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Planning Accelerated Life Tests for Burr Type X Failure Model with Type I Censoring

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Abstract

In this paper, we discuss the optimal accelerated life test plans for Burr type X distribution with log-linear model under periodic inspection and Type I censoring. We obtain the maximum likelihood estimators, the Fisher information and the asymptotic covariance matrix of the maximum likelihood estimators. Accelerated life test is optimized with respect to the low test stress and the proportion of test units allocated to the low test stress for given shape parameter. The asymptotic variance of the maximum likelihood estimator of qth quantile at the design stress is derived as an optimality criterion with equally spaced inspection times and the optimal allocation of units for two stress levels are determined. Optimality results show that the asymptotic variance of *q*th quantile at the design stress is insensitive to the number of inspection times and to misspecifications of guessed failure probabilities at design and high test stresses. Procedures for planning an accelerated life test, including selection of sample size, have been discussed through an example.

Keywords: Accelerated life testing, Burr type X distribution, Maximum likelihood estimation, Type I censoring, Optimization techniques.

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1. Introduction

There are many engineering situations where components in a system are designed to last a relatively long time under design conditions. These long life spans make it impractical to conduct life tests under design conditions. Accelerated life tests (ALTs) are used to estimate the lifetime of such highly reliable products within a reasonable testing time. In an ALT experiment, test units are run at higher than usual levels of stress to induce early failures. The test data obtained at the accelerated conditions are analyzed in terms of a model, and then extrapolated to the design stress to estimate the life distribution. Such ALTs have proven to be useful in many phases of product design and manufacture, from prototype testing to post production screening. Meeker and Escobar (1998) devoted three complete chapters to this topic and proper implementation of such tests. Nelson (1990) provided an extensive coverage of statistical models, test plans, and analytical procedures employed in accelerated life testing. For more details on research and issues in ALTs, see Meeker and Escobar (1993).

There are two inspection modes applied in ALT. One is the continuous inspection that results in exact failure times (see Chernoff (1962), Meeker (1984), Meeker and Nelson (1975), Nelson (1990), Nelson and Meeker (1978)). The other is the periodic inspection in which test units are inspected for failure at predetermined points in time. The periodic inspection is frequently employed by many authors (see Ahmad (2010), Ahmad and Islam (1996), Ahmad et al. (1994, 2006), Ehrenfeld (1962), Islam and Ahmad (1994), Meeker (1986), Yum and Choi (1989), Seo and Yum (1991)) because it requires less testing effort and is administratively more convenient. For more useful and up-to-date results in ALTs, see Nelson (2005), Yang (2007).

This paper considers planning ALT for items whose lifetime follows Burr type X failure model. Burr (1942) introduced twelve different form of cumulative distribution function for modeling lifetime data. Several authors consider different aspects of the Burr type X distribution (see Ahmad, et al. (1997), Jaheen (1996), Sartwi and Abu-Salih (1991), Raqab (1998), Surles and Padgett (1998, 2005), Raqab and Kundu (2005), Kundu and Raqab (2005)). Recently, Surles and Padgett (2001) and Ahmad, et al. (2009) showed that the Burr type X can be used quite effectively in reliability and survival analysis.

In this paper we develop ALT plans for the Burr type X distribution under Type I censoring and periodic inspection at two test stress levels. It is assumed that a log-linear model exists between the Burr type X scale parameter and the stresses and that the shape parameter is constant and is independent of the stresses. The unknown parameters in the log-linear model are estimated by maximum likelihood estimation (MLE) method. For the known shape parameter, the low test stress and associated proportion of test units are optimally determined at design stress. The optimal test plans are derived by minimizing the asymptotic variance of the maximum likelihood estimator of qth quantile at design stress. This paper is a generalization of Ahmad et al. (1994) work. It may also be viewed as an extension of Surles and Padgett (2001) work to the case of ALT with periodic inspection. A self-developed software program has been used to carry out the computations. The proposed method is limited to tests with only two acceleration levels, but we show it can be extended to multiple acceleration level situations.

Computational studies are conducted for various combinations of parameters to examine how the optimal plans vary with respect to these parameters at design and high-test stresses. Sensitivity analyses have also been performed for various combinations of parameters to assess the effect of inaccuracy to misspecification of guessed failure probabilities on the optimal plan at design and high-test stresses. Procedures for selecting a sample size and for planning an ALT are discussed with an example.

2. The Proposed Model and Test Method

A statistical model for an ALT consists of a life distribution that represents the scatter in product life and a relationship between distribution parameter and accelerating stress. Most previous work on optimal design of ALT assumes that the life distribution is either exponential or Weibull (see Ahmad (2010), Ahmad et al. (1994), Chernoff (1953, 1962), Ehrenfeld (1962), Islam and Ahmad (1994), Meeker (1984), Meeker and Escobar (1998), Meeker and Nelson (1975), Nelson (1990), Nelson and Meeker (1978), Park and Yum (1996), Yum and Choi (1989), Seo and Yum (1991)). The Burr type XII, Burr type III, Normal and log-normal models have also been used (see Ahmad and Islam (1996), Ahmad et al. (2006), Meeker (1984), Nelson and Kielpinski (1976)). We propose the Burr type X lifetime distribution that describes the failure mechanism of test units.

2.1 Assumptions

We make the following assumptions:

- Three test stress levels s_0, s_1, s_2 are used such that $s_0 < s_1 < s_2$, where s_0 is the design stress level representing use condition and s_1 and s_2 are higher than usual stresses representing accelerated conditions.
- The lifetimes (T) of test items at any stress s follow the Burr type X failure model. The probability density function of Burr type X failure model is given by

$$f(t) = 2(\sigma/\theta)(t/\theta)e^{-(t/\theta)^2}(1-e^{-(t/\theta)^2})^{\sigma-1}, \quad t \geq 0, \tag{1}$$

  where $\sigma > 0$ is shape parameter and $\theta > 0$ is a scale parameter.

  The reliability function of T is given by

$$S(t) = 1-(1-e^{-(t/\theta)^2})^{\sigma},$$

  and the failure rate function is given by

$$r(t) = \frac{2(\sigma/\theta)(t/\theta)e^{-(t/\theta)^2}(1-e^{-(t/\theta)^2})^{\sigma-1}}{1-(1-e^{-(t/\theta)^2})^{\sigma}}.$$

  Raqab and Kundu (2005) observed that for $\sigma \leq 1/2$ Burr type X density is a decreasing function and it is a right skewed unimodal function for $\sigma > 1/2$. It also observed that Burr type X failure rates have the different shapes depending on the value of σ. For $\sigma > 1/2$, it has increasing failure rate and for $\sigma \leq 1/2$, it is bathtub type. For more details properties of the Burr type X distribution, see Surles and Padgett (2005).
- The scale parameter θ is assumed to be a log-linear function of stress level s. That is,

$$\ln\theta = \beta_0 + \beta_1 s, \tag{2}$$

  where β_0 and β_1 are unknown parameters to be estimated. The above relationship is frequently used in ALT. This model includes the inverse power model and the Arrhenius relation rate model see, Islam and Ahmad (1994), Nelson (1990).

- The shape parameter σ is independent of stresses (constant for any stress).
- The lifetimes of test units at stress level s_i are independent and identically distributed.

2.2 Test Procedure

- The design stress (s_0) and high-test stress (s_2) are pre-specified, while the test stress (s_1) is to be optimally determined.
- Out of total N test items, the test items (n_i) allocated to s_i is given by

$$\alpha_i = n_i/N,\ \alpha_1 + \alpha_2 = 1 \text{ and } N = n_1 + n_2; \qquad \alpha_i > 0,\ i = 1, 2, \tag{3}$$

  where α_1 is to be optimally determined.
- At stress level s_i, the test items (n_i) are initially to be put on test and run until a pre-specified censoring time t_{ci}, see Fig. 1.
- The items are inspected periodically: at stress level s_i, at times $t_{i1}, t_{i2}, \ldots, t_{iK(i)}$, where $t_{i0} \equiv 0$, $t_{i,K(i)+1} \equiv \infty$ and $t_{i,K(i)}$ is the Type I censoring time, denoted by t_{ci}, see Fig. 1.
- At stress level s_i, the number of failures x_{ij} and corresponding probability of failures P_{ij} in the respective intervals ($t_{i,j-1}, t_{ij}$) are recorded for $i = 1, 2$ and $j = 1, 2, \ldots, K(i)+1$, see Fig. 1.
- The grouped data $\{x_{ij},\ i = 1, 2;\ j = 1, 2, \ldots, K(i)+1\}$ are used to estimate β_0 and β_1 in (2). The estimated relationship is then extrapolated to estimate mean lifetime or qth quantile at the design condition. At design condition s_0, (2) can be written as

$$\mu_0 = \ln\theta_0 = \beta_0 + \beta_1 s_0. \tag{4}$$

Note that the qth quantile (t_q) of the lifetime distribution at design stress s_0 and μ_0 are related as

$$y_q = \ln t_q = \beta_0 + \beta_1 s_0 + \frac{1}{2}\ln[-\ln(1-q^{1/\sigma})]. \tag{5}$$

Let $\hat{\beta}_0$ and $\hat{\beta}_1$ be the ML estimates of β_0 and β_1, respectively. Then,

$$\hat{\mu}_0 = \hat{\beta}_0 + \hat{\beta}_1 s_0, \tag{6}$$

and
$$\hat{y}_q = \hat{\mu}_0 + \frac{1}{2}\ln\{-\ln(1-q^{1/\sigma})\}. \tag{7}$$

2.3 Standardization

Without loss of generality, let with and without prime represent the original and standardized scale, respectively, then the stress level is standardized as follows (see Meeker (1984)):

$$s = (s' - s'_0)/(s'_2 - s'_0),$$

or equivalently,

$$s' = s(s_2' - s_0') + s_0' ,$$

so that $0 \le s \le 1$, design stress $s_0 = 0$, and high stress $s_2 = 1$. We also standardized all the time related variables with respect to censoring time t_c' (say $t_{c1} = t_{c2} = t_c'$). For instance, $t = t'/t_c'$ and $\theta = \theta'/t_c'$. Under the above standardization, the scale parameter is represented by $\theta = e^{(\beta_0' + \beta_1' s')}/t_c'$. Because $\theta = e^{(\beta_0 + \beta_1 s)}$, we have

$$\beta_0 = \beta_0' + \beta_1' s_0' - \ln t_c' , \; \beta_1 = \beta_1' (s_2' - s_0').$$

Then from equation (7) it can be shown that

$$\hat{y}_q' = \hat{\beta}_0 + \ln t_c' + \frac{1}{2}\ln\{-\ln(1 - q^{1/\sigma})\} .$$

Note that $t_c^{'}$ becomes 1 in the standardized time scale.

$$\hat{y}_q' = \hat{\beta}_0 + \frac{1}{2}\ln\{-\ln(1 - q^{1/\sigma})\} = \hat{y}_q .$$

Hence, no generality is lost under the above transformation.

2.4 Optimization Criterion

Nelson (1990) and Nelson and Kielpinski (1976) describe various criteria for determining optimal ALT plans. A common purpose of an ALT experiment is to estimate a particular quantile t_q in the lower tail of the failure-time distribution at use conditions. Thus our optimality criterion is to minimize $AsVar(\hat{y}_q)$, the asymptotic variance of the MLE of the logarithm of the target quantile at design conditions. q is often chosen to be a small number like .01 or .001.

2.5 Design Problem

The statistically optimal ALT plans under periodic inspection and Type I censoring can now be stated as: given N, s_0, s_2, σ, $\{t_{ci}\}_{i=1}^{2}$, and $\{K(i)\}_{i=1}^{2}$, determine α_1 and s_1 such that the asymptotic variance, $AsVar(\hat{y}_q)$, is minimized using equally spaced inspection times.

The design problem of an ALT under periodic inspection and Type I censoring may be extended to l stress levels and can be stated as: given N, s_0, s_l, σ, $\{t_{ci}\}_{i=1}^{l}$, and $\{K(i)\}_{i=1}^{l}$, determine $\{\alpha_i\}_{i=1}^{l-1}$ and $\{s_i\}_{i=1}^{l-1}$ such that the $AsVar(\hat{y}_q)$ is minimized using equally spaced inspection times.

3. Estimation of Parameters

There are several methods of estimation for censored data, which provide estimates of the parameters of the assumed log-linear model. The MLE method is used for the following reasons (see Ahmad and Islam (1996), Meeker and Nelson (1975), Nelson (1990)).

- It is easier to calculate the optimal plans by this method in comparison to linear estimation methods.
- This method provides asymptotically minimum variance estimates for large sample sizes. Also, for small sample sizes, ML estimates generally compare well with other estimates.

3.1 Maximum Likelihood Estimation

The likelihood function of the set of observations $\{x_{ij}\}_{j=1}^{K(i)+1}$ which are multinomially distributed with n_i and $\{P_{ij}\}_{j=1}^{K(i)+1}$ at stress level s_i, is given by

$$L' = \prod_{i=1}^{2} L_i' = \prod_{i=1}^{2} n_i! \left(\prod_{j=1}^{K(i)+1} x_{ij}! \right)^{-1} \left(\prod_{j=1}^{K(i)+1} P_{ij}^{x_{ij}} \right). \tag{8}$$

Taking logarithm of both the sides, we get

$$L = \ln L' = \sum_{i=1}^{2} \ln L_i' = C + \sum_{i=1}^{2} \sum_{j=1}^{K(i)+1} x_{ij} \ln P_{ij} , \tag{9}$$

where C is constant with respect to β_0 and β_1 and

$$P_{ij} = \left[1 - e^{-(t_{ij}/\theta_i)^2} \right]^{\sigma} - \left[1 - e^{-(t_{i,j-1}/\theta_i)^2} \right]^{\sigma}, \text{ for } i = 1, 2 \text{ and } j = 1, 2, \ldots, K(i)+1. \tag{10}$$

The ML estimates of β_0 and β_1 can be obtained by solving the following equations:

$$\frac{\partial L}{\partial \beta_0} = 0 \text{ and } \frac{\partial L}{\partial \beta_1} = 0 .$$

The above equations can be rewritten as

$$\frac{\partial L}{\partial \beta_0} = \sum_{i=1}^{2} \sum_{j=1}^{K(i)+1} \{x_{ij}(A_{i,j-1} - A_{ij})/P_{ij}\} = 0, \tag{11}$$

$$\frac{\partial L}{\partial \beta_1} = \sum_{i=1}^{2} s_i \sum_{j=1}^{K(i)+1} \{x_{ij}(A_{i,j-1} - A_{ij})/P_{ij}\} = 0, \tag{12}$$

where $A_{ij} = 2\sigma(t_{ij}/\theta_i)^2 e^{-(t_{ij}/\theta_i)^2}(1-e^{-(t_{ij}/\theta_i)^2})^{\sigma-1}$, for i = 1, 2 and j = 0, 1, 2, …, $K(i)$+1.

3.2 Fisher Information Matrix

The Fisher information matrix F (see Nelson (1990), Rao (1973)) for the optimal plan is

$$F = \begin{bmatrix} \sum_{i=1}^{2} E\left(-\frac{\partial^2 \ln L_i'}{\partial \beta_0^2}\right) & \sum_{i=1}^{2} E\left(-\frac{\partial^2 \ln L_i'}{\partial \beta_0 \partial \beta_1}\right) \\ \sum_{i=1}^{2} E\left(-\frac{\partial^2 \ln L_i'}{\partial \beta_1 \partial \beta_0}\right) & \sum_{i=1}^{2} E\left(-\frac{\partial^2 \ln L_i'}{\partial \beta_1^2}\right) \end{bmatrix} = N\,(f_{gh}), \quad g, h, = 0, 1, \tag{13}$$

where
$$f_{gh} = \sum_{i=1}^{2} \alpha_i \sum_{j=1}^{K(i)+1} \left(\frac{\partial P_{ij}}{\partial \beta_g}\right)\left(\frac{\partial P_{ij}}{\partial \beta_h}\right) \Big/ P_{ij}\,. \tag{14}$$

After some algebraic simplification, (14) becomes

$$f_{gh} = \sum_{i=1}^{2} \alpha_i s_i^{(g+h)} Q_i \qquad \text{, for } g, h = 0, 1, \tag{15}$$

where
$$Q_i = \sum_{j=1}^{K(i)+1} (A_{i,j-1} - A_{ij})^2 / P_{ij} \quad , \quad \text{for } i = 1, 2. \tag{16}$$

Note that
$$\frac{\partial P_{ij}}{\partial \beta_1} = s_i \frac{\partial P_{ij}}{\partial \beta_0}\,. \tag{17}$$

3.3 Asymptotic Variances of MLEs

The asymptotic covariance matrix V of the ML estimates $\hat{\beta}_0$ and $\hat{\beta}_1$ is the inverse of the Fisher information matrix F:

$$V = F^{-1} = \begin{bmatrix} AsVar(\hat{\beta}_0) & AsCov(\hat{\beta}_0, \hat{\beta}_1) \\ AsCov(\hat{\beta}_1, \hat{\beta}_0) & AsVar(\hat{\beta}_1) \end{bmatrix} = \frac{1}{N}(f_{gh})^{-1}, \; g, h = 0, 1. \tag{18}$$

The asymptotic variance (*AsVar*) of the MLE of $\hat{y}_q$ is

$$AsVar(\hat{y}_q) = (1, s_0)\ V\ (1, s_0)'$$

$$= N^{-1}\left(f_{00} f_{11} - f_{01}^2\right)^{-1}\left(f_{11} + s_0^2 f_{00} - 2s_0 f_{01}\right). \tag{19}$$

4. Optimal Plans

4.1 Optimization Method

The optimal plans are determined with the following simplified assumptions and standardization:

- Censoring times at s_1 and s_2 are the same, that is, $t_{c1} = t_{c2} = t_c$.
- The number of inspections at each stress level is the same, that is, $K(1) = K(2) = K$ (known).
- Parameters are standardized such that the common censoring time, as well as the high test stress becomes 1, and the design stress is 0. That is, $t_c = s_2 = 1$, and $s_0 = 0$. Such standardization does not alter the nature of our problem.

Based upon the above assumptions and standardization, (19) is reduced to

$$AsVar(\hat{y}_q) = AsVar(\hat{\beta}_0)$$

$$= N^{-1}\left(f_{00} f_{11} - f_{01}^2\right)^{-1} f_{11}. \tag{20}$$

After some algebraic simplification (20) becomes

$$AsVar(\hat{y}_q) = N^{-1}(s_1^2 Q_1\alpha_1 + Q_2(1-\alpha_1))/(Q_1 Q_2 (s_1 - 1)^2(-\alpha_1^2 + \alpha_1)), \tag{21}$$

or

$$AsVar(\hat{y}_q) = \frac{s_2^2}{\alpha_1 N Q_1 (s_1 - s_2)^2} + \frac{s_1^2}{\alpha_2 N Q_2 (s_1 - s_2)^2}. \tag{22}$$

Under the above assumptions, the optimal plan is developed by determining optimal values of s_1 and α_1, say s_1^* and α_1^* respectively, for given N, K, and σ by minimizing $AsVar(\hat{y}_q)$. This asymptotic variance depends on the unknown quantities β_0 and β_1, which are expressible in terms of the more approachable parameters P_u and P_h, where

$$P_u = P_{s_0}(T < t_c),$$

the probability that an item under design stress would fail before the censoring time and

$$P_h = P_{s_2}(T < t_c),$$

the probability that an item under high test stress would fail before the censoring time. From expert judgment the user supplies P_u and P_h, which determines β_0 and β_1. That is

$$\beta_0 = \frac{1}{2} \ln(-1/\ln(1-P_u^{1/\sigma})), \tag{23}$$

and

$$\beta_1 = \frac{1}{2} \ln(\ln(1-P_u^{1/\sigma})/\ln(1-P_h^{1/\sigma})). \tag{24}$$

For given values of K, P_u, P_h, and σ, optimal values of s_1 and α_1 are determined by the following two-step procedure that minimizes $AsVar(\hat{y}_q)$.

- We obtain the optimum values of $\alpha_i\ (i=1,2)$ by formulating the problem as the following nonlinear programming problem (NLPP):

$$\left.\begin{array}{ll} \text{Minimize} & Z = \sum_{i=1}^{2} \frac{A_i}{\alpha_i} \\ \text{subject to} & \sum_{i=1}^{2} \alpha_i = 1, \\ \text{and} & \alpha_i > 0, i = 1,2 \end{array}\right\} \tag{25}$$

where $A_1 = \dfrac{s_2^2}{NQ_1(s_1-s_2)^2}$ and $A_2 = \dfrac{s_1^2}{NQ_2(s_1-s_2)^2}$.

The restrictions $\alpha_i > 0$ are obvious because negative values of α_i are of no practical use.

Ignoring the restrictions $\alpha_i > 0$, we can use Lagrange multipliers technique to solve NLPP (25) for determining the optimum values of α_i. If these values (say) α_i^*, satisfy the ignored restrictions, the NLPP (25) is solved completely.

The Lagrangian function φ is defined as

$$\varphi(\alpha_i, \lambda) = \sum_{i=1}^{2} \frac{A_i}{\alpha_i} + \lambda\left(\sum_{i=1}^{2} \alpha_i - 1\right), \tag{26}$$

where λ is a Lagrange multiplier.

The necessary conditions for the solution of the problem are

$$\frac{\partial L}{\partial \alpha_i} = -\frac{A_i}{\alpha_i^2} + \lambda = 0\,;$$

which gives

$$\alpha_i = \sqrt{\frac{A_i}{\lambda}}\,; \qquad i = 1, 2 \tag{27}$$

and

$$\frac{\partial L}{\partial \lambda} = \sum_{i=1}^{2} \alpha_i = 1\,. \tag{28}$$

Solving (27) and (28) the optimum solution to NLPP (25) is given by

$$\alpha_1^* = \frac{s_2}{\sqrt{\left(s_1^2 Q_1 + s_2^2 Q_2\right)/Q_2}} \text{ and } \alpha_2^* = 1 - \alpha_1^*\,. \tag{29}$$

It can be verified that the objective function Z in (25) is convex and the constraint is linear. Therefore, the Kuhn-Tucker necessary conditions for the NLPP (25), which are also sufficient for the problem, are hold at the point $\left(\alpha_1^*, \alpha_2^*\right)$ given by (29). Hence, $\left(\alpha_1^*, \alpha_2^*\right)$ is optimum for NLPP (25).

- We imply a grid search with respect to s_1. We divide the range of standardized stress into 500 equal parts, and calculate α_1^* and $AsVar(\hat{y}_q)$ at each grid point, 0.000, 0.002, 0.004,…, 0.998. Among these grid points we select the one that yields the smallest value of $AsVar(\hat{y}_q)$.

When K tend to infinity (i.e. continuous inspection) the minimum $AsVar(\hat{y}_q)$ and corresponding s_1^* and α_1^* are determined by using the method described by Nelson and Meeker (1978). It can also be determined by a two-step procedure. Finally, the ratio of $AsVar(\hat{y}_q(K))$ to $AsVar(\hat{y}_q(\infty))$ is determined, A FORTRAN coded algorithm finds (s_1, α_1) that minimizes the $AsVar(\hat{y}_q)$ has been written and was run on the computer.

4.2 Sensitivity Analysis

To use an optimal plan, one must provide approximate values of the model parameters (or equivalently P_u and P_h). Chernoff (1953, 1962) call such a plan "locally optimal". Values that are appreciable in error may result in a plan that is far from optimal. This possibility can be checked if one examines the plans for different parameter values and suggested sensitivity analysis. For some selected values of P_u, P_h, and σ, a sensitivity analysis was conducted to see how misspecification of imputed failure probabilities affect $AsVar(\hat{y}_q)$. Let $\tilde{P}_u$ and $\tilde{P}_h$ be the guessed values of imputed failure probabilities. For these guessed values optimal s_1 (i.e. $\tilde{s}_1^*$) and α_1 (i.e. $\tilde{\alpha}_1^*$) are determined. The ratio of $AsVar(\hat{y}_q(\tilde{s}_1^*, \tilde{\alpha}_1^*))$ to minimum $AsVar(\hat{y}_q(s_1^*, \alpha_1^*))$ is calculated for various cases with $K = 2$.

4.3 Sample size

From (21), we observe that N appears in $AsVar(\hat{y}_q)$ as a scaling factor only; therefore, s_1^* and α_1^* do not depend on N. However, N affects the magnitude of $AsVar(\hat{y}_q)$, implying the need for its selection. To determine the sample size N, one requires that with high probability ϕ, $\hat{\theta}_0$ falls between θ_0/h and $\theta_0 h$ for a specified h (> 1) (see Meeker, (1986)). That is

$$P_r\{\theta_0 / h \le \hat{\theta}_0 \le \theta_0 h\} \ge \phi \,. \tag{30}$$

Note that (30) can be rewritten as

$$P_r\{\mu_0 - \ln h \le \hat{\mu}_0 \le \mu_0 + \ln h\} \ge \phi \,. \tag{31}$$

Then, the approximate sample size is obtained as

$$N^* \cong \frac{\mathrm{v}_0 \omega^2}{(\ln h)^2} \quad , \tag{32}$$

where v_0 is the asymptotic variance of $\hat{\mu}_0$ when $N = 1$ and ω is the $(1+\phi)/2$ quantile of the standard normal distribution.

5. Optimality Results

Optimal ALT results are presented in Tables 1-2 for various combinations of P_u, P_h, K, and σ. Tables 3-4 summarize results of sensitivity analysis for various values of P_u, P_h, and σ with $K = 2$. We have the following observations:

- For given values of P_u, P_h, and σ, $AsVar(\hat{y}_q)$ is almost same over the number of inspection (K). Thus increasing K has little effect on the asymptotic variance. This implies that the number of inspection need not be too large.
- For given value of σ, $AsVar(\hat{y}_q)$ is insensitive to K as P_u increases and/or P_h decreases.
- In all cases, the values of ratio in last column of tables indicate that there is no significant difference in $AsVar(\hat{y}_q)$ for variation in K, because it is much closer to 1. Therefore, unnecessarily large K is not needed, which is an encouraging result in terms of testing efforts.
- When P_u and P_h are fixed, s_1^* and α_1^* are fairly stable over K in all the cases under study.

- For selected values of σ and P_u, as P_h decreases, asymptotic variance of $\hat{\mu}_0$ increases and it become minimum when P_h is 0.99.
- For selected values of σ and P_h, as P_u increases, $AsVar(\hat{y}_q)$ decreases and it become minimum when P_u equals 0.1.
- In general, for given σ, s_1^* gets close to zero (the design stress) and α_1^* to 1, as P_u increases and/or P_h decreases. For instance, when $P_u = 0.1$ and $P_h \leq 0.99$, $s_1^* \cong 0$ and $\alpha_1^* \cong 1$. Similar trends are also observed when P_u is less than 0.1, for small values of P_h. This implies that there is almost no need for an ALT.
- For each P_u and P_h, asymptotic variance of $\hat{\mu}_0$ decreases as σ increases.
- For given σ, ratio values of sensitivity analysis indicate that the plan is robust against the likely departures of the true P_u and P_h from their guessed values for all K.
- The results of optimal plans obtained by Ahmad et al. (1994) becomes the particular case of the result obtained here for $\sigma = 1$ (Table 1).

6. Planning ALTs

We suggest the following procedure for planning an ALT.

- Provide pre-estimates of P_u, P_h, and σ and the ranges of their plausible values.
- Determine an optimal test plan using the pre-estimates.
- For given σ, conduct sensitivity analyses with respect to the plausible values of pre-estimates.
- Check the necessity of an ALT based upon s_1^* and α_1^*.
- Determine the sample size.

Table 1. Optimal ALT Plans when σ = 1.0.

P_u	P_h	β_0	β_1	K	s_1^*	α_1^*	$N\,AsVar(\hat{y}_q)$	*Ratio*
0.0001	0.99	4.605	-5.369	2	0.698	0.752	29.483	1.0744
				5	0.706	0.765	27.808	1.0134
				10	0.706	0.768	27.529	1.0032
				∞	0.706	0.770	27.441	1
	0.9	4.605	-5.022	2	0.708	0.787	41.060	1.0283
				5	0.710	0.794	40.114	1.0047
				10	0.710	0.795	39.373	1.0011
				∞	0.710	0.796	39.928	1
	0.5	4.605	-4.422	2	0.698	0.823	86.461	1.0029
				5	0.698	0.824	86.254	1.0005
				10	0.698	0.824	86.221	1.0001
				∞	0.698	0.824	86.211	1
	0.1	4.605	-3.480	2	0.630	0.849	317.219	1.0001
				5	0.630	0.849	317.203	1.0000
				10	0.630	0.849	317.200	1.0000
				∞	0.630	0.849	317.200	1
	0.01	4.605	-2.305	2	0.444	0.890	1318.717	1.0000
				5	0.444	0.890	1318.716	1.0000
				10	0.444	0.890	1318.716	1.0000
				∞	0.444	0.890	1318.716	1
0.001	0.99	3.454	-4.217	2	0.616	0.774	17.141	1.0676
				5	0.624	0.787	16.252	1.0122
				10	0.626	0.789	16.103	1.0029
				∞	0.626	0.790	16.056	1
	0.9	3.454	-3.871	2	0.620	0.809	23.131	1.0256
				5	0.624	0.814	22.649	1.0042
				10	0.624	0.815	22.577	1.0010
				∞	0.624	0.816	22.555	1
	0.5	3.454	-3.270	2	0.590	0.847	44.774	1.0025
				5	0.592	0.846	44.681	1.0004
				10	0.592	0.846	44.666	1.0001
				∞	0.592	0.846	44.662	1
	0.1	3.454	-2.328	2	0.446	0.888	129.820	1.0000
				5	0.446	0.888	129.815	1.0000
				10	0.446	0.888	129.814	1.0000
				∞	0.446	0.888	129.813	1
0.01	0.99	2.300	-3.064	2	0.472	0.817	8.113	1.0547
				5	0.484	0.826	7.768	1.0099
				10	0.486	0.828	7.710	1.0024
				∞	0.486	0.829	7.692	1
	0.9	2.300	-2.717	2	0.458	0.852	10.293	1.0200
				5	0.464	0.855	10.125	1.0033
				10	0.466	0.855	10.099	1.0008
				∞	0.466	0.855	10.091	1
	0.5	2.300	-2.117	2	0.368	0.898	16.644	1.0017
				5	0.368	0.899	16.621	1.0003
				10	0.368	0.899	16.617	1.0001
				∞	0.368	0.899	16.616	1

Table 2. Optimal ALT Plans when σ = 1.5.

P_u	P_h	β_0	β_1	K	s_1^*	α_1^*	$N\,AsVar(\hat{y}_q)$	*Ratio*
0.0001	0.99	3.070	-3.875	2	0.652	0.742	12.988	1.0889
				5	0.660	0.757	12.134	1.0173
				10	0.662	0.760	11.977	1.0042
				∞	0.662	0.761	11.927	1
	0.9	3.070	-3.564	2	0.668	0.776	18.386	1.0430
				5	0.672	0.785	17.753	1.0071
				10	0.674	0.785	17.658	1.0017
				∞	0.674	0.785	17.628	1
	0.5	3.070	-3.067	2	0.672	0.815	38.683	1.0064
				5	0.674	0.815	38.478	1.0011
				10	0.674	0.816	38.445	1.0003
				∞	0.674	0.816	38.435	1
	0.1	3.070	-2.361	2	0.620	0.845	141.717	1.0003
				5	0.620	0.845	141.680	1.0001
				10	0.620	0.845	141.674	1.0000
				∞	0.620	0.845	141.672	1
	0.01	3.070	-1.546	2	0.442	0.889	588.183	1.0000
				5	0.442	0.889	588.179	1.0000
				10	0.442	0.889	588.178	1.0000
				∞	0.442	0.889	588.178	1
0.001	0.99	2.300	-3.106	2	0.566	0.768	7.782	1.0801
				5	0.576	0.781	7.318	1.0156
				10	0.578	0.784	7.232	1.0038
				∞	0.578	0.785	7.205	1
	0.9	2.300	-2.795	2	0.578	0.799	10.621	1.0385
				5	0.582	0.808	10.293	1.0064
				10	0.584	0.808	10.243	1.0015
				∞	0.584	0.809	10.228	1
	0.5	2.300	-2.297	2	0.562	0.841	20.412	1.0056
				5	0.564	0.841	20.318	1.0010
				10	0.564	0.841	20.303	1.0002
				∞	0.564	0.841	20.298	1
	0.1	2.300	-1.592	2	0.436	0.886	58.627	1.0002
				5	0.436	0.886	58.615	1.0000
				10	0.436	0.886	58.613	1.0000
				∞	0.436	0.886	58.612	1
0.01	0.99	1.523	-2.329	2	0.422	0.816	3.871	1.0640
				5	0.434	0.826	3.684	1.0126
				10	0.436	0.828	3.649	1.0030
				∞	0.438	0.828	3.638	1
	0.9	1.523	-2.018	2	0.414	0.848	4.930	1.0297
				5	0.422	0.853	4.811	1.0050
				10	0.424	0.853	4.793	1.0012
				∞	0.424	0.853	4.787	1
	0.5	1.523	-1.520	2	0.340	0.897	7.841	1.0038
				5	0.340	0.898	7.817	1.0007
				10	0.340	0.898	7.813	1.0002
				∞	0.342	0.897	7.811	1

7. An Application

Suppose that an experimenter is planning to develop an ALT for a certain type of electrical capacitor with the use of temperature (or voltage) as an accelerating stress. The lifetimes of electrical capacitors are known to have a Burr type X failure distribution, and the log mean lifetime at the design stress is of interest. The design stress is characterized by 30 °C (or 20 V). The high stress level of temperature (or voltage) is pre-specified as 120 °C (or 400 V). The test duration (the censoring time t_c) is allowed for 1000 hours at each stress level. The experimenter first guesses the P_u, P_h and σ is 0.0001, 0.90 and 1.5, respectively.

Based upon the above information the optimal plan for $K = 2$ are computed to be (see Table 2):

$$\beta_0 = 3.070,\ \beta_1 = -3.564,\ s_1^* = 0.668,\ \alpha_1^* = 0.776 \text{, and } N.\ AsVar(\hat{y}_q) = 18.386. \tag{33}$$

Now, we want to calculate sample size by taking ϕ and h as 0.9 and 2.0, respectively. Using (32), the required sample size becomes approximately 104. Being conservative, the experimenter might want to determine the sample size for the worst case of this optimal plan where $AsVar(\hat{y}_q)$ = 588.183 for P_h = 0.01 and same P_u, σ, and K. We obtain a conservative sample size 3318.

Next, the experimenter guesses that P_u, P_h and σ are 0.0001, 0.90 and 1, respectively, then the optimal plan for $K = 2$ are computed as (see Table 1):

$$\beta_0 = 4.605,\ \beta_1 = -5.022,\ s_1^* = 0.708,\ \alpha_1^* = 0.787 \text{, and } N.\ AsVar(\hat{y}_q) = 41.060.$$

For the same ϕ and h, an approximate sample size obtained for this optimal plan is 232. Hence, if the shape parameter decreases an experimenter requires larger sample size.

Although the true values of P_u and P_h are different from their guessed values, suppose that the ranges of the plausible values of P_u, P_h, and σ are as follows:

$$0.00001 \le \tilde{P}_u \le 0.05,$$

$$0.006 \le \tilde{P}_h \le 0.999,$$

$$1 \le \sigma \le 2.$$

For the above plausible ranges of pre-estimates, sensitivity analyses are conducted. Tables 3-4 shows that sensitivity ratios are very close to 1, implying that the selected plan in (33) is generally robust against the likely departures of true P_u, P_h, and σ from their guessed values, except for the case where P_u is underestimated and P_h is overestimated. For instance, using the guessed values of $P_u = 0.0001$, $P_h = 0.90$, , and $\sigma = 1.5$ as $\tilde{P}_u = 0.0003$, $\tilde{P}_h = 0.70$, and $\sigma = 1.5$ (see Table 4) then the optimal plan for $K = 2$ has relatively increased in $AsVar(\hat{y}_q)$. The sensitivity is 1.0220 which means that the increase in $AsVar(\hat{y}_q)$ due to the uncertainties involved in estimating P_u, P_h, and σ is 2.20%. Also, the sensitivity value ranges from less than 1% to 15% approximately. In general, this variation may be tolerable.

Table 3. Sensitivities of $AsVar(\hat{y}_q)$ When $\sigma = 1.0$ with $K = 2$

P_u	P_h						
0.0001	0.99	$\tilde{P}_u / \tilde{P}_h$	0.9700	0.9800	0.990	0.995	0.9990
		0.00001	1.1150	1.1014	1.0861	1.0691	1.0519
		0.00003	1.0440	1.0361	1.0248	1.0192	1.0107
		0.00005	1.0208	1.0155	1.0085	1.0056	1.0034
		0.00010	1.0026	1.0011	1	1.0008	1.0069
		0.00020	1.0057	1.0074	1.0107	1.0151	1.0300
		0.00030	1.0200	1.0238	1.0296	1.0364	1.0568
		0.00050	1.0564	1.0588	1.0679	1.0823	1.1055
	0.90	$\tilde{P}_u / \tilde{P}_h$	0.7000	0.8000	0.900	0.950	0.9900
		0.00001	1.1097	1.1007	1.0823	1.0638	1.0446
		0.00003	1.0393	1.0341	1.0243	1.0181	1.0114
		0.00005	1.0169	1.0149	1.0087	1.0058	1.0067
		0.00010	1.0031	1.0014	1	1.0015	1.0126
		0.00020	1.0123	1.0087	1.0103	1.0160	1.0361
		0.00030	1.0324	1.0274	1.0279	1.0360	1.0618
		0.00050	1.0774	1.0658	1.0669	1.0784	1.1073
	0.50	$\tilde{P}_u / \tilde{P}_h$	0.3000	0.400	0.500	0.6000	0.7000
		0.00001	1.0652	1.0794	1.0888	1.0923	1.0948
		0.00003	1.0124	1.0206	1.0279	1.0299	1.0347
		0.00005	1.0017	1.0061	1.0102	1.0133	1.0147
		0.00010	1.0047	1.0008	1	1.0004	1.0015
		0.00020	1.0365	1.0208	1.0114	1.0075	1.0061
		0.00030	1.0737	1.0474	1.0325	1.0233	1.0203
		0.00050	1.1471	1.1036	1.0769	1.0620	1.0533
	0.10	$\tilde{P}_u / \tilde{P}_h$	0.0600	0.080	0.100	0.1200	0.1400
		0.00001	1.0826	1.1084	1.1267	1.1459	1.1601
		0.00003	1.0149	1.0278	1.0391	1.0520	1.0599
		0.00005	1.0013	1.0072	1.0146	1.0226	1.0299
		0.00010	1.0099	1.0016	1	1.0012	1.0031
		0.00020	1.0644	1.0340	1.0194	1.0101	1.0047
		0.00030	1.1298	1.0814	1.0524	1.0342	1.0234
		0.00050	1.2671	1.1790	1.1301	1.0968	1.0718
	0.01	$\tilde{P}_u / \tilde{P}_h$	0.0060	0.008	0.010	0.0120	0.0140
		0.00001	1.1685	1.2199	1.2604	1.2978	1.3306
		0.00003	1.0312	1.0579	1.0843	1.1092	1.1300
		0.00005	1.0024	1.0147	1.0312	1.0478	1.0632
		0.00010	1.0232	1.0042	1	1.0024	1.0076
		0.00020	1.1744	1.0869	1.0465	1.0232	1.0110
		0.00030	1.3942	1.2218	1.1359	1.0869	1.0561
		0.00050	1.8871	1.5968	1.3891	1.2749	1.1993
0.0010	0.99	$\tilde{P}_u / \tilde{P}_h$	0.9700	0.980	0.990	0.9950	0.9990
		0.00010	1.1637	1.1553	1.1323	1.1190	1.0925
		0.00030	1.0580	1.0501	1.0413	1.0349	1.0237
		0.00050	1.0238	1.0207	1.0154	1.0103	1.0074
		0.00100	1.0020	1.0007	1	1.0005	1.0048
		0.00200	1.0140	1.0146	1.0183	1.0228	1.0343
		0.00300	1.0439	1.0455	1.0519	1.0557	1.0716
		0.00500	1.1153	1.1181	1.1232	1.1332	1.1504

Table 3. Continued

P_u	P_h						
	0.90	$\tilde{P}_u / \tilde{P}_h$	0.7000	0.800	0.900	0.9500	0.9900
		0.00010	1.1514	1.1484	1.1342	1.1129	1.0873
		0.00030	1.0455	1.0464	1.0422	1.0349	1.0259
		0.00050	1.0163	1.0178	1.0152	1.0115	1.0112
		0.00100	1.0029	1.0011	1	1.0009	1.0085
		0.00200	1.0316	1.0232	1.0195	1.0214	1.0348
		0.00300	1.0792	1.0596	1.0510	1.0532	1.0700
		0.00500	1.1836	1.1477	1.1287	1.1264	1.1390
	0.50	$\tilde{P}_u / \tilde{P}_h$	0.3000	0.400	0.500	0.6000	0.7000
		0.00010	1.1067	1.1324	1.1540	1.1641	1.1730
		0.00030	1.0196	1.0345	1.0476	1.0591	1.0644
		0.00050	1.0021	1.0090	1.0174	1.0244	1.0300
		0.00100	1.0121	1.0019	1	1.0012	1.0036
		0.00200	1.0839	1.0435	1.0229	1.0126	1.0070
		0.00300	1.1711	1.1032	1.0641	1.0414	1.0298
		0.00500	1.3720	1.2365	1.1648	1.1189	1.0882
	0.10	$\tilde{P}_u / \tilde{P}_h$	0.0600	0.080	0.100	0.1200	0.1400
		0.00010	1.1677	1.2187	1.2588	1.2957	1.3208
		0.00030	1.0304	1.0567	1.0829	1.1075	1.1279
		0.00050	1.0025	1.0151	1.0302	1.0465	1.0639
		0.00100	1.0234	1.0043	1	1.0024	1.0077
		0.00200	1.1742	1.0883	1.0455	1.0224	1.0104
		0.00300	1.3917	1.2221	1.1355	1.0861	1.0552
		0.00500	1.9168	1.5895	1.3866	1.2676	1.1925
0.010	0.99	$\tilde{P}_u / \tilde{P}_h$	0.9700	0.980	0.990	0.9950	0.9990
		0.00100	1.2793	1.2629	1.2433	1.2272	1.2000
		0.00300	1.0902	1.0856	1.0763	1.0721	1.0605
		0.00500	1.0345	1.0318	1.0284	1.0245	1.0210
		0.01000	1.0011	1.0004	1	1.0003	1.0026
		0.02000	1.0414	1.0402	1.0381	1.0382	1.0417
		0.03000	1.1253	1.1160	1.1119	1.1080	1.1052
		0.05000	1.3358	1.3131	1.2929	1.2794	1.2603
	0.90	$\tilde{P}_u / \tilde{P}_h$	0.7000	0.800	0.900	0.9500	0.9900
		0.00100	1.2457	1.2559	1.2527	1.2394	1.2095
		0.00300	1.0637	1.0767	1.0821	1.0798	1.0714
		0.00500	1.0171	1.0245	1.0292	1.0302	1.0313
		0.01000	1.0086	1.0020	1	1.0008	1.0057
		0.02000	1.1096	1.0678	1.0411	1.0332	1.0320
		0.03000	1.2691	1.1829	1.1229	1.0988	1.0865
		0.05000	1.7033	1.4878	1.3327	1.2735	1.2196
	0.50	$\tilde{P}_u / \tilde{P}_h$	0.3000	0.400	0.500	0.6000	0.7000
		0.00100	1.2190	1.2838	1.3305	1.3721	1.3991
		0.00300	1.0379	1.0733	1.1085	1.1397	1.1624
		0.00500	1.0021	1.0184	1.0393	1.0616	1.0837
		0.01000	1.0403	1.0069	1	1.0040	1.0129
		0.02000	1.2905	1.1338	1.0622	1.0265	1.0094
		0.03000	1.4966	1.3468	1.1921	1.1070	1.0590
		0.05000	1.4968	1.4967	1.4966	1.3568	1.2251

Table 4. Sensitivities of $AsVar(\hat{y}_q)$ When $\sigma = 1.5$ with $K = 2$

P_u	P_h	$\tilde{P}_u / \tilde{P}_h$					
0.0001	0.99	$\tilde{P}_u / \tilde{P}_h$	0.9700	0.980	0.990	0.9950	0.9990
		0.00001	1.1315	1.1112	1.0903	1.0737	1.0481
		0.00003	1.0489	1.0406	1.0262	1.0181	1.0085
		0.00005	1.0248	1.0169	1.0098	1.0054	1.0027
		0.00010	1.0038	1.0012	1	1.0010	1.0084
		0.00020	1.0042	1.0070	1.0119	1.0185	1.0367
		0.00030	1.0191	1.0229	1.0315	1.0415	1.0666
		0.00050	1.0548	1.0616	1.0754	1.0903	1.1196
	0.90	$\tilde{P}_u / \tilde{P}_h$	0.7000	0.800	0.900	0.9500	0.9900
		0.00001	1.1447	1.1235	1.0889	1.0646	1.0356
		0.00003	1.0582	1.0455	1.0266	1.0153	1.0082
		0.00005	1.0299	1.0208	1.0088	1.0047	1.0069
		0.00010	1.0068	1.0030	1	1.0022	1.0182
		0.00020	1.0074	1.0064	1.0111	1.0216	1.0500
		0.00030	1.0220	1.0225	1.0317	1.0444	1.0807
		0.00050	1.0597	1.0620	1.0729	1.0908	1.1384
	0.50	$\tilde{P}_u / \tilde{P}_h$	0.3000	0.400	0.500	0.6000	0.7000
		0.00001	1.0880	1.0907	1.0924	1.0884	1.0790
		0.00003	1.0232	1.0271	1.0282	1.0264	1.0248
		0.00005	1.0056	1.0085	1.0106	1.0098	1.0092
		0.00010	1.0017	1.0003	1	1.0002	1.0010
		0.00020	1.0283	1.0171	1.0122	1.0117	1.0122
		0.00030	1.0620	1.0418	1.0339	1.0302	1.0304
		0.00050	1.1367	1.1005	1.0834	1.0731	1.0689
	0.10	$\tilde{P}_u / \tilde{P}_h$	0.0600	0.080	0.100	0.1200	0.1400
		0.00001	1.0892	1.1147	1.1269	1.1448	1.1518
		0.00003	1.0176	1.0287	1.0396	1.0490	1.0592
		0.00005	1.0021	1.0077	1.0150	1.0208	1.0274
		0.00010	1.0082	1.0014	1	1.0008	1.0024
		0.00020	1.0604	1.0333	1.0191	1.0113	1.0056
		0.00030	1.1248	1.0776	1.0522	1.0364	1.0255
		0.00050	1.2679	1.1793	1.1305	1.1009	1.0788
	0.01	$\tilde{P}_u / \tilde{P}_h$	0.0060	0.008	0.010	0.0120	0.0140
		0.00001	1.1716	1.2230	1.2634	1.2941	1.3260
		0.00003	1.0323	1.0593	1.0858	1.1075	1.1278
		0.00005	1.0027	1.0154	1.0305	1.0467	1.0618
		0.00010	1.0226	1.0040	1	1.0022	1.0079
		0.00020	1.1741	1.0887	1.0460	1.0240	1.0116
		0.00030	1.3900	1.2217	1.1385	1.0888	1.0576
		0.00050	1.8843	1.5996	1.3954	1.2753	1.2030
0.0010	0.99	$\tilde{P}_u / \tilde{P}_h$	0.9700	0.980	0.990	0.9950	0.9990
		0.00010	1.1791	1.1582	1.1359	1.1177	1.0880
		0.00030	1.0624	1.0544	1.0427	1.0337	1.0209
		0.00050	1.0271	1.0219	1.0149	1.0101	1.0061
		0.00100	1.0023	1.0009	1	1.0007	1.0054
		0.00200	1.0124	1.0147	1.0203	1.0247	1.0384
		0.00300	1.0417	1.0463	1.0526	1.0593	1.0786
		0.00500	1.1190	1.1218	1.1317	1.1417	1.1639

Table 4. Continued

P_u	P_h						
	0.90	$\tilde{P}_u / \tilde{P}_h$	0.7000	0.800	0.900	0.9500	0.9900
		0.00010	1.1862	1.1716	1.1355	1.1136	1.0762
		0.00030	1.0657	1.0577	1.0421	1.0320	1.0203
		0.00050	1.0287	1.0233	1.0156	1.0104	1.0079
		0.00100	1.0038	1.0016	1	1.0014	1.0110
		0.00200	1.0239	1.0183	1.0206	1.0249	1.0455
		0.00300	1.0670	1.0551	1.0563	1.0621	1.0833
		0.00500	1.1677	1.1433	1.1348	1.1425	1.1650
	0.50	$\tilde{P}_u / \tilde{P}_h$	0.3000	0.400	0.500	0.6000	0.7000
		0.00010	1.1333	1.1476	1.1555	1.1569	1.1515
		0.00030	1.0292	1.0427	1.0498	1.0539	1.0548
		0.00050	1.0050	1.0120	1.0173	1.0216	1.0245
		0.00100	1.0073	1.0014	1	1.0005	1.0020
		0.00200	1.0716	1.0391	1.0232	1.0160	1.0123
		0.00300	1.1603	1.0978	1.0684	1.0502	1.0403
		0.00500	1.3641	1.2385	1.1703	1.1314	1.1067
	0.10	$\tilde{P}_u / \tilde{P}_h$	0.0600	0.080	0.100	0.1200	0.1400
		0.00010	1.1788	1.2247	1.2637	1.2926	1.3158
		0.00030	1.0344	1.0618	1.0857	1.1066	1.1262
		0.00050	1.0031	1.0163	1.0317	1.0460	1.0605
		0.00100	1.0227	1.0039	1	1.0023	1.0073
		0.00200	1.1726	1.0885	1.0470	1.0246	1.0119
		0.00300	1.3951	1.2217	1.1369	1.0891	1.0594
		0.00500	1.9034	1.5997	1.3957	1.2772	1.2031
0.010	0.99	$\tilde{P}_u / \tilde{P}_h$	0.9700	0.980	0.990	0.9950	0.9990
		0.00100	1.2811	1.2653	1.2405	1.2195	1.1888
		0.00300	1.0940	1.0861	1.0774	1.0705	1.0567
		0.00500	1.0349	1.0323	1.0274	1.0237	1.0188
		0.01000	1.0009	1.0003	1	1.0002	1.0022
		0.02000	1.0426	1.0412	1.0384	1.0381	1.0427
		0.03000	1.1260	1.1195	1.1143	1.1093	1.1079
		0.05000	1.3525	1.3262	1.3025	1.2862	1.2684
	0.90	$\tilde{P}_u / \tilde{P}_h$	0.7000	0.800	0.900	0.9500	0.9900
		0.00100	1.2851	1.2779	1.2576	1.2311	1.1924
		0.00300	1.0802	1.0863	1.0806	1.0749	1.0647
		0.00500	1.0253	1.0297	1.0302	1.0276	1.0257
		0.01000	1.0063	1.0016	1	1.0008	1.0053
		0.02000	1.1014	1.0670	1.0435	1.0368	1.0358
		0.03000	1.2641	1.1820	1.1292	1.1060	1.0936
		0.05000	1.7269	1.5033	1.3511	1.2902	1.2328
	0.50	$\tilde{P}_u / \tilde{P}_h$	0.3000	0.400	0.500	0.6000	0.7000
		0.00100	1.2504	1.2979	1.3346	1.3580	1.3728
		0.00300	1.0469	1.0823	1.1087	1.1312	1.1476
		0.00500	1.0042	1.0211	1.0410	1.0587	1.0744
		0.01000	1.0360	1.0058	1	1.0032	1.0098
		0.02000	1.2945	1.1349	1.0642	1.0296	1.0127
		0.03000	1.4657	1.3600	1.1972	1.1148	1.0672
		0.05000	1.4659	1.4658	1.4657	1.3808	1.2443

8. Conclusions

Even through a lot of work has been done on optimal ALT plans, the computational techniques and results concerning asymptotically optimal ALT plans for Burr type X with log-linear model are new. In this paper, we have discussed optimal ALT plans for minimizing $AsVar(\hat{y}_q)$ under the assumptions of Burr type X distribution, periodic inspection, and Type I censoring with log-linear model. We have derived the optimal allocation of units for two stress levels using Lagrange multipliers technique.

In optimality results, we have indicated various patterns of optimal plans and shown that the number of inspections need not be large and the plan is insensitive to misspecification of guessed failure probabilities at the design and high stress levels. We have also observed that the schemes with equally spaced inspection times at each stress level are administratively convenient and statistically optimal. We conclude that the Burr type X failure model is widely and quit effectively lifetime distribution for ALT. Finally, we have used an example to illustrate the planning of an ALT.

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Appendix

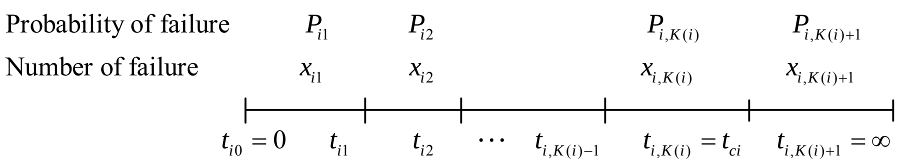

Figure 1. Structure of periodic inspection at the *i*th stress level