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On Integrability of a (2+1)-Dimensional Perturbed KdV Equation

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Abstract

A (2+1)-dimensional perturbed KdV equation, recently introduced by W.X. Ma and B. Fuchssteiner, is proven to pass the Painlevé test for integrability well, and its 4×4 Lax pair with two spectral parameters is found. The results show that the Painlevé classification of coupled KdV equations by A. Karasu should be revised.

Recently, Ma and Fuchssteiner [1] applied a new scheme of perturbation to the KdV equation and derived the following (2+1)-dimensional system:

$$u_t = u_{xxx} + 6uu_x, \tag{1}$$

$$v_t = v_{xxx} + 6\,(uv)_x + 3\left(u_{xx} + u^2\right)_y . \tag{2}$$

They also raised the question of its integrability.

In the present letter, we verify the integrability of the system (1)–(2) by means of the Painlevé test, show that this system possesses a 4×4 Lax pair with two spectral parameters, and then discuss our results in relation with one recent Painlevé classification of coupled KdV equations [9].

The singularity analysis may be applied in studying the integrability of nonlinear differential equations [2]. Let us apply the Weiss-Kruskal algorithm of singularity analysis [2]–[4] to the system (1)–(2). This system is a normal system [5] with non-characteristic hypersurfaces $\varphi\,(x,y,t) = 0$ satisfying the condition $\varphi_x \neq 0$ (we take $\varphi_x = 1$), and its general solution should contain six arbitrary functions of two variables each. Substituting $u = u_0\,(y,t)\,\varphi^\sigma + \cdots + u_r\,(y,t)\,\varphi^{\sigma+r} + \cdots$ and $v = v_0\,(y,t)\,\varphi^\tau + \cdots + v_r\,(y,t)\,\varphi^{\tau+r} + \cdots$ into (1)–(2), we find the admissible exponents σ and τ of the dominant behavior of solutions and the corresponding positions r of resonances. There is only one branch to be analyzed: $\sigma = -2$, $\tau = -3$, $u_0 = -2$, $\forall\ v_0\,(y,t)$, $r = \underline{-1}, 0, \underline{4}, 5, \underline{6}, 7$ (underlined are the positions of resonances of the KdV equation (1); all other branches are either special cases of this branch or Taylor expansions governed by the Cauchy-Kovalevskaya theorem). Then we substitute $u = \sum_{i=0}^{\infty} u_i(y,t)\varphi^{i-2}$ and $v = \sum_{i=0}^{\infty} v_i(y,t)\varphi^{i-3}$ into (1)–(2), find the recursion relations for u_i and v_i, and check the compatibility conditions at the resonances. All the compatibility conditions turn out to be satisfied *identically*. Thus, the analyzed system has passed the Painlevé test well, and we should expect its integrability.

Next we could see that the procedure of truncating singular expansions [6] turns out to be compatible for the system (1)–(2). Unfortunately, the explicit expressions for truncated expansions are too bulky, and the way they lead to the Lax pair is too artificial. We will show an easier way to the same Lax pair. Here we only have to note that *two* "spectral" parameters, α and β, appear in the truncated singular expansions:

$$\alpha = \alpha(y) : \forall\, \alpha, \qquad \beta = \beta(x,y) : \beta_x = -\alpha_y. \tag{3}$$

Let us remind that the system (1)–(2) is generated by the KdV equation (1) under the following perturbation (see [1] for details):

$$(x,t) \to (x,y,t)\,, \qquad y = \varepsilon x, \qquad u \to u + \varepsilon v, \qquad o\left(\varepsilon^2\right). \tag{4}$$

The Lax pair for the KdV equation (1) is well known [7]:

$$\Phi_x + A\Phi = 0, \qquad \Phi_t + B\Phi = 0, \qquad \Phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \tag{5}$$

$$A = \begin{pmatrix} 0 & u+\alpha \\ -1 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} -u_x & u_{xx} + 2u^2 - 2\alpha u - 4\alpha^2 \\ -2u + 4\alpha & u_x \end{pmatrix}, \tag{6}$$

$\alpha_x = \alpha_t = 0$. Let us see how the perturbation (4) acts on the Lax pair (5):

$$\Phi(x,t) \to \Phi\,(x,y,t) + \varepsilon\Psi\,(x,y,t)\,, \qquad \Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix},$$

$$\Phi_x \to \Phi_x + \varepsilon\left(\Phi_y + \Psi_x\right),$$

$$A \to A + \varepsilon M, \qquad \dim M = 2\times 2,$$

$$\Phi_x + A\Phi = 0 \to \begin{cases} \Phi_x + A\Phi = 0, \\ \Psi_x + \Phi_y + A\Psi + M\Phi = 0, \end{cases} \tag{7}$$

$$\Phi_t \to \Phi_t + \varepsilon\Psi_t,$$

$$B \to B + \varepsilon N, \qquad \dim N = 2\times 2,$$

$$\Phi_t + B\Phi = 0 \to \begin{cases} \Phi_t + B\Phi = 0, \\ \Psi_t + B\Psi + N\Phi = 0. \end{cases} \tag{8}$$

We can rewrite the linear systems (7) and (8) in the block form as follows:

$$\begin{pmatrix} \Phi \\ \Psi \end{pmatrix}_x + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \Phi \\ \Psi \end{pmatrix}_y + \begin{pmatrix} A & 0 \\ M & A \end{pmatrix}\begin{pmatrix} \Phi \\ \Psi \end{pmatrix} = 0, \tag{9}$$

$$\begin{pmatrix} \Phi \\ \Psi \end{pmatrix}_t + \begin{pmatrix} B & 0 \\ N & B \end{pmatrix} \begin{pmatrix} \Phi \\ \Psi \end{pmatrix} = 0. \tag{10}$$

Now we can find the explicit form of M and N, applying the perturbations (4) and $\alpha \to \alpha + \varepsilon\beta$ to A and B (6):

$$M = \begin{pmatrix} 0 & v+\beta \\ 0 & 0 \end{pmatrix}, \tag{11}$$

$$N = \begin{pmatrix} -u_y - v_x & v_{xx} + 2u_{xy} + 4uv - 2\alpha v - 2\beta u - 8\alpha\beta \\ -2v + 4\beta & u_y + v_x \end{pmatrix}. \tag{12}$$

Question: Should consider α and β as constants? Answer: No, we have to take the original conditions $\alpha_x = 0$ and $\alpha_t = 0$ and apply the perturbation (4) to them, assuming that $\alpha(x,t) \to \alpha(x,y,t) + \varepsilon\beta(x,y,t)$. In this way, we find that $\alpha_t = 0 \to \alpha_t = \beta_t = 0$ and $\alpha_x = 0 \to \{\alpha_x = 0, \beta_x = -\alpha_y\}$, i.e. we obtain exactly the conditions (3) for the two spectral parameters α and β. Moreover, we can check directly that the system (1)–(2) follows (as the compatibility condition) from the linear systems (9) and (10) if, and only if, α and β satisfy the conditions (3). Thus, the system (1)–(2) possesses the two-parameter 4×4 Lax pair

$$\Omega_x + S\Omega_y + P\Omega = 0, \qquad \Omega_t + Q\Omega = 0 \tag{13}$$

with the notations evident from (9), (10), (6), (11), (12) and (3). It remains a question whether some other remarkable objects of the system (1)–(2), such as solitons, conservation laws, as well as Miura and Bäcklund transformations, can also be derived *simply by applying the perturbation* (4) to the corresponding objects of the KdV equation (1).

The system (1)–(2) admits two interesting (1+1)-dimensional reductions. If we make $(x,y,t) \to (z,t)$ and choose $z = x$, we get from (1)–(2) the well-known integrable perturbed KdV equation [1, 8]

$$u_t = u_{zzz} + 6uu_z, \qquad v_t = v_{zzz} + 6(uv)_z. \tag{14}$$

Choosing $z = x + y$ and introducing $w = u + v$ for simplicity, we get the new system

$$u_t = u_{zzz} + 6uu_z, \qquad w_t = w_{zzz} + 6(uw)_z + 3u_{zzz}. \tag{15}$$

The Lax pair of (15) follows directly from (13) by the reduction. Moreover, since the systems (14) and (15) came from a system possessing the Painlevé property by reductions to its non-characteristic hypersurfaces, they possess the Painlevé property automatically. This is essential in relation with one recent Painlevé classification of coupled KdV equations [9]: neither (14) nor (15) appeared there as systems that had passed the Painlevé test for integrability. Consequently, the classification [9] should be revised. This work is in progress.

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