

Progressive Internal Gravity Waves With Bounded Upper Surface Climbing a Triangular Obstacle

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Abstract

In this paper we discuss a theoretical model for the interfacial profiles of progressive non-linear waves which result from introducing a triangular obstacle, of finite height, attached to the bottom below the flow of a stratified, ideal, two layer fluid, bounded from above by a rigid boundary. The derived equations are solved by using a nonlinear perturbation method. The dependence of the interfacial profile on the triangular obstacle size, as well as its dependence on some flow parameters, such as the ratios of depths and densities of the two fluids, have been studied.

1 Introduction

The determination of flow patterns over obstacles is a problem of a great interest, that attracted many scientists over the past decades. Lamb [6] was the first to give the essential features of the flow of an ideal fluid in an open channel in the presence of an obstruction in the channel. In 1955, Long [7] and then later on McIntyre [8] in 1972, considered the case of a steady and uniform stratification over obstacles of finite height, while Mei and LeMehaute [9] in 1966 studied the case of long waves in shallow water over an uneven bottom. The effect of the irregularities of the bottom, on gravity waves, has been studied by Kakutani [4] in 1971 via a reductive perturbation method. Recently, Kevorkian and Yu [5], in 1989, studied the behaviour of shallow water waves excited by a small amplitude bottom disturbance in the presence of a uniform incoming flow. In this paper we study a theoretical model for the interfacial profiles of progressive non-linear waves result from introducing a triangular obstacle, attached to the bottom below the flow of a stratified, ideal, two layer fluid, bounded from above by a rigid boundary. Our primary motivation

for the present investigation is calculate the shape of the interfacial waves, and to discuss the influence of both geometrical and flow parameters on the profiles. In section 2, we extended the mathematical technique applied by Helal & Molines [3], 1981, in determining the nonlinear free-surface and interfacial waves in a tank with flat horizontal bottom. Nonlinear perturbation technique is used, leading, in sections 3 and 4, to expression for interfacial wave, was derived in the form of expansions in powers of ε^2 , where ε is a small parameter that provides a measure of weakness of dispersion. Boutros, *et al* [1], in 1991, applied the same technique to study the internal waves over a ramp.

Finally, in section 5 we have discussed the effect of the density ratio, R , the thickness ratio, H , and the triangle height, L .

2 Formulation of the problem

Two-dimensional irrotational motion of a stably stratified two-layer fluid with a rigid upper boundary and a bottom surface in the form of a triangle with two inclination angles α and β . A cartesian coordinate system is defined with the origin at the bottom surface.

We assume that the motion is two-dimensional, and the fluid is inviscid, incompressible, and that the flow field due to the wave motion remains irrotational and consequently we can introduce velocity potentials of upper and lower layers are denoted by $\Phi^{*(i)}$, $i = 1, 2$, respectively. Moreover, let H_i^* , $\rho^{(i)}$; $i = 1, 2$ denote the thickness and densities of the upper and lower fluids, respectively, τ^* is the time, $Y^* = W^*(X^*)$ is the bed of the channel and $Y^* = f^*(X^*, \tau^*)$ is the interfacial disturbance from uniform condition. The component of gravity, vertically downwards, is g . The equations of motion are thus the Euler equations together with the continuity equation. All variables are non-dimensionalized by using the characteristic length H_2^* and time $(g/H_2^*)^{-1/2}$, and accordingly

$$U = U^*/[gH_2^*]^{1/2} \quad \text{and} \quad \Phi^{(i)} = \Phi^{*(i)}/(H_2^*[gH_2^*]^{1/2}).$$

Moreover, assuming that the fluids are in the undisturbed uniform state up/down stream at infinity, we impose the following boundary conditions with respect to X^*

$$\Phi_{X^*}^{*(i)} = U^*, \quad (i = 1, 2) \quad \text{as} \quad X^* \rightarrow \pm\infty.$$

An essential step which makes our problem easier in handling is to define an appropriate *stretching* of the horizontal coordinate while leaving the vertical coordinate unchanged due to the fact that the horizontal dimensions are much greater than the vertical dimensions, thus we define

$$x = \varepsilon X, \quad y = Y, \quad t = \varepsilon \tau, \tag{1}$$

where ε is a small parameter. Thus the basic equations for this system can be written as

$$\varepsilon^2 \Phi_{xx}^{(1)} + \Phi_{yy}^{(1)} = 0, \quad f < y < 1 + H, \quad -\infty < x < \infty, \tag{2}$$

$$\varepsilon^2 \Phi_{xx}^{(2)} + \Phi_{yy}^{(2)} = 0, \quad W < y < f, \quad -\infty < x < \infty, \tag{3}$$

with conditions

(i) Boundary conditions:

$$\left. \begin{aligned} \Phi_y^{(i)} &= \varepsilon f_t + \varepsilon^2 \Phi_x^{(i)} f_x, & (i = 1, 2) \\ R \left\{ \varepsilon \Phi_t^{(1)} + \frac{1}{2} \left[\varepsilon^2 (\Phi_x^{(1)})^2 + (\Phi_y^{(1)})^2 \right] + f - 1 \right\} &= \\ \left\{ \varepsilon \Phi_t^{(2)} + \frac{1}{2} \left[\varepsilon^2 (\Phi_x^{(2)})^2 + (\Phi_y^{(2)})^2 \right] + f - 1 \right\} & \end{aligned} \right\} \quad \text{at } y = f \quad (4)$$

$$\Phi_y^{(2)} = \varepsilon^2 \Phi_x^{(2)} W_x \quad \text{at } y = W(x) \quad (5)$$

$$\Phi_y^{(i)} = 0, \quad \text{at } y = 1 + H, \quad (6)$$

$$\varepsilon \Phi_x^{(i)} = 1, \quad (i = 1, 2) \quad \text{as } x \rightarrow \pm\infty \quad (7)$$

(ii) Initial condition:

at $t = 0$: the initial profile of the interfacial wave, denoted by $f(x, 0)$, is shown in Fig.1.

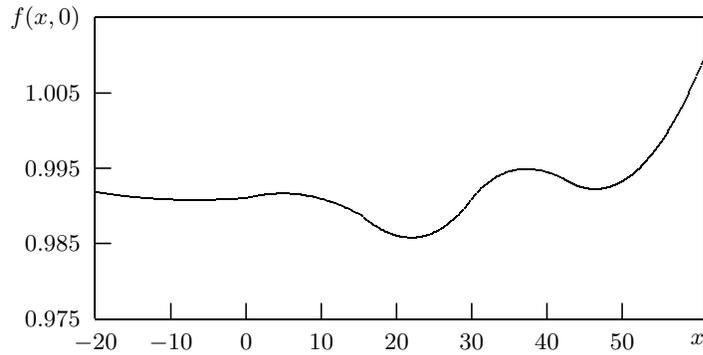


Fig.1. The initial waveform over a triangular obstacle with $x_m = 13$, $x_e = 30$, $L = 0.25$, $\alpha = 0.01923$, $\beta = -0.01471$

where the density ratio $R = \rho^{(1)}/\rho^{(2)}$ (less than unity) and the thickness ratio H are two characteristic parameters of the system, and $W(x)$ has the form

$$W(x) = ax + b, \quad (8)$$

where

$$(a, b) = \begin{cases} (0, 0) & x \leq 0 \\ (\alpha, 0) & 0 \leq x \leq x_m \\ (-\beta, \beta x_e) & x_m \leq x \leq x_e \\ (0, 0) & x > x_e \end{cases}$$

Since we consider weakly nonlinear waves, we expand the dependent variables as power series in the same parameter ε around the undisturbed uniform state, following Helal and Molines [3], we get

$$\begin{aligned} \Phi^{(i)} &= \sum_{n=0}^{\infty} \varepsilon^{2n-1} G_{2n-1}^{(i)}(x, y, t), & i = 1, 2 \\ f &= \sum_{n=0}^{\infty} \varepsilon^{2n} f_{2n}(x, y, t), & \text{with } f_0 = 1. \end{aligned}$$

The scale parameter ε , which is assumed to be small, provides a measure of weakness of dispersion.

The boundary conditions on the interface, equations (4), are expanded as a Taylor expansion of the type

$$[V]_{y=y_0+\varepsilon^2 A} = \sum_{n=0}^{\infty} \frac{(\varepsilon^2 A)^n}{n!} \left[\frac{\partial^n V}{\partial y^n} \right]_{y_0}. \quad (9)$$

When (1), (8), using the expansion (9), are inserted into equations (2)-(7) and powers of ε are sorted out, we get an ordered set of equations to be solved.

3 Orders of approximations

3.1 The first-order approximation

Equations of the first-order approximation, finally gives, for $i = 1, 2$

$$G_1^{(i)} = B^{(i)}(x, t),$$

where $B^{(i)}(x, t)$ are unknown functions to be determined.

3.2 The second-order approximation

From the equations obtained from the second-order approximation, we conclude that

$$B_x^{(i)} = 0, \quad (i = 1, 2) \quad \text{as } x \rightarrow \pm\infty$$

and

$$f_2(x, t) = \frac{1}{1-R} \left[RB_t^{(1)} - B_t^{(2)} \right].$$

3.3 The third- and fourth-order approximations

Equations of the third- and fourth-order approximation, finally gives, for $i = 1, 2$

$$G_3^{(i)} = -\frac{1}{2}y^2 B_{xx}^{(i)} + yC^{(i)}(x, t) + D^{(i)}(x, t), \quad (10)$$

where $C^{(i)}(x, t)$ and $D^{(i)}(x, t)$ are arbitrary functions satisfy the following boundary conditions:

$$C_x^{(i)} = 0 \quad (i = 1, 2) \quad \text{as } x \rightarrow \pm\infty, \quad (11)$$

$$C^{(2)}(x, t) = \left(WB_x^{(2)} \right)_x \quad \text{at } y = W(x), \quad (12)$$

$$D_x^{(i)} = 0 \quad (i = 1, 2) \quad \text{as } x \rightarrow \pm\infty. \quad (13)$$

Substituting equation (10) in the equations that are obtained from the third- and fourth-order approximation, we obtain

$$(H+1)B_{xx}^{(1)} - C^{(1)} = 0, \quad (14)$$

and for $i = 1, 2$

$$B_{xx}^{(i)} - C^{(i)} + \frac{1}{1-R} \left(RB_{tt}^{(1)} - B_{tt}^{(2)} \right) = 0. \quad (15)$$

From equations (12), (14), and (15) we get

$$\begin{aligned} \square_1 B^{(1)} &= B_{tt}^{(2)}, \\ \square_2 B^{(2)} &= RB_{tt}^{(1)}, \end{aligned} \quad (16)$$

where \square_1, \square_2 are the differential operators defined by

$$\begin{aligned} \square_1 &\equiv -H(1-R) \frac{\partial^2}{\partial x^2} + R \frac{\partial^2}{\partial t^2}, \\ \square_2 &\equiv -(1-R)(1-W) \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial t^2} + (1-R) \frac{\partial W}{\partial x} \frac{\partial}{\partial x}. \end{aligned} \quad (17)$$

From equations (16)-(17) we can get, after getting rid of $B^{(1)}$ and substituting for $W(x)$, the following differential equation for the unknown function $B^{(2)}(x, t)$

$$\begin{aligned} -H(1-R)(1-b-ax)B_{xxxx}^{(2)} + [H + R(1-b-ax)]B_{xxtt}^{(2)} \\ -RaB_{xtt}^{(2)} + 3Ha(1-R)B_{xxx}^{(2)} = 0 \end{aligned} \quad (18)$$

and for $f_4(x, t)$ we can get the following relation

$$\begin{aligned} f_4(x, t) = \frac{1}{1-R} \left\{ R \left[-\frac{1}{2}B_{xxt}^{(1)} + C_t^{(1)} + D_t^{(1)} + \frac{1}{2} \left(B_x^{(1)} \right)^2 \right] \right. \\ \left. + \frac{1}{2}B_{xxt}^{(2)} - C_t^{(2)} - D_t^{(2)} - \frac{1}{2} \left(B_x^{(2)} \right)^2 \right\}. \end{aligned}$$

3.4 The fifth- and sixth-order approximations

Equations of the fifth- and sixth-order approximation lead to, for $i = 1, 2$

$$G_5^{(i)} = \frac{y^4}{24} B_{xxxx}^{(i)} - \frac{y^3}{6} C_{xx}^{(i)}(x, t) - \frac{y^2}{2} D_{xx}^{(i)}(x, t) + yE^{(i)}(x, t) + F^{(i)}(x, t), \quad (19)$$

where $E^{(i)}(x, t)$ and $F^{(i)}(x, t)$ are arbitrary functions, satisfy the following conditions:

$$E_x^{(i)} = 0 \quad (i = 1, 2), \quad \text{as } x \rightarrow \pm\infty \quad (20)$$

and at $y = W(x)$

$$E^{(2)}(x, t) = \left(-\frac{W^3}{3!} B_{xxx}^{(2)} + \frac{W^2}{2!} C_x^{(2)}(x, t) + W D_x^{(2)} \right)_x, \quad (21)$$

$$F_x^{(i)} = 0 \quad (i = 1, 2) \quad \text{as } x \rightarrow \pm\infty. \quad (22)$$

Introducing equations (10)-(19) in the boundary conditions, we have the following relations:

$$\frac{(H+1)^3}{3!} B_{xxxx}^{(1)} - \frac{(H+1)^2}{2!} C_{xx}^{(1)} - (H+1)D_{xx}^{(1)} + E^{(1)} = 0 \quad (23)$$

and for $i = 1, 2$

$$\begin{aligned} & \frac{1}{3!} B_{xxxx}^{(i)} - \frac{1}{2!} C_{xx}^{(i)} - D_{xx}^{(i)} + E^{(i)} + \frac{1}{1-R} \left[(B_t^{(2)} - RB_t^{(1)}) B_{xx}^{(i)} \right. \\ & \quad \left. + (B_{xt}^{(2)} - RB_{xt}^{(1)}) B_x^{(i)} - \frac{1}{2} B_{xxt}^{(2)} + C_{tt}^{(2)} + D_{tt}^{(2)} \right. \\ & \quad \left. - R \left(-\frac{1}{2} B_{xxt}^{(1)} + C_{tt}^{(1)} + D_{tt}^{(1)} \right) + B_x^{(2)} B_{xt}^{(2)} - RB_x^{(1)} B_{xt}^{(1)} \right] = 0. \end{aligned} \quad (24)$$

Thus the problem is now reduced to solving equations (14) and (15) for $B^{(i)}$ and $C^{(i)}$ and next equations (20), (21) and (24) for $D^{(i)}$ and $E^{(i)}$, where $i = 1, 2$.

4 Case of progressive wave

It must be remarked that our procedure is valid as long as $a \gg \varepsilon^2$, otherwise a two-parameter analysis has to be carried out. Moreover, we shall invoke the smallness of a and write perturbation expansions for $B^{(i)}$, ($i = 1, 2$), in the form

$$B^{(i)} = B_0^{(i)} + aB_1^{(i)} + a^2B_2^{(i)} + \dots \quad (25)$$

Substituting (25) in (18) and equating coefficients of $a^{(j)}$, $j = 0, 1, 2, \dots$ we get the following system of differential equations

$$\square B_j^{(2)} = \Lambda B_{j-1}^{(2)} \quad (j = 0, 1, 2, \dots), \quad B_{-1}^{(2)} = 0, \quad (26)$$

where \square, Λ are two differential operators defined as

$$\begin{aligned} \square & \equiv -H(1-R)(1-b) \frac{\partial^4}{\partial x^4} + [H + R(1-b)] \frac{\partial^4}{\partial x^2 \partial t^2}, \\ \Lambda & \equiv -xH(1-R) \frac{\partial^4}{\partial x^4} + xR \frac{\partial^4}{\partial x^2 \partial t^2} - 3H(1-R) \frac{\partial^3}{\partial x^3} + R \frac{\partial^3}{\partial x \partial t^2}. \end{aligned}$$

Equation (26), for $j = 0$, has the following general solution, for the case of pure progressive waves,

$$B_0^{(i)} = B_0^{(i)}(\xi)$$

with

$$\xi = x - \gamma t, \quad \gamma^2 = \frac{H(1-b)(1-R)}{H + (1-b)R}.$$

From equations (8), (12), and (25) we get

$$C^{(2)} = \sum_{n=0}^{\infty} a^n \left[bB_{n,xx}^{(2)} + \left(xB_{n-1,x}^{(2)} \right)_x \right], \quad B_{-1}^{(2)} = 0. \quad (27)$$

Again substituting equations (25), (27) in equation (15) we get, after equating coefficients of a^n , $n = 0, 1, 2, \dots$

$$B_{0,x}^{(2)} = \lambda B_{0,x}^{(1)}, \quad B_{1,x}^{(2)} = \frac{x}{1-b} B_{0,x}^{(2)} + \lambda B_{1,x}^{(1)},$$

where

$$\lambda = \frac{H}{1-b}.$$

The elimination of $E^{(1)}$ in equations (23) and (24) gives, for “ a ”, the following system of differential equations

$$\begin{aligned} \left(H - \frac{\gamma^2 R}{1-R}\right) D_{\xi\xi}^{(1)} + \frac{\gamma^2}{1-R} D_{\xi\xi}^{(2)} &= P_1 B_{0,\xi\xi\xi\xi}^{(1)} + Q_1 B_{0,\xi}^{(1)} B_{0,\xi\xi}^{(1)}, \\ \frac{\gamma^2 R}{1-R} D_{\xi\xi}^{(1)} + \left(1-b - \frac{\gamma^2}{1-R}\right) D_{\xi\xi}^{(2)} &= P_2 B_{0,\xi\xi\xi\xi}^{(1)} + Q_2 B_{0,\xi}^{(1)} B_{0,\xi\xi}^{(1)}, \end{aligned} \quad (28)$$

where

$$\begin{aligned} P_1 &= \frac{-H(2H^2 + 6H + 3)}{6} + \frac{\gamma^2[\lambda(1-2b) + R(2H+1)]}{2(1-R)}, \\ P_2 &= \frac{\lambda}{6}(1-3b+2b^3) + \frac{\gamma^2}{2(1-R)}[(2b-1)\lambda - R(2H+1)], \\ Q_1 &= \frac{\gamma}{1-R}(\lambda^2 + 2\lambda - 3R), \quad Q_2 = \frac{\gamma}{1-R}[R(2\lambda+1) - 3\lambda^2]. \end{aligned}$$

For the non-trivial solution of $D_{\xi\xi}^{(1)}$ and $D_{\xi\xi}^{(2)}$, the following differential equation for $B_0^{(1)}$ should be satisfied:

$$M_1 B_{0,\xi\xi\xi\xi}^{(1)} + M_2 B_{0,\xi}^{(1)} B_{0,\xi\xi}^{(1)} = 0, \quad (29)$$

where

$$M_1 = \left(1-b - \frac{\gamma^2}{1-R}\right) P_1 - \frac{\gamma^2}{1-R} P_2, \quad M_2 = \left(1-b - \frac{\gamma^2}{1-R}\right) Q_1 - \frac{\gamma^2}{1-R} Q_2.$$

Define

$$\Gamma = B_{0,\xi}^{(1)}. \quad (30)$$

Thus equation (29), by virtue of equation (30), will be transformed to the Boussinesq equation

$$M_1 \Gamma_{\xi\xi\xi} + M_2 \Gamma \Gamma_{\xi} = 0. \quad (31)$$

Helal & Molines [3] mentioned that the general solution of equation (31) was found by Byrd and Friedmann [2] to be, in terms of the Jacobi elliptic function $\text{sn}(u, k)$, as

$$B_{0,\xi}^{(1)} = Y_1 \left[1 - \frac{3k^2}{k^2+1} \text{sn}^2(\delta\xi, k^2) \right],$$

where Y_1 is the greatest of the roots of the polynomial resulting from integrating equation (31) twice and k is the modulus of the Jacobean elliptic function, and

$$\delta = \frac{1}{2} \left(-\frac{3AY_1}{k^2+1} \right)^{1/2}.$$

For small values of k the above elliptic function could be calculated in terms of trigonometric functions, see Milne-Thomson [11], thus we have

$$B_{0,\xi}^{(1)} = Y_1 \left\{ 1 - \frac{3k^2}{k^2 + 1} \left[\left(\frac{1}{2} + \frac{k^2}{8} + \frac{k^4}{16} \right) + \frac{k^4 - 64}{128} \cos 2\delta\xi - \frac{8k^2 + k^4}{64} \cos 4\delta\xi - \frac{k^4}{128} \cos 6\delta\xi - \delta\xi \left\{ \left(\frac{k^2}{2} + \frac{k^4}{8} \right) \sin 2\delta\xi + \frac{k^4}{16} \sin 4\delta\xi \right\} + \delta^2 \xi^2 \left\{ \frac{k^4}{8} + \frac{k^4}{8} \cos 2\delta\xi \right\} \right] \right\}. \quad (32)$$

Substituting in equation (26), for $B_{0,x}^{(2)}$ and $B_{0,t}^{(2)}$, we get the following fourth-order linear partial differential equation

$$\begin{aligned} \square B_1^{(2)} &= \sum_{n=1}^3 (A_n x \sin 2n\delta\xi + A_{n+6} \cos 2n\delta\xi) \\ &+ \delta\xi \sum_{n=1}^2 (A_{n+3} x \cos 2n\delta\xi + A_{n+10} \sin 2n\delta\xi) \\ &+ \delta^2 \xi^2 (A_6 x \sin 2\delta\xi + A_{13} \cos 2\delta\xi) + A_{10}, \end{aligned} \quad (33)$$

where the coefficients A_1, A_2, \dots, A_{13} are given at the end of the paper, as Appendix 1.

Solving equation (33) for the unknown $B_1^{(2)}$, following Miller [10], and calculating $B_{1,t}^{(2)}$ we get

$$\begin{aligned} B_{1,t}^{(2)} &= B_{0,t}^{(2)} + r_1 t^3 + r_2 x^2 t + (r_3 + r_4 x^2 + r_5 x t + r_6 t^2) \sin 2\delta\xi \\ &+ (r_7 + r_8 x^2 + r_9 x t + r_{10} t^2) \sin 4\delta\xi + r_{11} \sin 6\delta\xi \\ &+ (r_{12} + r_{13} x + r_{14} t + r_{15} x^3 + r_{16} x^2 t + r_{17} x t^2 + r_{18} t^3) \cos 2\delta\xi \\ &+ (r_{19} x + r_{20} t) \cos 4\delta\xi + (r_{21} x + r_{22} t) \cos 6\delta\xi, \end{aligned} \quad (34)$$

where the coefficients r_1, r_2, \dots, r_{22} are also given at the end of the paper, as Appendix 2.

Taking into consideration the value of $B_{0,x}^{(1)}$ from equation (32), we can get $B_{0,x}^{(2)}$ and thus, using (34) for $B_{1,t}^{(2)}$ we can get $B_{1,t}^{(1)}$

$$B_{1,t}^{(1)} = \frac{1}{\lambda} \left(B_{1,t}^{(2)} - \frac{x}{1-a} B_{0,t}^{(2)} \right).$$

In order to account for the nonlinear effects the $O(\varepsilon^4)$ equations have to be considered as well. Thus bearing in mind the linear system of equations (28), the principal and secondary determinants of this system, we come to the result that

$$D_t^{(i)} = 0, \quad (i = 1, 2).$$

Hence $f_4(x, t)$ may be rewritten in the simplified form

$$\begin{aligned} f_4(x, t) &= \frac{1}{2(1-R)} \left\{ \left((\lambda - R) + 2(1+H)R - 2\lambda(ax+b) + \frac{ax\lambda(1-2b)}{1-b} \right) B_{0,txt}^{(1)} \right. \\ &+ (R(2H+1) + \lambda(1-2b)) B_{1,txt}^{(1)} + \left(2(R-\lambda^2) - \frac{2ax\lambda^2(ax+2)}{1-b} \right) (B_{0,x}^{(1)})^2 \\ &\left. + a^2(R-a)(B_{1,x}^{(1)})^2 + \frac{2a\lambda b}{b-1} B_{0,xt}^{(1)} + 2a \left(R - \lambda^2 - \frac{ax\lambda^2}{1-b} \right) B_{0,x}^{(1)} B_{1,x}^{(1)} \right\}. \end{aligned} \quad (35)$$

Hence $f(x, t)$ will take the form

$$f(x, t) = 1 + \varepsilon^2 \left\{ \frac{(R - \lambda)(b - 1) + \lambda bx}{(1 - R)(b - 1)} B_{0,t}^{(1)} + \frac{a(R - \lambda)}{1 - R} B_{1,t}^{(1)} \right\} + \varepsilon^4 f_4(x, t) + O(\varepsilon^6),$$

where $f_4(x, t)$ is given by (35) and $B_{0,t}^{(1)}$ and $B_{1,t}^{(1)}$ are given by (32) and (34) respectively.

5 Presentation of results and discussion

The number of terms which has been obtained seems to be a good measure for the purpose of illustrating the effect of the parameters the density ratio, R , the thickness ratio, H , and the obstacle height, L . The error, difference between the fourth and second order approximations, in the interfacial profile for the two approximations is of order 10^{-6} . Thus we limit our calculations up to the second-order approximation, as well as we considered the following values for the description of the triangular obstacle: $x_m = 13$ and $x_e = 30$.

We studied the effect of the density ratio, R , on the wave profiles at the interfacial surface. Three values of R have been considered, namely $R = 0.7, 0.8,$ and 0.9 for fixed values of $H, L,$ and t . It is clear that as R decreases, there is a kind of violent oscillations in the obstacle region. This phenomena vanishes gradually as “ R ” increases. An important remark must be mentioned is that, for the interfacial wave in the downstream region, the period of oscillation is much longer for the case when the two fluids are of very nearly equal density than that of significant different densities. This is due to the fact that the presence of the upper fluid has the effect of decreasing the velocity of propagation of the wave which consequently causes the decrease of the potential energy of a given deformation of the interface as well as the increase of the inertia. This result comes in good agreement with Lamb [6], who gave a marvelous natural example for such a phenomena, occurring near the mouths of some of the Norwegian fiord, when there is a layer of fresh water over salt water.

The interfacial wave profiles, $f(x, t)$, has been studied for different values of the thickness ratio, H , namely $H = 0.3, 0.5,$ and 0.6 while the other parameters $R, L,$ and t are fixed. It is clear that as H increases, there is an increase in the amplitude of the wave along the obstacle interval, as well as an increase in the wave length.

We study the effect of changing the triangle height, L . Three values of L have been considered, namely $L = 0.1, 0.2,$ and 0.25 for fixed values of $R, H,$ and t . For the interfacial wave, as L increases a kind of violent disturbance in the wave profile appears, starting by a sudden increase in the profile, ending by a steep decrease at the beginning of the downstream interval. The behaviour of that solution can be interpreted, following Kakutani [4], as follows: a given smooth waveform will propagate along the characteristic curves, gradually steepen its shape due to nonlinear interactions, and then the dispersive term will begin to play its role to balance this steeping.

Appendix 1

$$A_1 = W_1 \left(-4 + 6k^2 + \frac{1}{16}k^4 \right) \quad A_2 = W_1(2k^4 - 8k^2) \quad A_3 = W_1 \left(-\frac{27}{16}k^4 \right)$$

$$A_4 = W_1(4k^2 - 2k^4) \quad A_5 = 4W_1k^4 \quad A_6 = W_1k^4$$

where

$$W_1 = [\gamma^2 - H(1 - R)] \left(-\frac{3Y_1 k^2 \delta^3}{k^2 + 1} \right)$$

and

$$A_7 = W_2 \left(2 - 2k^2 - \frac{9}{32}k^4 \right) \quad A_8 = W_2 \left(2k^2 - \frac{1}{4}k^4 \right) \quad A_9 = \frac{9}{32}W_2 k^4$$

$$A_{10} = \frac{1}{4}W_2 k^4 \quad A_{11} = W_2 \left(2k^2 - \frac{1}{2}k^4 \right) \quad A_{12} = W_2 k^4 \quad A_{13} = -\frac{1}{2}W_2 k^4$$

where

$$W_2 = [\gamma^2 - 3H(1 - R)] \left(-\frac{3Y_1 k^2 \delta^2}{k^2 + 1} \right)$$

and

$$A_{14} = H(1 - R)(b - 1) \quad A_{15} = H + 1 - b \quad A_{16} = \frac{1}{4\gamma\delta}(2A_1 - A_4 - A_6)$$

$$A_{17} = \frac{1}{2\gamma}\delta A_{12} \quad A_{18} = \frac{1}{2\gamma}(A_{11} - A_{13}) \quad A_{19} = \frac{1}{16\gamma\delta}(4A_2 - A_5)$$

$$A_{20} = \frac{1}{4\gamma}A_{12} \quad A_{21} = \frac{1}{6\gamma\delta}A_3 \quad A_{22} = \frac{1}{4\gamma\delta}(A_{13} - 2A_7 - A_{11})$$

$$A_{23} = -\frac{1}{2\gamma}(A_4 + A_6) \quad A_{24} = -\frac{1}{2\gamma}\delta A_{13} \quad A_{25} = -\frac{1}{16\gamma\delta}(4A_8 + A_{12})$$

$$A_{26} = -\frac{1}{4\gamma}A_5 \quad A_{27} = -\frac{1}{6\gamma\delta}A_9 \quad A_{28} = \frac{1}{A_{15} - 2\gamma^2} \quad A_{29} = -\frac{2}{\gamma}A_{14}A_{28}$$

$$A_{30} = -2A_{28} \quad A_{31} = A_{15}A_{28} \quad A_{32} = -\frac{1}{\gamma}A_{15}A_{28} \quad A_{33} = \gamma A_{15}A_{28}$$

$$A_{34} = \frac{1}{2\gamma}A_{14}A_{28} \quad A_{35} = 3\gamma A_{28} \quad A_{36} = \frac{1}{2\gamma}A_{15}A_{28} \quad A_{37} = \frac{1}{2\gamma}A_{28}$$

$$A_{38} = (\gamma^2 A_{15} + 2A_{14})A_{28} \quad A_{39} = \frac{1}{2\gamma}(6A_{14} + \gamma^2 A_{15})A_{28} \quad A_{40} = 2A_{15}A_{28}$$

$$A_{41} = \frac{1}{4\gamma\delta}(2A_1 - A_4 - A_6 + 2\delta[A_{11} - A_{13}]) \quad A_{42} = -\delta A_6 \quad A_{43} = \frac{1}{2}\gamma\delta A_6$$

$$A_{44} = \frac{1}{2}(A_{13} - A_{11}) \quad A_{45} = \frac{1}{16\gamma\delta}(4A_2 - A_5 + 4\delta A_{12}) \quad A_{46} = -\frac{1}{4}A_{12}$$

$$A_{47} = -\frac{1}{2\gamma}(A_4 + A_6 + \delta A_{13}) \quad A_{48} = \frac{1}{2}(A_4 + A_6 + 2\delta A_{13}) \quad A_{49} = -\frac{1}{2}\gamma\delta A_{13}$$

$$\begin{aligned}
A_{50} &= \frac{1}{4}A_5 & A_{51} &= 3A_{38}^2A_{35} & A_{52} &= 3A_{38}^2A_{40} & A_{53} &= 6A_{38}(A_{30}A_{39} + A_{32}A_{40}) \\
A_{54} &= 6A_{38}A_{30}A_{35} & A_{55} &= 6A_{38}(A_{30}A_{40} + A_{32}A_{35}) & A_{56} &= -3A_{35}^2A_{38} \\
A_{57} &= -3A_{38}(A_{40}^2 + 2A_{39}A_{35}) & A_{58} &= 2A_{35}A_{40}A_{38} \\
A_{59} &= A_{29} + 2A_{31}A_{38} - 2A_{39}A_{40} + 3A_{38}^2A_{32} + A_{57} & A_{60} &= A_{30} - A_{35}^2 \\
A_{61} &= A_{31} + 2A_{38}A_{32} - A_{40}^2 - 2A_{39}A_{35} + 3A_{38}^2A_{30} + A_{58} \\
A_{62} &= A_{32} + 2A_{38}A_{30} - 2A_{35}A_{40} + A_{56} \\
A_{63} &= 2(A_{38}A_{36} + A_{32}A_{40} + A_{32}A_{39}) - A_{40} - 6A_{39}A_{35}A_{40} + A_{53} \\
A_{64} &= A_{36} + 2(A_{31}A_{35} + A_{32}A_{40}) - 3A_{35}(A_{35}A_{39} + A_{40}^2) + A_{55} \\
A_{65} &= 2(A_{30}A_{40} + A_{32}A_{35}) + A_{54} - 3A_{35}^2A_{40} & A_{66} &= A_{39} + 2A_{38}A_{40} + A_{51} \\
A_{67} &= A_{40} + 2A_{38}A_{35} & A_{68} &= 2A_{38}A_{39} + A_{52} \\
A_{69} &= A_{25} + \frac{1}{16\delta^2}(2A_{61}A_{26} + A_{62}A_{50}) & A_{70} &= A_{50} + 2A_{38}A_{26} \\
A_{71} &= \frac{1}{2}A_{38}(A_{50} + 2A_{38}A_{26}) & A_{72} &= \frac{1}{4\delta}(A_{35}A_{50} + 2A_{26}A_{67}) \\
A_{73} &= \frac{1}{4\delta}(2A_{66}A_{26} + A_{67}A_{50}) & A_{74} &= A_{22} + \frac{1}{2\delta^2}(A_{60}A_{49} + A_{61}A_{47} + 2A_{62}A_{48}) \\
A_{75} &= A_{48} + 2A_{38}A_{47} & A_{76} &= A_{49} + \frac{1}{2}A_{38}(A_{48} + 2A_{38}A_{47}) \\
A_{77} &= \frac{1}{2\delta}(A_{35}A_{48} + 2A_{67}A_{47}) & A_{78} &= \frac{1}{\delta}(A_{35}A_{49} + 2A_{67}A_{48} + A_{66}A_{47}) \\
A_{79} &= \frac{1}{4\delta}(A_{35}A_{46} + A_{67}A_{45}) & A_{80} &= A_{41} + \frac{1}{2\delta^2}(A_{60}A_{43} + 3A_{61}A_{17} + A_{62}A_{42}) \\
A_{81} &= A_{42} + 3A_{38}A_{17} & A_{82} &= A_{43} + A_{38}A_{42} + 3A_{38}^2A_{17} \\
A_{83} &= A_{44} + A_{38}A_{41} + \frac{1}{2\delta^2}(A_{61}A_{42} + A_{62}A_{43}) \\
A_{84} &= \frac{1}{3}(A_{38}A_{43} + A_{38}^2A_{42}) + A_{38}^3A_{17} \\
A_{85} &= \frac{1}{4\delta^3}(3A_{63}A_{17} + A_{64}A_{42} + A_{65}A_{43}) + \frac{1}{2\delta}(A_{35}A_{44} + A_{67}A_{41}) \\
A_{86} &= \frac{1}{\delta}(3A_{66}A_{17} + A_{35}A_{43} + A_{67}A_{42}) & A_{87} &= \frac{1}{2\delta}(A_{66}A_{42} + A_{67}A_{43} + 3A_{68}A_{17}) \\
A_{88} &= \frac{1}{2\delta}(A_{35}A_{42} + 3A_{67}A_{17})
\end{aligned}$$

Appendix 2

$$\begin{aligned}
r_1 &= -\frac{1}{6}A_{10} & r_2 &= \frac{1}{8\delta^3}A_{37}[2\gamma\delta(A_{74} - A_{85}) - A_{78} - A_{83}] \\
r_3 &= \frac{1}{8\delta^3}A_{37}[2\gamma\delta(A_{47} - A_{88}) - A_{81}] & r_4 &= \frac{1}{4\delta^3}A_{37}[\gamma\delta(A_{75} - A_{86}) - A_{82}] \\
r_5 &= \frac{1}{8\delta^3}A_{37}[2\gamma\delta(A_{76} - A_{87}) - 3A_{84}] \\
r_6 &= \frac{1}{64\delta^3}A_{37}[4\gamma\delta(A_{69} - A_{79}) - A_{73} - A_{46} - A_{38}A_{45}] & r_7 &= \frac{1}{16\delta^2}\gamma A_{37}A_{26} \\
r_8 &= \frac{1}{16\delta^2}\gamma A_{37}A_{70} & r_9 &= \frac{1}{16\delta^2}\gamma A_{37}A_{71} \\
r_{10} &= \frac{1}{216\delta^3}A_{37}[\gamma(6\delta A_{27} - A_{67}) - A_{38}A_{21}] & r_{11} &= \frac{1}{4\delta^2}A_{37}A_{80} \\
r_{12} &= \frac{1}{8\delta^3}A_{37}(2\gamma\delta A_{77} + A_{75} - A_{86}) & r_{13} &= \frac{1}{4\delta^3}A_{37}[\gamma\delta(A_{78} + A_{83}) + A_{76} - A_{87}] \\
r_{14} &= \frac{1}{4\delta^2}\gamma A_{37}A_{88} & r_{15} &= \frac{1}{4\delta^2}\gamma A_{37}A_{81} & r_{16} &= \frac{1}{4\delta^2}\gamma A_{37}A_{82} \\
r_{17} &= \frac{1}{4\delta^2}\gamma A_{37}A_{84} & r_{18} &= \frac{1}{64\delta^3}A_{37}[A_{70} + 4\gamma\delta(A_{72} + A_{45})] \\
r_{19} &= \frac{1}{32\delta^3}A_{37}[A_{71} + 2\gamma\delta(A_{73} + A_{46} + A_{38}A_{45})] & r_{20} &= \frac{1}{36\delta^2}\gamma A_{37}A_{21} \\
r_{21} &= \frac{1}{36\delta^2}\gamma A_{37}A_{38}A_{21} & r_{22} &= \frac{1}{36\delta^2}\gamma A_{38}A_{21}A_{43}
\end{aligned}$$

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