2), we apply the idea proposed in [4, 5, 6]. Namely, we assume that the functions $V_k = V_k(t, \vec{x})$ are new dependent variables on equal conditions with the function u. In other words, we seek for symmetry operators of equation (2) in the form

$$X = \xi^\mu \partial_{x_\mu} + \eta \partial_u + \rho^k \partial_{V_k}, \tag{3}$$

where ξ^μ, η, ρ^k are real functions of t, $\vec{x}$, u, $\vec{V}$. Applying the Lie algorithm [7, 8, 9], we find that the unknown functions ξ^μ, η, ρ^k have the form

$$\begin{aligned} &\xi^0 = 2A(t), \qquad \xi^k = \dot{A}(t)x_k + B^{kl}(t)x_l + U^k(t),\\ &\rho^k = B^{kl}(t)V_l - \ddot{A}(t)x_k - \dot{B}^{kl}(t)x_l - \dot{U}^k(t) - \dot{A}(t)V_k, \qquad \eta = C_1 u + C_2, \end{aligned} \tag{4}$$

where A, B^{kl}, $(k, l = \overline{1,n},\ k \neq l)$, $B^{kl} = -B^{lk}$, U^k $(k = \overline{1,n})$ are arbitrary smooth real functions of t; C_1, C_2 are arbitrary constants. Thus, the following assertion is true:

Theorem 1 *The equation of convection diffusion (2) in the class of operators (3) is invariant under the infinite–dimensional Lie algebra with infinitesimal operators*

$$\begin{aligned} &Q_A = 2A(t)\partial_t + \dot{A}(t)x_r\partial_{x_r} - [\ddot{A}(t)x_r + \dot{A}(t)V_r]\partial_{V_r},\\ &Q_{kl} = B^{kl}(t)\,[x_l\partial_{x_k} - x_k\partial_{x_l} + V_l\partial_{V_k} - V_k\partial_{V_l}] - \dot{B}^{kl}(t)(x_l\partial_{V_k} - x_k\partial_{V_l}),\\ &Q_a = U^a(t)\partial_{x_a} - \dot{U}^a(t)\partial_{V_a}, \quad a = \overline{1,n},\\ &Z_1 = u\partial_u, \qquad Z_2 = \partial_u, \end{aligned} \tag{5}$$

where we mean summation from 1 *to* n *over the repeated index* r *and no summation over indices* k, l, *and* a.

Remark 1. Infinite–dimensional algebra (5) includes the Galilei operator Q_a. This operator generates the following transformations:

$$\begin{cases} t \to \widetilde{t} = t,\\ x_b \to \widetilde{x}^b = x_b + \alpha_b U^b(t)\delta_{ab},\\ u \to \widetilde{u} = u,\\ V^b \to \widetilde{V}^b = V_b - \alpha_b \dot{U}^b(t)\delta_{ab}, \end{cases} \tag{6}$$

where α_b is an arbitrary real parameter of transformations, δ_{ab} is the Kronecker symbol, there is summation from 1 to n over the repeated index b and no summation over the repeated index a. We see that the function u is not changed under the action of this operator. This fact is essentially different from the Galilei transformations for the standard free heat equation

$$\frac{\partial u}{\partial t} - \lambda \triangle u = 0, \tag{7}$$

where the Galilei operator has the form

$$G_a = t\partial_{x_a} - \frac{1}{2\lambda}x_a u\partial_u. \tag{8}$$

For operator (8), the function u is changed as follows:

$$u \to \widetilde{u} = u \exp\left(-\frac{x_a\alpha_a}{2\lambda} - \frac{t(\alpha_a)^2}{4\lambda}\right), \tag{9}$$

Thus, the operators Q_a and G_a are essentially different representations of the Galilei operator.

Let us now investigate the symmetry of systems including equation (2) and additional conditions for the potentials. Note that in [3], the authors find a nontrivial symmetry of the nonlinear Fokker–Planck equation by imposing the additional conditions for coefficient functions.

Let the additional conditions for the potentials V_k be the Euler equations. In other words, consider the following system:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \lambda \triangle u = V_k \dfrac{\partial u}{\partial x_k}, \\ \dfrac{\partial V_k}{\partial t} - \lambda_1 V_l \dfrac{\partial V_k}{\partial x_l} = 0, \ \ k = \overline{1,n}. \end{cases} \tag{10}$$

Symmetry of the nonlinear system (10) essentially depends on the value of the parameter λ_1. There are two different cases.

The first case. $\lambda_1 = 1$.

In this case, system (10) in the class of operators (3) is invariant under the Lie algebra with the basis operators

$$\begin{aligned} &P_0 = \partial_t, \qquad P_a = \partial_{x_a}, \qquad J_{ab} = x_a\partial_{x_b} - x_b\partial_{x_a} + V_a\partial_{V_b} - V_b\partial_{V_a}, \\ &\widetilde{G}_a = t\partial_{x_a} - \partial_{V_a}, \qquad D = 2t\partial_t + x_k\partial_{x_k} - V_k\partial_{V_k}, \\ &A = t^2\partial_t + tx_k\partial_{x_k} - (x_k + tV_k)\partial_{V_k}, \qquad Z_1 = u\partial_u, \qquad Z_2 = \partial_u. \end{aligned} \tag{11}$$

The Galilei operator $\widetilde{G}_a$ generates the following finite transformatios:

$$\begin{cases} t \to \widetilde{t} = t, \\ x_b \to \widetilde{x}^b = x_b + t\alpha_b\delta_{ab}, \\ V_b \to \widetilde{V}^b = V_b - \alpha_b\delta_{ab}, \\ u \to \widetilde{u} = u, \end{cases} \tag{12}$$

where we mean summation from 1 to n over the repeated index b.

Conclusion 1. Thus, the scalar function u, unlike the heat equation, is not changed under the Galilei transformations.

The second case. $\lambda_1 \neq 1$.

In this case, the invariance algebra of system (10) is essentially more restricted and does not include the Galilei operator and the projective one. In other words, for $\lambda_1 \neq 1$ in the class of operators (3), system (10) is invariant under the Lie algebra with basis elements P_0, P_a, J_{ab}, D, Z_1, Z_2 of the form (11).

The first case is essentially more interesting and important that the second one. Therefore, in what follows, we consider system (10) in the case where $\lambda_1 = 1$.

Consider now system (10), where the Euler equations have the right-hand sides of the form $F(u)\dfrac{\partial u}{\partial x_k}$, i.e., the following nonlinear system:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \lambda \triangle u = V_k \dfrac{\partial u}{\partial x_k}, \\ \dfrac{\partial V_k}{\partial t} - V_l \dfrac{\partial V_k}{\partial x_l} = F(u)\dfrac{\partial u}{\partial x_k}, \quad k = \overline{1,n}, \end{cases} \tag{13}$$

where $F(u)$ is a smooth function of u. Let us carry out symmetry classification of system (13), i.e., determine all classes of functions $F(u)$, which admit a nontrivial symmetry of system (13). We consider the following six cases:

Case 1. $F(u)$ is an arbitrary smooth function.
System (13) is invariant under the Galilei algebra

$$AG(1,n) =< P_0,\ P_a,\ J_{ab},\ \widetilde{G}_a >, \tag{14}$$

where the basis operators have the form (11).

Case 2. $F = C\exp(\kappa u)$ (κ and C are arbitrary constants, $\kappa \neq 0,\ C \neq 0$).
In this case, the symmetry of system (13) is more extended and includes algebra (14) and the dilation operator

$$D^{(1)} = 2t\partial_t + x_k\partial_{x_k} - V_k\partial_{V_k} - \frac{2}{\kappa}\partial_u.$$

Case 3. $F = Cu^\kappa$ (κ and C are arbitrary constants, $\kappa \neq 0,\ \kappa \neq 1,\ C \neq 0$).
In this case, system (13) is invariant under the extended Galilei algebra (14) with the dilation operator

$$D^{(2)} = 2t\partial_t + x_k\partial_{x_k} - V_k\partial_{V_k} - \frac{2}{\kappa+1}u\partial_u.$$

Case 4. $F = \dfrac{C}{u}$ (C is an arbitrary constant, $C \neq 0$).
The maximal invariance algebra is

$$< P_0,\ P_a,\ J_{ab},\ \widetilde{G}_a,\ Z_1 >,$$

where $Z_1 = u\partial_u$.

Case 5. $F = C$ (C is an arbitrary constant, $C \neq 0$).
The maximal invariance algebra is

$$< P_0,\ P_a,\ J_{ab},\ \widetilde{G}_a,\ D^{(2)},\ Z_2 >,$$

where $Z_2 = \partial_u$. In this case, the dilation operator $D^{(2)}$ has the form

$$D^{(2)} = 2t\partial_t + x_k\partial_{x_k} - V_k\partial_{V_k} - 2u\partial_u.$$

Case 6. $F = 0$.
In this case, system (13) admits the widest invariance algebra, namely,

$$< P_0,\ P_a,\ J_{ab},\ \widetilde{G}_a,\ D,\ A,\ Z_1,\ Z_2 >,$$

where the dilation operator D and the projective operator A have the form (11).

Conclusion 2. It is important that system (13) is invariant under the Galilei transformations for an arbitrary smooth function $F(u)$. It should be stressed once more that, unlike the standard heat equation, the function u is not changed under the Galilei transformations.

Consider other examples of systems of the equation of convection diffusion and additional conditions for the potentials V_k.

Let the functions V_k satisfy the heat equation, i.e., we investigate the following system:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \lambda \triangle u = V_k \dfrac{\partial u}{\partial x_k}, \\ \dfrac{\partial V_k}{\partial t} - \lambda_1 \triangle V_k = 0, \quad k = \overline{1,n}, \end{cases} \tag{15}$$

where $\lambda_1 \neq 0$ is an arbitrary real parameter.

Theorem 2 *System (14) in the class of operators (3) is invariant under the Lie algebra with the basis operators*

$$P_0,\ P_a,\ J_{ab},\ D,\ Z_1,\ Z_2$$

of the form (11).

The case where the functions V_k satisfy the Laplace equation is more important:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \lambda \triangle u = V_k \dfrac{\partial u}{\partial x_k}, \\ \triangle V_k = 0, \quad k = \overline{1,n}. \end{cases} \tag{16}$$

Theorem 3 *System of equations (16) in the class of operators (3) is invariant under the infinite–dimensional Lie algebra with the basis operators*

$$Q_A,\ Q_{kl},\ Q_a,\ Z_1,\ Z_2$$

of the form (5).

Note that the symmetry of system (16) is the same as the symmetry of equation (2). In other words, the conditions $\triangle V_k = 0$ do not contract the symmetry of the equation of convection diffusion.

2 The Schrödinger Equation with Convection Term

Consider the Schrödinger equation with convection term

$$i\frac{\partial \psi}{\partial t} + \lambda \Delta \psi = V_k \frac{\partial \psi}{\partial x_k}, \tag{17}$$

where $\psi = \psi(t, \vec{x})$ and $V_k = V_k(t, \vec{x})$ $(k = \overline{1,n})$ are *complex* functions. For extension of symmetry, we regard the functions V_k as dependent variables. Note that the requirement that the functions V_k are complex is essential for the symmetry of (17).

Let us investigate the symmetry of (17) in the class of first-order differential operators

$$X = \xi^\mu \partial_{x_\mu} + \eta \partial_\psi + \eta^* \partial_{\psi^*} + \rho^k \partial_{V_k} + \rho^{*k} \partial_{V_k^*}, \tag{18}$$

where $\xi^\mu, \eta, \eta^*, \rho^k, \rho^{*k}$ are functions of t, $\vec{x}$, ψ, ψ^*, $\vec{V}$, $\vec{V}^*$.

Theorem 4 *Equation (17) is invariant under the infinite–dimensional Lie algebra with the infinitesimal operators*

$$
\begin{aligned}
&Q_A = 2A\partial_t + \dot{A}x_r\partial_{x_r} - i\ddot{A}x_r(\partial_{V_r} - \partial_{V_r^*}) - \dot{A}(V_r\partial_{V_r} + V_r^*\partial_{V_r^*}),\\
&Q_{kl} = B_{kl}(x_l\partial_{x_k} - x_k\partial_{x_l} + V_l\partial_{V_k} - V_k\partial_{V_l} + V_l^*\partial_{V_k^*} - V_k^*\partial_{V_l^*}) -\\
&\qquad - i\dot{B}_{kl}(x_l\partial_{V_k} - x_k\partial_{V_l} - x_l\partial_{V_k^*} + x_k\partial_{V_l^*}),\\
&Q_a = U^a\partial_{x_a} - i\dot{U}^a(\partial_{V_a} - \partial_{V_a^*}),\\
&Z_1 = \psi\partial_\psi,\ Z_2 = \psi^*\partial_{\psi^*},\ Z_3 = \partial_\psi,\ Z_4 = \partial_{\psi^*},
\end{aligned}
\tag{19}
$$

where A, B^{kl} $(k < l,\ k,l = \overline{1,n})$, U^a $(a = \overline{1,n})$ *are arbitrary smooth functions of* t, $B^{kl} = -B^{lk}$, *we mean summation over the index* r *and no summation over indices* a, k, *and* l.

This theorem is proved by the standard Lie algorithm in the class of operators (18).

Note that algebra (19) includes as a particular case the Galilei operator of the form:

$$
\widetilde{G}_a = t\partial_{x_a} - i\partial_{V_a} + i\partial_{V_a^*}. \tag{20}
$$

This operator generates the following finite transformations:

$$
\begin{cases}
x_b \to \widetilde{x}_b = x_b + \beta_b t\delta_{ab},\\
t \to \widetilde{t} = t,\\
\psi \to \widetilde{\psi} = \psi,\ \psi^* \to \widetilde{\psi}^* = \psi^*,\\
V_b \to \widetilde{V}_b = V_b - i\beta_b\delta_{ab},\ V_b^* \to \widetilde{V}_b^* = V_b^* + i\beta_b\delta_{ab},
\end{cases}
$$

where β_b is an arbitrary real parameter and we mean summation from 1 to n over the repeated index b. Note that the wave function ψ *is not changed* for these transformations. Operator (20) is essentially different from the standard Galilei operator

$$
G_a = t\partial_{x_a} + \frac{i}{2\lambda}x_a(\psi\partial_\psi - \psi^*\partial_{\psi^*}). \tag{21}
$$

of the free Schrödinger equation ($V_k = 0$). Note that we cannot derive operator (21) from algebra (19). Thus, we have two essentially different representations of the Galilei operator: (20) for the Schrödinger equation with convection term and (21) for the free Schrödinger equation.

Remark 2. If we assume that the functions V_k are real in equation (17) and study symmetry in the class of operators

$$
X = \xi^\mu\partial_{x_\mu} + \eta\partial_\psi + \eta^*\partial_{\psi^*} + \rho^a\partial_{V_a}, \tag{22}
$$

where the unknown functions $\xi^\mu, \eta, \eta^*, \rho^a$ depend on $t, \vec{x}, \psi, \psi^*, \vec{V}$, then the maximal invariance algebra of equation (17) is sufficiently restricted. Namely, in the class of operators (22), equation (17) is invariant under the Lie algebra with the basis operators

$$
\begin{aligned}
&P_0,\quad P_a,\quad J_{ab} = x_a\partial_{x_b} - x_b\partial_{x_a} + V_a\partial_{V_b} - V_b\partial_{V_a},\\
&D = 2t\partial_t + x_r\partial_{x_r} - V_r\partial_{V_r},\quad Z_1 = \psi\partial_\psi,\quad Z_2 = \psi^*\partial_{\psi^*},\quad Z_3 = \partial_\psi,\ Z_4 = \partial_{\psi^*}.
\end{aligned}
$$

Thus, in the case of real functions V_k, equation (17) *is not invariant under the Galilei transformations.*

Consider now the system of equation (17) with the additional condition for the potentials V_k, namely, the complex Euler equations:

$$\left\{ \begin{array}{l} i\dfrac{\partial \psi}{\partial t} + \lambda \Delta \psi = V_k \dfrac{\partial \psi}{\partial x_k}, \\ i\dfrac{\partial V_k}{\partial t} - V_l \dfrac{\partial V_k}{\partial x_l} = F(|\psi|)\dfrac{\partial \psi}{\partial x_k}. \end{array} \right. \tag{23}$$

Here, ψ and V_k are complex dependent variables of t and $\vec{x}$, F is a smooth function of $|\psi|$. The coefficients of the second equation of (23) provide the broad symmetry of this system.

Let us investigate symmetry classification of system (23). Consider the following five cases.

Case 1. F is an arbitrary smooth function.
The maximal invariance algebra is $< P_0, P_a, J_{ab}, G_a >$, where

$$J_{ab} = x_a \partial_{x_b} - x_b \partial_{x_a} + V_a \partial_{V_b} - V_b \partial_{V_a} + V_a^* \partial_{V_b^*} - V_b^* \partial_{V_a^*},$$
$$\widetilde{G}_a = t\partial_{x_a} - i\partial_{V_a} + i\partial_{V_a^*}.$$

Case 2. $F = C|\psi|^k$ (C is an arbitrary complex constant, $C \neq 0$, k is an arbitrary real number, $k \neq 0$ and $k \neq -1$).
The maximal invariance algebra is $< P_0, P_a, J_{ab}, \widetilde{G}_a, D^{(1)} >$, where

$$D^{(1)} = 2t\partial_t + x_r \partial_{x_r} - V_r \partial_{V_r} - V_r^* \partial_{V_r^*} - \frac{2}{1+k}(\psi \partial_\psi + \psi^* \partial_{\psi^*}).$$

Case 3. $F = \dfrac{C}{|\psi|}$ (C is an arbitrary complex constant, $C \neq 0$).
The maximal invariance algebra is $< P_0, P_a, J_{ab}, \widetilde{G}_a, Z = Z_1 + Z_2 >$, where

$$Z = \psi \partial_\psi + \psi^* \partial_{\psi^*}, \ Z_1 = \psi \partial_\psi, \ Z_2 = \psi^* \partial_{\psi^*}.$$

Case 4. $F = C \neq 0$ (C is an arbitrary complex constant).
The maximal invariance algebra is $< P_0, P_a, J_{ab}, \widetilde{G}_a, D^{(1)}, Z_3, Z_4 >$, where

$$Z_3 = \partial_\psi, \ Z_4 = \partial_{\psi^*}.$$

Case 5. $F = 0$.
The maximal invariance algebra is $< P_0, P_a, J_{ab}, \widetilde{G}_a, D, A, Z_1, Z_2, Z_3, Z_4 >$, where

$$D = 2t\partial_t + x_r \partial_{x_r} - V_r \partial_{V_r} - V_r^* \partial_{V_r^*},$$
$$A = t^2 \partial_t + t x_r \partial_{x_r} - (i x_r + t V_r)\partial_{V_r} + (i x_r - t V_r^*)\partial_{V_r^*}.$$

Thus, system (23) is invariant under the Galilei transformations generated by operator (20) for an arbitrary function $F(|\psi|)$.

Let us now apply these results to obtain invariant solutions of system (23) with $\lambda = 1$ in two-dimensional space–time in the case where $F(|\psi|) = 0$:

$$\left\{ \begin{array}{l} i\dfrac{\partial \psi}{\partial t} + \dfrac{\partial^2 \psi}{\partial x^2} = V \dfrac{\partial \psi}{\partial x}, \\ i\dfrac{\partial V}{\partial t} - V\dfrac{\partial V}{\partial x} = 0. \end{array} \right. \tag{24}$$

The invariance algebra of system (24) includes the translation operators, Galilei, dilation, and projective operators:

$$P_0 = \partial_t, \quad P_1 = \partial_x, \quad \widetilde{G} = t\partial_x - i\partial_V + i\partial_{V^*}, \quad D = 2t\partial_t + x\partial_x - V\partial_V - V^*\partial_{V^*},$$
$$A = t^2\partial_t + tx\partial_x - (ix + tV)\partial_V + (ix - tV^*)\partial_{V^*}.$$

1). The one-dimensional subalgebra $\widetilde{G} + \alpha P_0$ is associated with the symmetry ansatz

$$\begin{cases} \psi = \varphi(2\alpha x - t^2), \\ V = -\dfrac{i}{\alpha}t + U(2\alpha x - t^2). \end{cases} \tag{25}$$

Ansatz (25) reduces system (24) to the following system of ordinary differential equations:

$$\begin{cases} 2\alpha\varphi'' = U\varphi', \\ \dfrac{1}{\alpha} - 2\alpha UU' = 0, \end{cases} \tag{26}$$

where $\varphi' \equiv \dfrac{\partial\varphi}{\partial\omega}$, $\omega = 2\alpha x - t^2$. The general solution of system (26) has the form

$$U = \sqrt{C_1 + \frac{1}{\alpha^2}\omega}, \qquad \varphi = C_2 \int \exp\left\{\frac{\alpha}{3}(C_1 + \frac{1}{\alpha^2}\omega)^{3/2}\right\} d\omega + C_3, \tag{27}$$

where C_1, C_2, C_3 are arbitrary constans. Thus, we obtain the partial solution of system (24), where ψ has the form (27), and

$$V = -\frac{i}{\alpha}t + \sqrt{C_1 + \frac{1}{\alpha^2}\omega}.$$

2). The subalgebra

$$\widetilde{G} + \alpha(Z_3 + Z_4) = t\partial_x - i\partial_V + i\partial_{V^*} + \alpha(\partial_\psi + \partial_{\psi^*})$$

is associated with the symmetry ansatz

$$\begin{cases} \psi = \alpha\dfrac{x}{t} + \varphi(t), \\ V = -i\dfrac{x}{t} + U(t). \end{cases} \tag{28}$$

Ansatz (28) reduces system (24) to the following system of ordinary differential equations:

$$\begin{cases} i\dot{\varphi} = \dfrac{\alpha}{t}U, \\ \dot{U} + \dfrac{U}{t} = 0 \end{cases}$$

with the general solution of the form

$$U = \frac{C_1}{t}, \qquad \varphi = i\frac{C_1\alpha}{t} + C_2,$$

where C_1, C_2 are arbitrary constants. Thus, we get the partial solution of system (24):

$$V = -i\frac{x}{t} + \frac{C_1}{t}, \qquad \psi = \alpha\frac{x}{t} + i\frac{C_1\alpha}{t} + C_2.$$

3). The subalgebra

$$\widetilde{G}+\alpha(Z_1+Z_2)=t\partial_x-i\partial_V+i\partial_{V^*}+\alpha(\psi\partial_\psi+\psi^*\partial_{\psi^*})$$

is associated with the symmetry ansatz

$$\left\{\begin{array}{l}\psi=\exp\left(\alpha\dfrac{x}{t}\right)\varphi(t),\\ V=-i\dfrac{x}{t}+U(t).\end{array}\right. \tag{29}$$

Ansatz (29) reduces system (24) to the following system of ordinary differential equations:

$$\left\{\begin{array}{l}i\dot{\varphi}+\dfrac{\alpha^2}{t^2}\varphi=U\dfrac{\alpha}{t}\varphi,\\ \dot{U}+\dfrac{U}{t}=0\end{array}\right.$$

with the general solution

$$U=\frac{C_1}{t},\qquad \varphi=C_2\exp\left(\frac{i}{t}C_1\alpha-\frac{i\alpha^2}{t}\right),$$

where C_1, C_2 are arbitrary constants. Thus, we get the partial solution of system (24):

$$V=-i\frac{x}{t}+\frac{C_1}{t},\qquad \psi=C_2\exp\left(\frac{\alpha x}{t}+\frac{i}{t}C_1\alpha-\frac{i\alpha^2}{t}\right).$$

4). The subalgebra

$$A+\alpha i(Z_1-Z_2)=t^2\partial_t+tx\partial_x-(ix+tV)\partial_V+(ix+tV^*)\partial_{V^*}+i\alpha(\psi\partial_\psi-\psi^*\partial_{\psi^*})$$

is associated with the symmetry ansatz

$$\left\{\begin{array}{l}\psi=\exp\left(-i\dfrac{\alpha}{t}\right)\varphi\left(\dfrac{x}{t}\right),\\ V=-i\dfrac{x}{t}+\dfrac{1}{t}U\left(\dfrac{x}{t}\right).\end{array}\right. \tag{30}$$

Ansatz (30) reduces system (24) to the following system of ordinary differential equations:

$$\left\{\begin{array}{l}U=0,\\ \varphi''-\alpha\varphi=0.\end{array}\right.$$

where $\varphi''\equiv\dfrac{\partial^2\varphi}{\partial\omega^2}$, $\omega=\dfrac{x}{t}$. Consider the following two cases:

4a). $\alpha>0$.

In this case, system (24) has the following solution:

$$V=-i\frac{x}{t},\qquad \psi=\exp\left(-i\frac{\alpha}{t}\right)\left[C_1\exp\left(\sqrt{\alpha}\frac{x}{t}\right)+C_2\exp\left(-\sqrt{\alpha}\frac{x}{t}\right)\right],$$

where C_1, C_2 are arbitrary constants.

4b). $\alpha < 0$.
In this case, system (24) has the following solution:

$$V = -i\frac{x}{t}, \psi = \exp\left(-i\frac{\alpha}{t}\right)\left[C_1\cos\left(\sqrt{-\alpha}\frac{x}{t}\right) + C_2\sin\left(\sqrt{-\alpha}\frac{x}{t}\right)\right],$$

where $C_1,\ C_2$ are arbitrary constants.

5). The one-dimensional algebra

$$A + \alpha(Z_3 + Z_4) = t^2\partial_t + tx\partial_x - (ix + tV)\partial_V + (ix + tV^*)\partial_{V^*} + \alpha(\partial_\psi + \partial_{\psi^*})$$

is associated with the symmetry ansatz

$$\begin{cases} \psi = -\dfrac{\alpha}{t} + \varphi\left(\dfrac{x}{t}\right), \\ V = -i\dfrac{x}{t} + \dfrac{1}{t}U\left(\dfrac{x}{t}\right), \end{cases} \tag{31}$$

which reduces system (24) to the following one:

$$\begin{cases} U = 0, \\ \varphi'' + i\alpha = 0. \end{cases}$$

where $\varphi'' \equiv \dfrac{\partial^2\varphi}{\partial\omega^2}, \omega = \dfrac{x}{t}$. Solving this system, we obtain the exact solution of system (24):

$$V = -i\frac{x}{t}, \qquad \psi = -\frac{\alpha}{t} - i\frac{\alpha}{2}\frac{x^2}{t^2} + C_1\frac{x}{t} + C_2,$$

where $C_1,\ C_2$ are arbitrary constants.

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