

On the Moyal Quantized BKP Type Hierarchies

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Abstract

Quantization of BKP type equations are done through the Moyal bracket and the formalism of pseudo-differential operators. It is shown that a variant of the dressing operator can also be constructed for such quantized systems.

Quantization of integrable system is one of the most important aspect of present day research on nonlinear systems. In two dimensions, a well established methodology was suggested by Faddeev and his collaborators [1], which goes by the name of Quantum Inverse Spectral Transform (QISM) [1]. On the other hand, no such technique is known (which utilises the inverse scattering framework) for integrable systems in three dimensions. Recently the formalism of pseudo-differential operators was used extensively to study such systems in 3 (three) dimensions [2] and an effective way was found to study the Bi-Hamiltonian structures of such three dimensional systems. An ingenious way to derive the quantum version for the KP system was suggested by Kupershmidt [3] using the tools of p pseudo-differential algebra in conjunction with the idea of the Moyal bracket [4]. The basic idea is that the quantization is a deformation of the classical situation. Here, in this communication, we study a quantization of the BKP like hierarchy [5] using the Moyal bracket approach. It is demonstrated that such systems can possess an infinite number of conservation laws; essential for the complete integrability. Furthermore, we also show that an extension of the usual dressing operator [6] is possible even for these quantized or deformed systems.

A quantization is a rule which assigns to every polynomial in variables p and q , a polynomial in operators p and q . The rule must be a linear map satisfying a few natural properties.

For any two operators f and g depending upon the canonical set of variables (p_i, q_i) , the Moyal quantization dictates that the product be defined according to the following rule:

$$f * g = \exp \left[\varepsilon \left(\frac{\partial}{\partial p_f} \frac{\partial}{\partial q_g} - \frac{\partial}{\partial q_f} \frac{\partial}{\partial p_g} \right) \right] (fg), \quad (1)$$

where the operator:

$$v = \frac{\partial}{\partial p_f} \frac{\partial}{\partial q_g} - \frac{\partial}{\partial q_f} \frac{\partial}{\partial p_g} \quad (2)$$

acts according to the rule

$$v(fg) = \frac{\partial f}{\partial p} \frac{\partial g}{\partial q} - \frac{\partial f}{\partial q} \frac{\partial g}{\partial p}. \quad (3)$$

For our situation, f and g are some pseudo-differential operators. Before proceeding to the actual problem of the quantized BKP equation, we give below some algebraic formulae which will be useful later.

Using equation (1) we get

$$\begin{aligned} p^i * a &= \sum_{s=0}^i \binom{i}{s} \varepsilon^s a^{(s)} p^{i-s}, \\ a * p^i &= \sum_{s=0}^i \binom{i}{s} (-\varepsilon)^s a^{(s)} p^{i-s}, \\ bp^i * a &= \sum_{s=0}^i \binom{i}{s} \varepsilon^s a^{(s)} bp^{i-s}, \\ p^i * ap^m &= \sum_{s=0}^i \binom{i}{s} \varepsilon^s a^{(s)} p^{i+m-s}, \\ ap^m * p^i &= \sum_{s=0}^i \binom{i}{s} (-\varepsilon)^s a^{(s)} p^{i+m-s}, \\ bp^i * ap^m &= \sum_{s,k=0}^i \frac{\varepsilon^s}{s} \binom{s}{k} \binom{i}{s-k} (s-k) \binom{m}{k} k(-1)^k b^{(k)} a^{(s-k)} p^{i+m-s}. \end{aligned} \quad (4)$$

Here $a^{(k)} = \partial^k a$. etc., a, b are any two arbitrary functions of q only, and ∂^k represents k -th order derivative with respect to q , that is $\partial^k / \partial q^k$. The Moyal commutator is defined as:

$$[f, g] = f * g - g * f. \quad (5)$$

For example, it can be easily seen that equation (4) leads to

$$\begin{aligned} [p^i, a] &= 2 \sum_{k=0}^i \binom{i}{2k+1} a^{2k+1} a^{(2k+1)} p^{i-2k-1}, \\ [bp^i, a] &= 2 \sum_{k=0}^i \binom{i}{2k+1} \varepsilon^{2k+1} a^{(2k+1)} bp^{i-2k-1}, \end{aligned} \quad (6)$$

and similarly for others.

In the above expressions $\binom{c}{d}$ denotes a Binomial coefficient with the following useful properties:

$$\binom{-i-1}{k_0} = \begin{cases} \binom{i+2k}{2k}, & \text{when } k_0 = 2k \\ (-1) \binom{i+2k+1}{2k+1}, & \text{when } k_0 = 2k+1. \end{cases} \quad (7)$$

For positive values c, d these are

$$\frac{c!}{d!(c-d)!}.$$

The standart pseudo-differential operator ($\bar{\Psi}$ DO) is written as:

$$L = \partial + \sum_{m=1} a_m \partial^{-m}, \quad (8)$$

where a_m are functions of $(x, y$ and $t)$. In the following we shall rewrite L as

$$L = p + \sum_{m=1} a_m p^{-m}. \quad (9)$$

The deformed or quantized hierarchy of equations are to be obtained from the Lax equation:

$$L_t = [P_+, L] = P_+ * L - L * P_+. \quad (10)$$

Here $P_+ = [L^m]_+$, where m is any positive integer and $(+)$ denotes the only terms with positive powers to be retained. One must also note that

$$L^m = L * L * L * L * \dots * L \quad (m \text{ times}). \quad (11)$$

For the construction of integrable systems, the time flows can be easily build up with the help of equation (11). For example,

$$\begin{aligned} L^{*p} &= \left(p + \sum_{m=1} a_m p^{-m} \right) * \left(p + \sum_{l=1} a_l p^{-l} \right) \\ &= p^2 + 2 \sum_m a_m p^{-m+1} + \sum_{s,k} \varepsilon^s (-1)^k \binom{-m}{s-k} \binom{-i}{k} a_m^{(k)} a_i^{(s-k)} p^{-m-i-s} \end{aligned} \quad (12)$$

or

$$\begin{aligned} L^{*2} &= p^2 + 2a_1 + 2a_2 p^{-1} + \dots \\ L^{*3} &= p^3 + 3a_1 p + 3a_2 + (3a_3 + 3a_1^2 + \varepsilon^2 a_1^{(2)}) p^{-1} + \dots \\ L^{*4} &= p^4 + 4a_1 p^2 + 4a_2 p + (4a_3 + 6a_1^2 + \varepsilon^2 4a_1^{(2)}) \\ &\quad + (4a_4 + 12a_1 a_2 + 4\varepsilon^2 a_2^{(2)}) p^{-1} + \dots \\ L^{*5} &= p^5 + 5a_1 p^3 + 5a_2 p^2 + (5a_3 + 10a_1^2 + 10\varepsilon^2 a_2^{(2)}) p \\ &\quad + (5a_4 + 20a_1 a_2 + 10\varepsilon^2 a_2^{(2)}) \\ &\quad + [5a_5 + 20a_1 a_3 + 10a_2^2 + 10a_1^3 + \varepsilon(8a_4^{(1)} + 24a_1 a_2^{(1)} + 24a_2 a_1^{(1)}) \\ &\quad + \varepsilon^2(10a_3^{(2)} + 20a_1 a_1^{(2)} + 10a_1^{(1)} a_1^{(1)}) + \varepsilon^3 8a_2^{(3)} + \varepsilon^4 a_1^{(4)}] p^{-1} + \dots \end{aligned} \quad (13)$$

The quantized BKP system is then defined through the Lax equations

$$\frac{\partial L}{\partial y} = [L_+^3, L]_*, \quad \frac{\partial L}{\partial t} = [L_+^5, L]_*. \quad (14)$$

Using the above expressions we immediately get:

$$\begin{aligned}
a_{1y} &= \varepsilon(6a_3^{(1)} + 12a_1a_1^{(1)}) + 2\varepsilon^3a_1^{(3)}, \\
a_{2y} &= \varepsilon(6a_4^{(1)} + 12a_1^{(1)}a_2 + 12a_1a_2^{(1)}) + 2\varepsilon^3a_2^{(3)}, \\
a_{3y} &= \varepsilon(6a_5^{(1)} + 6a_1a_3^{(3)} + 18a_1^{(1)}a_3 + 12a_2a_2^{(1)}) \\
&\quad + \varepsilon^3(2a_3^{(3)} + 6a_1^{(3)}a_1 + 6a_1^{(2)}a_1^{(1)}), \\
a_{4y} &= (6a_6^{(1)} + 6a_1a_4^{(1)} + 24a_1^{(1)}a_4 + 18a_2^{(1)}a_3) \\
&\quad + \varepsilon^3(2a_4^{(3)} + 18a_1^{(2)}a_2^{(1)} + 24a_1^{(3)}a_2 + 6a_1a_2^{(3)}).
\end{aligned} \tag{15}$$

Similar equations for time evolution. Actually, (14) leads to an infinite set of coupled nonlinear equations in 3 dimensions. To restrict it to a BKP hierarchy we impose the constraint that $(L^*)_+^{2n+1}$ will not contain any constant level term. Whence we get $a_2 = a_4 = 0$, along with

$$a_{3y} = \varepsilon(6a_5^{(1)} + 6a_1a_3^{(1)} + 18a_1^{(1)}a_3) + \varepsilon^3(2a_3^{(3)} + 6a_1^{(3)}a_1 + 6a_1^{(2)}a_1^{(1)}), \tag{16}$$

$$\begin{aligned}
a_{1t} &= 2\varepsilon^5a_1^{(5)} + \varepsilon^3(20a_3^{(3)} + 40a_1^{(3)}a_1 + 80a_1^{(2)}a_1^{(1)}) \\
&\quad + \varepsilon(10a_5^{(1)} + 40a_3^{(1)}a_1 + 40a_3a_1^{(1)} + 60a_1^2a_1^{(1)}).
\end{aligned} \tag{17}$$

By eliminating a_3 and a_5 we obtain

$$\begin{aligned}
a_{1t} &= -\frac{32}{9}\varepsilon^5a_1^{(5)} + \varepsilon^3\left[-\frac{40}{3}a_1^{(3)}a_1 - \frac{100}{3}a_1^{(1)}a_1^{(2)}\right] + \varepsilon^2\frac{20}{9}a_{1y}^{(2)} \\
&\quad - 10\varepsilon a_1^2a_1^{(1)} + \frac{5}{3}(a_1a_{1y} + a_1^{(1)}\partial^{-1}a_{1y}) + \frac{5}{18\varepsilon}\partial^{-1}a_{1y},
\end{aligned} \tag{18}$$

which is the required nonlinear equation in (2+1)-dimensions for $a_1(xyt)$. One can observe that this equation is also fifth order in derivative with respect to x as in the case of the usual BKP system.

It is possible to generate other types of nonlinear system from different choices of y and t flows in equation (14). For example, consider

$$\begin{aligned}
\frac{\partial L}{\partial y} &= [L^{*2}, L]_*, \\
\frac{\partial L}{\partial t} &= [L^{*5}, L]_*,
\end{aligned} \tag{19}$$

where equations (15) are modified to:

$$\begin{aligned}
a_{1y} &= 4\varepsilon a_2^{(1)}, \\
a_{2y} &= 4\varepsilon a_3^{(1)} + 4\varepsilon a_1a_1^{(1)}, \\
a_{3y} &= 4\varepsilon a_4^{(1)} + 8\varepsilon a_1^{(1)}a_2, \\
a_{4y} &= 4\varepsilon a_5^{(1)} + 12\varepsilon a_1^{(1)}a_3 + 4\varepsilon^3a_1a_1^{(3)}
\end{aligned} \tag{20}$$

along with

$$\begin{aligned}
a_{1t} &= 2\varepsilon^5a_1^{(5)} + \varepsilon^3(20a_3^{(3)} + 40a_1^{(3)}a_1 + 80a_1^{(2)}a_1^{(1)}) \\
&\quad + \varepsilon(10a_5^{(1)} + 40a_3^{(1)}a_1 + 30a_3a_1^{(1)} + 40a_2a_2^{(1)} + 60a_1^2a_1^{(1)}).
\end{aligned} \tag{21}$$

It is interesting to note that one can eliminate all the variables other than a_1 , without imposing any extra condition on L_+^{2n+1} , since equation (20) yields:

$$\begin{aligned} a_2 &= \frac{1}{4\varepsilon} \partial^{-1} a_{1y}, \\ a_3 &= \frac{1}{16\varepsilon^2} \partial^{-2} a_{1yy} - \frac{1}{2} a_1^2, \\ a_4 &= \frac{1}{4\varepsilon} \partial^{-1} a_{3y} - 2\partial^{-1} (a_1^{(1)} a_2). \end{aligned} \quad (22)$$

Thus, from (21) we at once obtain

$$\begin{aligned} a_{1t} &= 2\varepsilon^5 a_1^{(5)} + \varepsilon^3 \left[10a_1 a_1^{(3)} + 20a_1^{(1)} a_1^{(2)} \right] + \varepsilon \left[15a_1^2 a_1^{(1)} + \frac{5}{4} a_{1yy}^{(1)} \right] \\ &\quad + \frac{5}{8\varepsilon} \left[3\partial^{-1} (a_{1y} a_{1y} + a_{1y} a_{1yy}) + 2a_{1y} \partial^{-1} a_{1y} - 3a_1^{(1)} \partial^{-2} a_{1yy} + 4a_1 \partial^{-1} a_{1yy} \right] \\ &\quad - \frac{5}{128\varepsilon^3} \partial^{-3} a_{14y}. \end{aligned} \quad (23)$$

In each case, as in the case of usual $\overline{\Psi}$ DO approach, the conserved quantities are obtained as $\text{Res.}(L^{2n+1})$, where Res. denote the coefficient of p^{-1} . We have checked that such residues turn out to be combinations of total derivatives with respect to (x, y, t) .

We now report on intriguing fact about such Moyal quantized systems. This is related to the so-called modified system. In an interesting paper it was observed by Kupershmidt that modified equations can be generated if one uses

$$\frac{dL}{dt} = [(L^m)_{\geq 1}, L] \quad (24)$$

instead of $(L^m)_{\geq 0}$. (In some cases one can also consider $(L^m)_{\geq 2}$.) Here we observe that, if we consider

$$\frac{\partial L}{\partial y} = [L_{\geq 1}^3, L]_* , \quad \frac{\partial L}{\partial t} = [L_{\geq 1}^5, L]_* , \quad (25)$$

and using

$$\begin{aligned} (L^3)_{\geq 1} &= p^3 + 3a_1 p, \\ (L^{*5})_{\geq 1} &= p^5 + 5a_1 p^3 + 5a_2 p^2 + (5a_3 + 10a_1^2 + \varepsilon^2 10a_1^{(2)}) p \end{aligned} \quad (26)$$

in these equations, we get

$$\begin{aligned} a_2^{(1)} &= 0, \\ a_{1y} &= 6\varepsilon a_3^{(1)} + 12\varepsilon a_1 a_1^{(1)} + 2\varepsilon^3 a_1^{(3)}, \\ a_{2y} &= 6\varepsilon a_4^{(1)} + 12\varepsilon a_2 a_1^{(2)}, \\ a_{3y} &= 6\varepsilon a_5^{(1)} + 6\varepsilon a_1 a_3^{(1)} + 18\varepsilon a_1^{(1)} a_3 + \varepsilon^3 (2a_3^{(3)} + 6a_1 a_1^{(3)} + 6a_1^{(1)} a_1^{(2)}) \end{aligned} \quad (27)$$

alongwith

$$\begin{aligned} a_{1t} &= 2\varepsilon^5 a_1^{(5)} + \varepsilon^3 (20a_3^{(3)} + 40a_1 a_1^{(3)} + 80a_1^{(1)} a_1^{(2)}) \\ &\quad + \varepsilon (40a_1 a_3^{(1)} + 40a_1^{(1)} a_3) + 60a_4^2 a_1^{(1)} + 10a_5^{(1)}. \end{aligned}$$

Again eliminating variables other than a_1 , we get

$$\begin{aligned} a_{1t} = & 2\varepsilon^5 a_1^{(5)} + \varepsilon^3 (20a_3^{(3)} + 40a_1^{(3)} a_1 + 80a_1^{(2)} a_1^{(1)}) \\ & + \varepsilon (40a_1 a_3^{(1)} + 40a_1^{(1)} a_3 + 60a_1^2 a_1^{(1)} + 10a_5^{(1)}). \end{aligned} \quad (28)$$

So we get back equation (21), but this time without any restriction on the constant level term of $(L^*)^{2n+1}$.

From the structure of the nonlinear equations discuseed so far, it appears that one can define an analogue of a dressing operator even in the Moyal quantized systems.

Let us set

$$s = 1 + \sum_{m=1}^{\infty} \omega_m p^{-m} \quad (29)$$

for the deressing operator. Now

$$s^{-1} = 1 + \sum_{m=1}^{\infty} u_m p^{-m}, \quad (30)$$

with $s + s^{-1} = 1$, leads to

$$(u_m + \omega_m) p^{-m} + \varepsilon^s (-1)^k \binom{-m}{j-k} \binom{-j}{k} \omega_m^{(k)} u_j^{(s-k)} p^{-m-j} = 0, \quad (31)$$

from which one can express the u 's in terms of the ω 's. For example,

$$\begin{aligned} u_1 &= -\omega_1, \\ u_2 &= -\omega_2 + \omega_1^2, \\ u_3 &= -\omega_3 - \omega_1^2 + 2\omega_1 \omega_2, \dots, \text{ etc.} \end{aligned} \quad (32)$$

Now we demand that

$$s * p * s^{-1} = L = p + \sum_{m=1}^{\infty} a_m p^{-m}, \quad (33)$$

which immediately leads to

$$\begin{aligned} a_1 &= -2\varepsilon \omega_1^{(1)}, \\ a_2 &= \varepsilon (-2\omega_2^{(1)} + 2\omega_1 \omega_1^{(1)}), \\ a_3 &= u_4 + \omega_4 + \omega_1 u_3 + \omega_2 u_2 + \omega_3 u_1 \\ &\quad + \varepsilon (u_3^{(1)} - \omega_3^{(1)} + \omega_1^{(1)} u_2 - \omega_2 u_1^{(1)}) + \varepsilon^2 \omega_1^{(1)} u_1^{(1)}, \dots, \text{ etc.} \end{aligned} \quad (34)$$

On the other hand, if we compute $s * p^3 * s^{-1}$ and $s * p^5 * s^{-1}$ via the rules (4), we get

$$\begin{aligned} s * p^3 * s^{-1} &= \left(1 + \sum_m \omega_m p^{-m} \right) * p^3 * \left(1 + \sum_l u_l p^{-l} \right) \\ &= p^3 + (\omega_1 + u_1) p^2 + (u_2 + \omega_2 + \omega_1 u_1 + 3\varepsilon u_1^{(1)} - 3\varepsilon \omega_1^{(1)}) p + \dots \end{aligned} \quad (35)$$

Whence, using equations (32) and (34), we get

$$\begin{aligned} (s * p^3 * s^{-1})_+ &= p^3 + 3a_1 p + (3\varepsilon u_2^{(1)} - 3\varepsilon \omega_2^{(1)}) \\ &= p^3 + 3a_1 p + 3a_2 \\ &= L^3 \quad (\text{as given in equation (13)}). \end{aligned} \quad (36)$$

Using the same methodology, but with a more laborious computation, we can prove that

$$(s * p^5 * s^{-1})_+ = p^5 + 5a_1p^3 + 5a_2p^2 + (5a_3 + 10a_1^2 + 10a_1^2a_1^{(2)})p + (5a_4 + 20a_1a_2 + 10\varepsilon^2a_2^{(2)}). \quad (37)$$

In the above analysis we have shown that nonlinear systems in (2+1)-dimensions, which are in the category of the BKP equation, can be quantized using the pseudo-differential operators and Moyal bracket formalism. They are completely integrable since there are an infinite number of conserved quantities with them.

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