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On Asymptotic Nonlocal Symmetry of Nonlinear Schrödinger Equations

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Abstract

A concept of asymptotic symmetry is introduced which is based on a definition of symmetry as a reducibility property relative to a corresponding invariant ansatz. It is shown that the nonlocal Lorentz invariance of the free-particle Schrödinger equation, discovered by Fushchych and Segeda in 1977, can be extended to Galilei-invariant equations for free particles with arbitrary spin and, with our definition of asymptotic symmetry, to many nonlinear Schrödinger equations. An important class of solutions of the free Schrödinger equation with improved smoothing properties is obtained.

1 Introduction

It is well-known that the maximal Lie invariance algebra of the free linear Schrödinger equation in three spatial dimensions

$$\left(i\hbar\partial_t + \frac{\hbar^2}{2m}\Delta\right)\psi = 0, \tag{1}$$

is the Schrödinger algebra $sch(1,3)$, a Lie algebra which contains the Galilei and dilation algebras as well as some special conformal transformations [1–3]. It was quite surprising therefore, when Fushchych and Segeda [4] showed that equation (1) is also invariant under an algebra of nonlocal pseudodifferential operators which is isomorphic to the Lie algebra $so(1,3)$ of the three-dimensional homogeneous Lorentz group.

The purpose of the present work is to establish similar results concerning nonlocal symmetry for some other equations of interest in mathematical physics. In particular, it will be shown that linear Schrödinger equations with linear and quadratic potentials with arbitrary time-dependent coefficients and Hurley's Galilei-invariant wave equations [5], which describe free nonrelativistic quantum particles with arbitrary spin, are invariant under related algebras of nonlocal pseudodifferential operators. The results on nonlocal symmetries for linear Schrödinger equations with the potentials described above follows from the fact that these equations can be transformed to the free Schrödinger equation (1) and, therefore, inherit the nonlocal symmetry of the latter equation. The generators of the respective nonlocal Lie algebras for these Schrödinger equations necessarily have different representations than the generators of the algebras corresponding to equation (1), but the algebras associated, respectively, with these different equations are isomorphic.

Analogous results to those for the linear Schrödinger equations mentioned above can be obtained for some nonlinear Schrödinger equations (NSEs). Specifically, a subclass of the family of NSEs proposed and discussed by Doebner and Goldin over the past several years [6, 7] in connection with representations of diffeomorphism groups and the corresponding Lie algebras of vector fields, as well as some related equations studied by Auberson and Sabatier [8] have been shown to be linearizable; i.e., they can be mapped by means of an appropriate change of variables to linear Schrödinger equations. In particular, the potential in the latter equation can be chosen to be identically zero so that the solutions of these nonlinear equations can be related to the nonlocal invariant solutions of equation (1). These results make the situation for these linearizable NSEs analogous to the situation for linear Schrödinger equations with time-dependent linear and quadratic potentials, where the isomorphism of the respective Lie algebras was first proved by Niederer [9] for the linear Schrödinger equation with the usual (time-independent) harmonic oscillator potential. In the cases examined by Niederer, however, the Lie algebras were those corresponding to the Schrödinger and oscillator groups whereas, in the present paper, we consider isomorphisms for nonlocal Lie algebras.

Since the symmetries that we discuss are nonlocal, Lie's approach is not adequate to deal with them. A generalization of Lie's method, suitable for linear partial differential equations (PDEs), was suggested in [10] and is based upon the following commutator form of invariance condition.

Definition 1.1. *An operator Q is a symmetry of a linear system of PDEs*

$$Lu = 0 \tag{2}$$

if and only if $[L, Q]u = 0$ for each solution u of (2).

Note that there is no restriction on the order of the operator Q (unlike the situation with Lie's methods [11]). In fact, Q need not be a differential operator, but may also be a pseudodifferential or integral operator. This non-Lie approach proved to be very effective, and wide classes of new symmetries were discovered for many equations of mathematical physics [12].

In the present paper we suggest an alternative symmetry criterion of invariance for nonlocal operators which is also suitable for nonlinear equations.

Definition 1.2. *We will say that an operator Q is a symmetry of a system of PDEs if and only if a corresponding ansatz, obtained as a solution of the equation*

$$Qu = 0, \tag{3}$$

reduces the given system of PDEs.

Remark. When Q is a (linear) pseudodifferential operator, one can construct an invariant ansatz (solution of (3)) by means of a Fourier transform, as shown in [13] (see also [3], Sec. 5.11).

We prove that certain classes of nonlinear Schrödinger equations (NSEs) have a similar invariance when the time is larger than the squared modulus of the spatial variables in a certain well-defined sense. This "asymptotic symmetry" is based upon Definition 1.2 when there does not exist an exact reduction of the given system of nonlinear PDEs by an invariant ansatz, but there does exist such a reduction in a well-defined asymptotic sense. (The precise definition will be given in Section 4.) This asymptotic symmetry situation is somewhat reminiscent of the situation encountered in scattering theory in which a solution of a nonfree equation approaches a solution of the corresponding free equation as $t \to \pm\infty$ (cf. [14]). This scenario has been established for some NSEs with power-type nonlinearities [15]. In the present case, certain NSEs have a nonlocal asymptotic symmetry which is an exact symmetry of the free Schrödinger equation.

As we shall discuss in more detail later, a number of authors have given Lie symmetry analyses for the NSEs that we consider [3, 11]. However, since the symmetries of interest in the present paper are nonlocal. Lie's algorithm cannot be applied to them and a different approach must be used to study them.

The paper is organized in the following manner. In Section 2 we discuss the results for the free Schrödinger equation in further detail, extending the three-dimensional results of [4] to any spatial dimension $n \geq 2$ and extending the nonlocal Lorentz-invariant solutions of the free Schrödinger equation to a larger class of symmetries. Nonlocal symmetries of the type discussed in the present paper do not exist when $n = 1$. They are phenomena peculiar to higher spatial dimensions. Similar results are obtained for the three-dimensional Galilei-invariant free-particle wave equations of Hurley [5] for nonrelativistic free quantum particles with arbitrary spin. We have found a new representation of the Lorentz algebra for particles with arbitrary spin whose time evolution is described by these equations. In particular, the spin contributions to the angular momentum operators do not satisfy the familiar angular momentum commutation relations, even though the total angular momentum operators satisfy the correct commutation relations with themselves and with the nonlocal generators.

In Section 3 we show that, since linear Schrödinger equations with linear and quadratic (harmonic oscillator) potentials with arbitrary time-dependent coefficients can be transformed to the free Schrödinger equation (1), they inherit the nonlocal symmetry of the free equation. The corresponding nonlocal Lie algebras are isomorphic. Similar results are then discussed for the linearizable Doebner-Goldin and Auberson-Sabatier NSEs.

In Section 4 we prove that several classes of NSEs possess nonlocal symmetries in an asymptotic sense when the time variable is sufficiently large. The collection of NSEs considered include several classes of standard type with power-type nonlinearities as well as the families of NSEs proposed and discussed by Doebner and Goldin [6, 7], Bialynicki-Birula and Mycielski [16], and Kostin [17]. Finally, Section 5 consists of some concluding remarks.

2 Free-particle equations

We first discuss the exact nonlocal symmetry of the free-particle Schrödinger equation, and then consider analogous exact nonlocal symmetries for the Galilei-invariant wave equations of Hurley.

Free-particle Schrödinger equations

We define the standard operators $p_0 \equiv i\hbar\partial_t$, $p_j \equiv -i\hbar\partial_{x_j}$ $(j = 1, 2, \ldots, n)$, and set $L_S \equiv p_0 - \dfrac{p_j p_j}{2m}$ where, in the latter definition, the convention of summing over repeated indices is used. Then the free-particle Schrödinger equation (1) takes the form $L_S\psi = 0$. Consider the operators $(j, k = 1, 2, \ldots, n \geq 2)$:

$$J_{jk} = x_j p_k - x_k p_j, \tag{4a}$$

and

$$J_{0j} = \frac{1}{2m}(f(p)G_j + G_j f(p)), \tag{4b}$$

where

$$G_j = tp_j - mx_j, \tag{4c}$$

$p \equiv \left(\sum\limits_{j=1}^{n} p_j p_j\right)^{1/2}$, and f denotes a smooth function.

One easily verifies that the Schrödinger operator L_S is invariant under the operators (4) since

$$[L_S, J_{jk}] = 0 = [L_S, J_{0j}], \qquad j, k = 1, 2, \ldots, n. \tag{5}$$

The operators (4) satisfy the following commutation relations:

$$[J_{jk}, J_{qr}] = i\hbar(\delta_{rk}J_{jq} - \delta_{qk}J_{jr} + \delta_{qj}J_{kr} - \delta_{jr}J_{kq}), \tag{6a}$$

$$[J_{jk}, J_{0q}] = i\hbar(\delta_{qj}J_{0k} - \delta_{kq}J_{0j}), \tag{6b}$$

$$[J_{0j}, J_{0k}] = -i\hbar\frac{ff'}{p}J_{jk}, \tag{6c}$$

$(j, k, q, r = 1, 2, \ldots, n \geq 2)$ where $f' \equiv f'(p) = \dfrac{df(p)}{dp}$. For $f(p) = p$, relations (6) are commutation relations of the Lotentz Lie algebra [4]. When $f(p) \neq p$, the invariance relations (5) are still valid. In this case, the operators $\{J_{jk}, J_{0q}; j, k, q = 1, 2, \ldots, n\}$ still form a Lie algebra isomorphic to the Lorentz algebra when $f = \sqrt{p^2 + \text{const}}$. For other choices of f, the operators $\{J_{jk}, J_{0q}\}$ do not form a Lie algebra although it is possible that they could be embedded in a Lie algebra of larger dimension than the Lorentz algebra. We will not consider this approach in the present paper.

Using the symmetries (5), one can derive fundamental solutions ψ of $L_S\psi = 0$ which are invariant under the operators (4) by requiring that $J_{jk}\psi = 0$, $J_{0q}\psi = 0$ $(j,k,q = 1,2,\ldots,n)$. In terms of the Fourier transform $\widetilde{\psi}$ of ψ , this procedure leads to:

$$\widetilde{\psi}(k,t) = cf(k)^{-1/2}\exp\left(-\frac{itk^2}{2m\hbar}\right), \tag{7}$$

where $k \equiv \left(\sum_{j=1}^{n} k_j k_j\right)^{1/2}$ and c denotes a complex constant. This result was first obtained (for the case $f(k) = k$) in [13]. In order to calculate the inverse Fourier transform of (7), it is convenient to specify the function f. A convenient set of choices is:

Case I: $f(k) = k^\alpha,\ \ 0 < \alpha < 2n.$

This reduces to the Lorentz case when $\alpha = 1$. One finds from (7) (see [18] for a derivation in the case $\alpha = 1$):

$$\begin{aligned}\psi(x,r) &= (2\pi\hbar)^{-\frac{n}{2}}\int_{\mathbb{R}^n}\exp\left(\frac{i}{\hbar}k\cdot x\right)\widetilde{\psi}(k,t)\,d^nk\\ &= \left(\frac{m}{2\pi i\hbar t}\right)^{\frac{n}{2}-\frac{\alpha}{4}}\frac{\Gamma\left(\frac{n}{2}-\frac{n}{4}\right)}{\Gamma\left(\frac{n}{2}\right)}\,{}_1F_1\left(\frac{n}{2}-\frac{\alpha}{4};\frac{n}{2};\frac{im|x|^2}{2\hbar t}\right)\end{aligned} \tag{8}$$

(valid in all dimensions $n \geq 2$ if $0 < \alpha < 2n$) in terms of the confluent hypergeometric function ${}_1F_1$, and we have chosen the constant c in (7) so that (8) reduces to the standard Galilei-invariant fundamental solution

$$\psi_g(x,t) = \left(\frac{m}{2\pi i\hbar t}\right)^{\frac{n}{2}}\exp\left(\frac{im|x|^2}{2\hbar t}\right) \tag{9}$$

when $\alpha = 0$. For the case $n = 3$ and $\alpha = 1$, one may use the recurrence relations for the confluent functions and the well-known relations

$${}_1F_1\left(\nu+\frac{1}{2};2\nu+1;2iz\right) = \Gamma(1+\nu)e^{iz}\left(\frac{z}{2}\right)^{-\nu}J_\nu(z)$$

to write (8) in terms of Bessel functions (see [13, 3]):

$$\psi(x,t) = \frac{i^{\frac{1}{4}}\pi^{\frac{3}{4}}}{2}\left(\frac{m}{2\pi i\hbar t}\right)^{\frac{3}{2}}\exp\left(\frac{im|x|^2}{4\hbar t}\right)|x|^{\frac{1}{2}}\left[J_{-\frac{1}{4}}\left(\frac{m|x|^2}{4\hbar t}\right) + iJ_{\frac{3}{4}}\left(\frac{m|x|^2}{4\hbar t}\right)\right], \tag{10}$$

Similarly, when $\alpha = n \geq 2$ the same relation between ${}_1F_1$, and J_ν given above can be used to write (8) in the following Bessel function form:

$$\psi(x,t) = \frac{\sqrt{\pi}}{2^{\frac{n}{2}-1}}\left(\frac{m}{2\pi i\hbar t}\right)^{\frac{n}{4}}\exp\left(\frac{im|x|^2}{4\hbar t}\right)\left(\frac{m|x|^2}{8\hbar t}\right)^{-\left(\frac{n}{4}-\frac{1}{2}\right)}J_{\frac{n}{4}-\frac{1}{2}}\left(\frac{m|x|^2}{4\hbar t}\right),$$

where the following well-known relation between Γ functions has also been used:

$$\Gamma(z)\Gamma\left(z+\frac{1}{2}\right) = \frac{\sqrt{\pi}\Gamma(2z)}{2^{2z-1}}.$$

We note that, when n is a multiple of 4, the Bessel function in the above expression can be further simplified to a combination of sine, cosine, and algebraic functions.

The fact that the fundamental solutions (8) have a slower time decay, $t^{-\left(-\frac{n}{2}-\frac{\alpha}{4}\right)}$ than the corresponding Galilean result (9), which is seen to be $t^{-\frac{n}{2}}$, was discussed in [18] for the case $\alpha = 1$. The point was made that, although this property makes (8) unacceptable for use in quantum mechanics (when used as a kernel analogously to the usual use of the Galilean fundamental solution (9)) because the standard probability interpretation for the wave functions of free nonrelativistic particles is not obtained; its use may actually be more desirable than (9) for mathematical applications because of the smoothing properties which the linear operators that are constructed from them possess. (Analogous operators constructed from (9) do not have such smoothing properties.)

One advantage of considering the cases $\alpha \neq 1$ of Case I is that the corresponding convolution mappings formed from (8) or (10) have improved smoothing properties as α increases. Thus, generalizing the discussion in [18] for the case $\alpha = 1$, we define the mappings $G_n : g \to G_n(\alpha)g$:

$$(G_n(\alpha)g)(x,t) = (2\pi\hbar)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \exp\left(\frac{i}{\hbar}k\cdot x\right)\left(|k|^2\right)^{-\frac{\alpha}{4}} \exp\left(-\frac{it|k|^2}{2m\hbar}\right)\widetilde{g}(k)\,dk,$$

and deduce that the maps $G_n(\alpha)$ are smoothing in the sense that $G_n g$ have $\dfrac{\alpha}{2}$ (distribution) derivatives if $g \in L^2\left(\mathbb{R}^n\right)$ $(n \geq 2)$.

The above argument shows that the smoothing properties of mappings on L^2 constructed with the fundamental solutions (8) or (10) increase as α increases. However, since $f(p) = p^\alpha$ of Case I only increases algebraically with α, and α is bounded above by $2n$, the smoothing properties are limited. This suggests that one consider functions f of exponential type in the variable k^2:

Case II: $f(k) = \exp\left(2\beta|k|^2\right)$, $(n \geq 2)$.

In this case, the corresponding mappings G_n have much improved smoothing properties because $G_n : L^2\left(\mathbb{R}^n\right) \to L^q\left(\mathbb{R}^n\right)$ for all $q \in [2,\infty)$. In addition, it turns out that the corresponding fundamental solutions (i.e., the inverse Fourier transforms of (7)) have the same asymptotic time decay as $t \to +\infty$ as the Galilean fundamental solutions (9):

$$\psi(x,t) = c\left(2\hbar\beta + \frac{it}{m}\right)^{-\frac{n}{2}} \exp\left(-\frac{|x|^2}{4\hbar^2\left(\beta + \frac{it}{2m\hbar}\right)}\right). \tag{11}$$

The fact that the functions (11) have the same asymptotic decay as the Galilean fundamental solutions (9) is explained by the fact that the former correspond to imaginary time translations of the latter: $\psi(x,t) = \psi_g(x, t - 2mi\hbar\beta)$ with the choices $c = (2\pi\hbar)^{-\frac{n}{2}}$ for the constant in (11). The increase in smoothness of the mappings G_n associated with (11) relative to those associated with (9) is intimately connected with the fact that the time translations leading from (9) to (11) are imaginary.

Free-particle wave equations for arbitrary spin

According to the principles of quantum mechanics, a wave function which represents a free nonrelativistic particle with spin $s = \dfrac{1}{2}, 1, \dfrac{3}{2}, \ldots$ should have $2s+1$ components. The equations obtained by Hurley [5]:

$$L_H\psi \equiv \begin{pmatrix} i\hbar\partial_t I_{2s+1} & -\dfrac{1}{\hbar s}S\cdot p & -\dfrac{1}{\hbar s}K^*\cdot p \\ \dfrac{1}{2m\hbar s}S\cdot p & I_{2s+1} & 0_{2s+1,2s-1} \\ \dfrac{i}{2m\hbar s}K\cdot p & 0_{2s-1,2s+1} & I_{2s-1} \end{pmatrix} \begin{pmatrix} \psi \\ \chi \\ \Omega \end{pmatrix} = 0, \tag{12}$$

are a system of $6s+1$ equations in which the first row in (12) gives the equations of motion for the $(2s+1)$-component wave function ψ and the remaining two rows are equations of constraint defining the redundant functions χ ($2s+1$ components) and Ω ($2s-1$ components) in terms of ψ. S_j and K_j ($j = 1,2,3$) are spin matrices with dimensions $(2s+1)\times(2s-1)$, $(2s-1)\times(2s+1)$, respectively, and can be chosen to satisfy the following relations [5] (with K^* the adjoint of K and ε_{ijk} the completely antisymmetric Levi-Civita symbol):

$$S_iS_j + K_i^*K_j = is\hbar\varepsilon_{ijk}S_k + \hbar^2 s^2\delta_{ij}. \tag{13}$$

The symbol I_m denotes the m-dimensional unit matrix, and 0_{ab} denotes the zero matrix with a rows and b columns. Equations (12) reduce to the equations derived by Levy-Leblond [19] when $s = \frac{1}{2}$ and by Hagen [20] when $s = 1$. They are Galilei invariant by construction. For convenience in the demonstration of nonlocal invariance of equations (12) in the discussion to follow, we have used a different normalization than Hurley.

By substituting the equations of constraint (the second and third rows of (12)) into the equations of motion for ψ (first row of (12)), and using (13), one finds that each component of ψ satisfies equation (1). In order to establish the nonlocal symmetry of equations (12), we define new representations of the commutation relations (6). Thus, in place of the operators (4), we define (with $j,k = 1,2,3$):

$$\widetilde{J}_{jk} = (x_jp_k - x_kp_j)I_{6s+1} - \frac{1}{m}(\lambda_jp_k - \lambda_kp_j), \tag{14a}$$

and

$$\widetilde{J}_{0j} = \frac{1}{2m}(f(p)\widetilde{G}_j + \widetilde{G}_jf(p)), \tag{14b}$$

where

$$\widetilde{G}_j = (tp_j - mx_j)I_{6s+1} + \lambda_j, \tag{14c}$$

with

$$\lambda_j = \begin{pmatrix} 0_{2s+1,2s+1} & 0_{2s+1,2s+1} & 0_{2s+1,2s-1} \\ \dfrac{1}{2s}S_j & 0_{2s+1,2s+1} & 0_{2s+1,2s-1} \\ \dfrac{1}{2s}K_j & 0_{2s-1,2s+1} & 0_{2s-1,2s-1} \end{pmatrix}. \tag{14d}$$

Then, using (13), one verifies that solutions of (12) are invariant under the operators (14) in the sense that $[L_H,\vartheta]\psi = 0$ when $L_H\psi = 0$, with $\vartheta = \widetilde{J}_{jk}, \widetilde{J}_{0j}$ ($j,k = 1,2,3$) and that

$\{\widetilde{J}_{jk}, \widetilde{J}_{0j}; j, k = 1, 2, 3\}$ satisfy the commutation relations (6) with $\{J_{jk}, J_{0j}\}$ replaced by $\{\widetilde{J}_{jk}, \widetilde{J}_{0j}\}$, respectively.

We can now construct solutions of (12) in the form

$$\psi = \text{ column } \left(\psi, -\frac{i}{2m\hbar s} S \cdot p\psi, -\frac{i}{2m\hbar s} K \cdot p\psi\right) \tag{15}$$

with $\psi(x,t) = \beta(t)F(x,t)$, where $\beta(t)$ has $2s+1$ components and $F(x,t)$ is a (scalar) solution of (1). Then, inserting (15) into (12) and using (13), we infer that the components of β must be constant. In particular, $F(x,t)$ can be taken as the invariant solution (10). However, in order to write the solutions (15) in complete detail, one needs representations for the spin matrices S_j and K_j $(j = 1, 2, 3)$. Examples of these are given in [5].

Finally, we note that the operators $\widetilde{J}_{0j}$ defined in (14b)–(14d) are pseudodifferential operators and that the terms $-\dfrac{1}{m}(\lambda_j p_k - \lambda_k p_j)$ $(j, k = 1, 2, 3)$ in (14a), which play the role of spin angular momenta in the present case, are noncanonical in the sense that they do not satisfy the usual angular momentum commutation relations. Thus, they do not generate the usual representations of the three-dimensional rotation group corresponding to spin s. Nevertheless, as we noted above, the operators $\{\widetilde{J}_{jk}, \widetilde{J}_{0j}; j, k = 1, 2, 3\}$ satisfy the correct commutation relations.

3 Nonfree equations possessing exact nonlocal Lorentz symmetries

In this section we discuss how the exact nonlocal symmetry of free-particle equations, discussed in the preceding section, can be extended to some nonfree equations which describe interactions between particles. The existence of nonlocal symmetries for these equations follows from the fact that appropriate Lie algebras can be constructed which are isomorphic to the Lie algebra formed by the operators in equations (6) with $\dfrac{ff'}{p} = 1$. We discuss this situation for two cases: (1) linear Schrödinger equations with linear and quadratic potentials (with arbitrary time-dependent coefficients), and (2) some classes of nonlinear Schrödinger equations.

Linear Schrödinger equations with linear and quadratic potentials

The symmetry properties of these equations have been well-studied, especially in one space dimension. We refer to a recent discussion of the latter case in the context of coherent states and squeezed states [21], from which many references may be traced.

Our approach to these equations is based on the fact that they can be transformed to the free Schrödinger equation (1). The results are analogous to those obtained by Niederer [9] for linear Schrödinger equations with time-independent harmonic oscillator potentials which showed that the Lie algebras of the oscillator and Schrödinger groups are isomorphic. Our transformation results generalize the treatment of Niederer and are a direct extension of those of Truax [22], Bluman [23], and of Bluman and Shtelen [24] for the case of one spatial dimension. However, our objective is different than that of the authors cited above in that we are interested in algebras of nonlocal symmetries rather than in algebras of point transformations.

For Schrödinger equations of the form:

$$i\hbar\partial_t\psi(x,t) + \frac{\hbar^2}{2m}\Delta\psi(x,t) = \big(a(t)|x|^2 + b_j(t)x_j + c(t)\big)\,\psi(x,t), \tag{16}$$

with $|x|^2 = \sum_{j=1}^{n} x_j x_j$ for $n \geq 1$, where a, b_j $(j = 1, 2, \ldots, n)$, and c are arbitrary functions of t, we define the following transformation:

$$\psi(x,t) = \exp\left(-\frac{i}{\hbar}\big(A(t)|x|^2 + B_j(t)x_j + C(t)\big)\right) u(y,\tau), \tag{17a}$$

with

$$\tau(t) = \int^t \sigma^2(\mu)\,d\mu, \qquad y_j = \sigma(t)x_j + \rho_j(t), \tag{17b}$$

where $\sigma(t)$, $A(t)$, $C(t)$, $\rho_j(t)$, and $B_j(t)$ $(j = 1, 2, \ldots, n)$ are to be expressed in terms of $a(t)$, $b_j(t)$ $(j = 1, 2, \ldots, n)$, and $c(t)$. By substitution of (17) into (16), one can choose the former coefficients so that (16) reduces to equation (1) for the function $u(y,\tau)$:

$$i\hbar\partial_\tau u(y,\tau) + \frac{\hbar^2}{2m}\Delta_y u(y,\tau) = 0.$$

For the standard harmonic oscillator: $a = \frac{1}{2}m\omega^2$, $b_j = 0$ $(j = 1, 2, \ldots, n)$, $c = 0$; (17) reduces to the transformation obtained by Niederer:

$$\psi(x,t) = (\sec(\omega t))^{\frac{n}{2}} \exp\left(-\frac{im\omega}{2\hbar}\tan(\omega t)|x|^2\right) u(y,\tau), \tag{18}$$

with $y_j = \sec(\omega t)x_j$ $(j = 1, 2, \ldots, n)$ and $\tau = \frac{1}{\omega}\tan(\omega t)$. By taking $u(y,\tau)$ to be the Galilean fundamental solution (9) expressed in the y, τ variables, one obtains the standard fundamental solution for the harmonic oscillator ([9] and [25], p.63). In a similar manner, we can use the mapping (18) to obtain nonlocal invariant solutions by taking for $u(y,\tau)$ the nonlocal invariant solution (8) of equation (1):

$$u(y,\tau) = \left(\frac{m}{2\pi i\hbar\tau}\right)^{\frac{n}{2}-\frac{\alpha}{4}} \frac{\Gamma\left(\frac{n}{2}-\frac{\alpha}{4}\right)}{\Gamma\left(\frac{n}{2}\right)}\, {}_1F_1\left(\frac{n}{2}-\frac{\alpha}{4}; \frac{n}{2}; \frac{im|y|^2}{2\hbar\tau}\right), \qquad n \geq 2,$$

and obtain

$$\psi(x,t) = \left(\frac{m\omega}{2\pi i\hbar\sin(\omega t)}\right)^{\frac{n}{2}} \left(\frac{m\omega\tan(\omega t)}{2\pi i\hbar}\right)^{-\frac{\alpha}{4}} \frac{\Gamma\left(\frac{n}{2}-\frac{\alpha}{4}\right)}{\Gamma\left(\frac{n}{2}\right)}$$
$$\times \exp\left(-\frac{im\omega}{2\hbar}\tan(\omega t)|x|^2\right) {}_1F_1\left(\frac{n}{2}-\frac{\alpha}{4}; \frac{n}{2}; \frac{im\omega|x|^2}{\hbar\sin(2\omega t)}\right).$$

Remark. An explicit form for the generators of the Lorentz algebra for the harmonic oscillator is as follows:

$$\hat{J}_{0j} = \frac{1}{2m\omega}\left(\hat{p}\hat{G}_j + \hat{G}_j\hat{p}\right), \qquad \hat{p} \equiv \left(\sum_{j=1}^{n} \hat{p}_i\hat{p}_i\right)^{\frac{1}{2}}, \quad n \geq 2,$$

where

$$\hat{p}_j = \cos(\omega t)p_j - m\omega\sin(\omega t)x_j,$$

and

$$\hat{G}_j = -\sin(\omega t)p_j - m\omega\cos(\omega t)x_j, \qquad j = 1, 2, \ldots, n,$$

in terms of the same coodinates x_j, and momentum operators p_j used in Section 2. We use the same angular momentum operators J_{ik} $(j, k = 1, 2, \ldots, n)$ as defined in (4a) and obtain the following commutation relations in place of (6b), (6c) for the free case:

$$[J_{jk}, \hat{J}_{0q}] = i\hbar(\delta_{qj}\hat{J}_{0k} - \delta_{qk}\hat{J}_{0j}),$$
$$[\hat{J}_{0j}, \hat{J}_{0k}] = -i\hbar J_{jk}, \qquad j, k = 1, 2, \ldots, n \geq 2.$$

We derive similar results for linear potentials by setting $a = 0$, $c = 0$, $b_j \neq 0$ $(j = 1, 2, \ldots, n \geq 1)$ and obtain mappings of solutions of (1) onto solutions of (16):

$$\psi(x, t) = \exp\left(-\frac{i}{\hbar}\left(tb_jx_j + \frac{|b|^2t^3}{6m}\right)\right)u(y, \tau);$$

$$y_j = \sigma x_j + \frac{\sigma t^2}{2m}b_j, \qquad (j = 1, 2, \ldots, n) \qquad \tau = \sigma^2 t.$$

$\sigma = \text{const}$; and mappings of nonlocal invariant fundamental solutions (with $n \geq 2$):

$$u(y, \tau) = \left(\frac{m}{2\pi i\hbar\sigma^2 t}\right)^{\frac{n}{2}-\frac{\alpha}{4}} \frac{\Gamma\left(\frac{n}{2} - \frac{\alpha}{4}\right)}{\Gamma\left(\frac{n}{2}\right)}\, {}_1F_1\left(\frac{n}{2} - \frac{\alpha}{4}; \frac{n}{2}; \frac{im|x|^2}{2\hbar t} + \frac{it}{2\hbar}b_jx_j + \frac{i|b|^2t^3}{8m\hbar}\right),$$

$$\psi(x, t) = \left(\frac{m}{2\pi i\hbar\sigma^2 t}\right)^{\frac{n}{2}-\frac{\alpha}{4}} \frac{\Gamma\left(\frac{n}{2} - \frac{\alpha}{4}\right)}{\Gamma\left(\frac{n}{2}\right)} \exp\left(-\frac{i}{\hbar}\left(tb_jx_j + \frac{|b|^2t^3}{6m}\right)\right)$$
$$\times\, {}_1F_1\left(\frac{n}{2} - \frac{\alpha}{4}; \frac{n}{2}; \frac{im|x|^2}{2\hbar t} + \frac{it}{2\hbar}b_jx_j + \frac{i|b|^2t^3}{8m\hbar}\right),$$

Similar results can be given for the case of a harmonic oscillator driven by an external force $f_j(t)$ $(j = 1, 2, \ldots, n)$, for which $a = \frac{1}{2}m\omega^2 = \text{const}$ and $b_j = -f_j$, whose Galilean fundamental solutions were given in [25]. However, since the expressions obtained are somewhat unwieldy, we shall not give those results here.

Nonlinear Schrödinger equations with exact nonlocal symmetry

Doebner and Goldin [6, 7] considered a class of nonlinear Schrödinger equations which were suggested by their studies of dissipative quantum theory based on group theoretic considerations relating to groups of diffeomorphisms on Euclidean spaces and the corresponding Lie algebras (algebras of vector fields). Related equations were considered earlier by Sabatier [26] and subsequently by Auberson and Sabatier [8] and by Auberson [27].

We first consider a subclass of the Doebner-Goldin (DG) equations which are linearizable in the sense that they can be mapped to linear Schrödinger equations by point transformations. The nonlocal symmetry of the linearizable DG equations follows from the isomorphy of the Lie algebras of these equations with the nonlocal Lie algebras of linear Schrödinger equations.

The DG equations can be written in the form:

$$i\hbar\psi_t = \left(-\frac{\hbar^2}{2m}\Delta + V(x,t)\right)\psi + \frac{i\hbar D}{2}R_2(\psi,\overline{\psi})\psi + \hbar D'\sum_{j=1}^{5} c_j R_j(\psi,\overline{\psi})\psi, \tag{19}$$

where D and D' denote constant diffusion coefficients, and the real-valued nonlinear functionals $R_j(\psi,\overline{\psi})$ $(j = 1, 2, 3, 4, 5)$ are given by:

$$\begin{gathered} R_1(\psi,\overline{\psi}) = \frac{\nabla\cdot\widetilde{j}}{\rho}, \qquad R_2(\psi,\overline{\psi}) = \frac{\Delta\rho}{\rho}, \qquad R_3(\psi,\overline{\psi}) = \frac{\widetilde{j}^2}{\rho^2}, \\ R_4(\psi,\overline{\psi}) = \frac{\widetilde{j}\cdot\nabla\rho}{\rho^2}, \qquad R_5(\psi,\overline{\psi}) = \frac{(\nabla\rho)^2}{\rho^2}, \end{gathered} \tag{20}$$

where $\rho = \overline{\psi}\psi$, and $\widetilde{j}$ is related to the usual probability current density j by

$$\widetilde{j} = \frac{m}{\hbar}j = \frac{1}{2i}(\overline{\psi}\nabla\psi - \psi\nabla\overline{\psi}).$$

Lie symmetry analyses of these equations have been discussed by several authors [28–30]. References to discussions of analogous equations by other authors can be found in [7].

It was pointed out by Doebner and Goldin that the subfamily of their equations (19), (20) defined by the following relations between the coefficients D, D', c_j $(j = 1, 2, 3, 4, 5)$:

$$D = D'c_1 = -D'c_4; \qquad D'(c_2 + 2c_5) = D'c_3 = 0, \tag{21}$$

are linearizable and that the corresponding solutions can be constructed from solutions of linear Schrödinger equations by means of "nonlinear gauge transformations" (see also [28]). The nonlinear gauge transformations of DG are given by:

$$\psi \to \psi' = N(\psi) = |\psi| \exp\Big(i[\gamma \ln|\psi| + \Lambda \operatorname{Arg}\psi]\Big), \tag{22}$$

with γ, Λ real numbers (and $\Lambda \neq 0$). Doebner and Goldin show that, given the relations (21) among the coefficients, if ψ is a solution of (19), (20); then $\psi' = N(\psi)$ is a solution of the following linear Schrödinger equation:

$$\frac{i\hbar}{\Lambda}\psi'_t = -\frac{\hbar^2}{2m\Lambda^2}\Delta\psi' + V(x,t)\psi', \tag{23}$$

when

$$\Lambda = \left(1 - \frac{4m}{\hbar} D' c_2 - \frac{4m^2 D^2}{\hbar^2}\right)^{-\frac{1}{2}}$$

and

$$\gamma = -\frac{2mD\Lambda}{\hbar}, \tag{24}$$

provided that $\frac{4m}{\hbar} D' c_2 + \frac{4m^2 D^2}{\hbar^2} < 1$. Since the gauge transformations satisfy the group law $N_{\gamma_1,\Lambda_1} \bullet N_{\gamma_2,\Lambda_2} = N_{\gamma_1+\Lambda_1\gamma_2,\Lambda_1\Lambda_2}$, the gauge transformations inverse to (22) are given by:

$$\psi = N^{-1}(\psi') = |\psi'| \exp\left(i\left[-\Lambda^{-1}\gamma \ln|\psi'| + \Lambda^{-1} \operatorname{Arg}\psi'\right]\right). \tag{25}$$

This mapping is analogous to the mapping (17) which transforms solutions of the free Schrödinger equation to solutions of the linear Schrödinger equation with linear or quadratic potentials.

For the case when the potential V is identically zero, ψ' is a solution of the free Schrödinger equation with the mass m replaced by the "effective mass" $m\Lambda$. For example, we may take the solution analogous to (8):

$$\psi'(x,t) = \left(\frac{m\Lambda}{2\pi i\hbar t}\right)^{\frac{n}{2}-\frac{\alpha}{4}} \frac{\Gamma\left(\frac{n}{2}-\frac{\alpha}{4}\right)}{\Gamma\left(\frac{n}{2}\right)} {}_1F_1\left(\frac{n}{2}-\frac{\alpha}{4}; \frac{n}{2}; \frac{im\Lambda|x|^2}{2\hbar t}\right) \tag{26}$$

for $0 < \alpha < 2n$ and $n \geq 2$. More generally, one can also consider mappings from solutions of the DG equations (19), (20) to solutions of a linear Schrödinger equation with one of the potentials discussed in the first part of this section and then map to solutions of the free Schrödinger equation by using the results discussed there. The composition of this sequence of mappings yields transformations of solutions of (19), (20) to solutions of free Schrödinger equations without the necessity of assuming that the linear potential in the DG equations is identically zero.

Auberson and Sabatier [8] (AS) considered the following NSE (for convenience, we set $\hbar = 1$ and $2m = 1$):

$$i\psi_t(x,t) = (-\Delta + V)\psi(x,t) + s\frac{\Delta|\psi|}{\psi}\psi(x,t), \tag{27}$$

where s is a real parameter. For $s < 1$ AS use the following linearization transformation:

$$\psi = |\psi| \exp(-i\theta), \qquad t = (1-s)^{-\frac{1}{2}} t', \qquad \theta(x,t) = (1-s)^{\frac{1}{2}} \theta'(x,t'), \tag{28}$$

which transforms equation (27) to the following linear Schrödinger equation:

$$i\psi'_{t'}(x,t') = -\Delta\psi'(x,t') + (1-s)^{-1} V(x)\psi'(x,t')$$

for the quantity

$$\psi'(x,t') = \left|\psi\left(x, (1-s)^{-\frac{1}{2}} t'\right)\right| \exp(-i\theta'(x,t')). \tag{29}$$

AS also linearize (27) when $s \geq 1$, but the linear equations thereby obtained are not Schrödinger equations so we shall not discuss them.

If we consider invariant solutions analogous to (26) for (29):

$$\psi'(x,t') = (4\pi i\hbar t')^{-\left(\frac{n}{2}-\frac{\alpha}{4}\right)} \frac{\Gamma\left(\frac{n}{2}-\frac{\alpha}{4}\right)}{\Gamma\left(\frac{n}{2}\right)}\, {}_1F_1\left(\frac{n}{2}-\frac{\alpha}{4}; \frac{n}{2}; \frac{i|x|^2}{4t'}\right),$$

then the corresponding invariant solutions of (27) are obtained from (28).

As a consequence of the analogies between the inverse gauge transformation (25) of the DG equations or of the linearization transformation (29) of the AS equation with the transformation (17) between solutions of the free Schrödinger equation and solutions of linear Schrödinger equations with linear or quadratic potentials, we see that the symmetry algebras corresponding, respectively, to the free Schrödinger and to the linearizable DG or AS equations are isomorphic.

4 Asymptotic symmetry

For general nonlinear Schrödinger equations, one does not expect the nonlocal symmetries for the equations discussed in the preceding sections to extend in an exact form because those results depended on the fact that the equations were either linear or linearizable. Lie's algorithm [3, 11] gives a general method for the investigation of local symmetries and solutions of differential equations, including nonlinear ones; and this method has been applied to many types of equations, both linear and nonlinear. However, as we have noted in the Introduction, since the symmetries that we are discussing are nonlocal and are defined in terms of pseudodifferential, rather than differential, operators; Lie's approach and related techniques are not adequate to deal with them. Because of this difficulty of extending nonlocal symmetries to general nonlinear equations, we propose to use a definition of symmetry based on a reducibility property (see Definition 1.2 and the discussion below). When there is no exact reducibility, we introduce a weaker concept of *asymptotic symmetry.*

Power-type nonlinearities

Consider an NSE of the form

$$i\partial_t\psi + \frac{1}{2m}\Delta\psi = F\left(\psi, \overline{\psi}, \partial_{x_j}\psi, \partial_{x_j}\overline{\psi}, \partial^2_{x_i x_j}\psi, \partial^2_{x_i x_j}\overline{\psi}\right) \tag{30}$$

$(i, j = l, 2, \ldots, n)$, where the nonlinear function F depends in general on the solution ψ, its complex conjugate $\overline{\psi}$, derivatives of these functions through the second order and, unless a statement is made to the contrary, we set $\hbar = 1$ in the present section.

Following Definition 1.2, we will say that an operator Q is a symmetry of equation (30) if and only if the corresponding ansatz, obtained as a solution of (3), reduces (30) to a system of PDEs in fewer independent variables or, as a limiting case, to a system of ordinary differential equations (ODEs). This approach is especially useful when one wants to extend symmetries of a system of linear equations (in the present case the free Schrödinger equation (1)) to a system of nonlinear equations.

By extending an argument in [13] for Case I with $\alpha = 1$, we conclude that the following ansatz is invariant under the algebra (4):

$$\psi(x,t) = \phi(t)g(t,x), \tag{31}$$

where ϕ is an arbitrary function of t and g has the form (8) (or (10) in the special case when $n = 3$ and $\alpha = 1$). According to Definition 1.2, we say that *equation (30) is invariant under the algebra* (4) if and only if the ansatz (31) reduces (30) to an ODE for the function $\phi(t)$. In the following definition, we introduce a concept of *asymptotic symmetry* when an exact reduction does not exist, but a reduction does exist in an asymptotic sense.

Definition 4.1. *We will say that equation (30) has the asymptotic symmetry (4) if and only if the ansatz (31) reduces (30) to an ODE for $\phi(t)$ in the asymptotic region $m|x|^2 \ll 2t$.*

In this section we will first discuss NSEs with power-type nonlinearities and then consider several cases of derivative nonlinearities. We first treat the case in which the nonlinear term in (30) is of the form:

$$F = \lambda(\overline{\psi}\psi)^k\psi, \tag{32}$$

where k denotes a positive real number (not necessarily an integer) and λ denotes a complex (coupling) constant. Lie symmetry analyses of equations of the form (30), (32) have been discussed by many authors (see [3] for a summary). In addition, many authors have investigated the existence of solutions to such equations in various Banach and Hilbert spaces. For dimensions $n \geq 3$, many of these results require that $0 < k < \dfrac{2n}{n-2}$ because the proofs use the Sobolev embedding theorem. See [15] for a summary. Our results are not subject to this restriction.

To investigate the asymptotic symmetry of (30), (32), we look for a solution of the form (31) where $g(x,t)$ is a solution of the free Schrödinger equation defined by (8):

$$g(x,t) = \left(\frac{m}{2\pi it}\right)^{\frac{n}{2}-\frac{\alpha}{4}} \frac{\Gamma\left(\frac{n}{2}-\frac{\alpha}{4}\right)}{\Gamma\left(\frac{n}{2}\right)}\, {}_1F_1\left(\frac{n}{2}-\frac{\alpha}{4}; \frac{n}{2}; \frac{im|x|^2}{2t}\right), \tag{33}$$

with $0 < \alpha < 2n$ for spatial dimensions $n \geq 2$. (If $n = 3$ and $\alpha = 1$ we may, of course, use (10) instead.) Then, substituting these expressions into (30) and (32), assuming that ϕ depends only on t, and using the fact that g satisfies the free Schrödinger equation $i\partial_t g = -\dfrac{1}{2m}\Delta g$, we obtain the following equation for ϕ:

$$\phi_t = -i\lambda(\overline{\phi}\phi)^k(\overline{g}g)^k\phi, \qquad k > 0. \tag{34}$$

Since ϕ is assumed to depend only on t, the above derivation is only consistent if the quantities $(\overline{g}g)^k$ are independent of the spatial coordinates x_j $(j = 1, 2, \ldots, n)$. This is not true in general, so we consider the limit of large t or, more precisely, values of the variables x_j $(j = 1, 2, \ldots, n)$ and t such that $m|x|^2 \ll 2t$. Then, using the small-argument expansions

$$ {}_1F_1(a; c; z) = 1 + \frac{a}{c}z + \frac{a(a+1)}{c(c+1)}\frac{z^2}{2} + \cdots \tag{35}$$

for the confluent functions, we obtain the following asymptotic result for $\overline{g}g$:

$$\overline{g}g \cong \left(\frac{m}{2\pi t}\right)^{n-\frac{\alpha}{2}} \left(\frac{\Gamma\left(\frac{n}{2}-\frac{\alpha}{4}\right)}{\Gamma\left(\frac{n}{2}\right)}\right)^2 \left(1+O\left(\left(\frac{m|x|^2}{2t}\right)^2\right)\right), \tag{36}$$

and (34) becomes, to leading order in the quantity $\frac{m|x|^2}{2t}$,

$$\phi_t \cong -i\lambda \left(\frac{m}{2\pi t}\right)^{(n-\frac{\alpha}{2})k} \left(\frac{\Gamma\left(\frac{n}{2}-\frac{\alpha}{4}\right)}{\Gamma\left(\frac{n}{2}\right)}\right)^{2k} (\overline{\phi}\phi)^k\phi. \tag{37}$$

From the form of equation (37), we find that its solution must have the form $\phi(t) = \beta\exp(-i\hbar(t))$ with β a real constant and $h(t)$ a real-valued function of t. Substitution of this expression into (37) yields an equation for $h(t)$ which has the following solution: ($\overline{\omega}$ a real constant)

$$h(t) = \lambda \left(\frac{m}{2\pi}\right)^{(n-\frac{\alpha}{2})k} \left(\frac{\Gamma\left(\frac{n}{2}-\frac{\alpha}{4}\right)\beta}{\Gamma\left(\frac{n}{2}\right)}\right)^{2k} \chi_k(t) + \overline{\omega}, \tag{38}$$

where $\chi_k(t) = t^{-(n-\frac{\alpha}{2})k+1}$ when $k \neq \left(n-\frac{\alpha}{2}\right)^{-1}$ and $= \ln t$ when $k = \left(n-\frac{\alpha}{2}\right)^{-1}$. The asymptotic solution to (30), (32) is now obtained by substitution into (31):

$$\psi(x,t) \cong \beta\exp(-i\hbar(t))g(x,t)$$

with g given by (33) and h given by (38).

We note that the asymptotic result for $|\psi| = (\overline{\psi}\psi)^{\frac{1}{2}}$ corresponds (apart from the constant β) to the asymptotic result for the solution $g(x,t)$ of the free Schrödinger equation, whereas the asymptotic value of $\operatorname{Arg}\psi = \frac{1}{2i}\ln\left(\frac{\psi}{\overline{\psi}}\right)$ contains effects of the nonlinear terms (32).

Similar results can also be obtained when the nonlinear term in (30) is a linear combination of power-type nonlinearities such as, for example:

$$F = -a_0\psi - a_1(\overline{\psi}\psi)\psi - a_2(\overline{\psi}\psi)^2\psi, \tag{39}$$

where a_j, $(j = 0,1,2)$ are real constants, and $a_2 \neq 0$. Lie symmetries of equations (30), (39) were discussed by Gagnon and Winternitz [31].

The Doebner-Goldin and related equations

We next consider asymptotic symmetry results for the DG equation (19), (20) and then discuss the relationship between these results and some others for these equations.

To show that the DG equations have asymptotic symmetry in the sense of Definition 4.1, we follow the procedure used for NSEs with power-type nonlinities and look for a solution ψ of (19), (20) of the form (31) with $g(x,t)$ defined by (33) or (10) with the appropriate powers of $\hbar$ again inserted. We obtain the following linear equation for ψ by virtue of the homogeneity property of the nonlinear functionals R_j $(j = 1,2,3,4,5)$:

$$i\hbar\phi_t = V\varphi + \frac{i\hbar D}{2}R_2(g,\overline{g})\phi + \hbar D'\sum_{j=1}^{5} c_jR_j(g,\overline{g})\phi. \tag{40}$$

Using the derivative relations $\frac{d}{dz}\,{}_1F_1(a;c;z) = \frac{a}{c}\,{}_1F_1(a{+}1;c{+}1;z)$ and the small-argument expansions (35) for the confluent functions, we obtain the following asymptotic results for the $R_j(g,\overline{g})$ $(j = 1,2,3,4,5)$:

$$R_1(g,\overline{g}) = \left(1 - \frac{\alpha}{2n}\right)\frac{mn}{\hbar t}\left(1 + O\left(\left(\frac{m|x|^2}{2\hbar t}\right)^2\right)\right),$$

$$R_2(g,\overline{g}) = -\left(1 - \frac{\alpha}{2n}\right)\frac{2\alpha}{n}\frac{m}{\hbar t}\frac{m|x|^2}{2\hbar t}\left(1 + O\left(\frac{m|x|^2}{2\hbar t}\right)\right),$$

$$R_3(g,\overline{g}) = \left(1 - \frac{\alpha}{2n}\right)^2\frac{2m}{\hbar t}\left(\frac{m|x|^2}{2\hbar t}\right) + O\left(\left(\frac{m|x|^2}{2\hbar t}\right)^2\right),$$

$$R_4(g,\overline{g}) = -\left(1 - \frac{\alpha}{2n}\right)^2\frac{2\alpha}{n(n+2)}\frac{2m}{\hbar t}\left(\frac{m|x|^2}{2\hbar t}\right)^2\left(1 + O\left(\frac{m|x|^2}{2\hbar t}\right)\right),$$

$$R_5(g,\overline{g}) = \frac{\left(1 - \frac{\alpha}{2n}\right)^2\alpha^2}{n^2(n+2)^2}\frac{2m}{\hbar t}\left(\frac{m|x|^2}{2\hbar t}\right)^3\left(1 + O\left(\frac{m|x|^2}{2\hbar t}\right)\right),$$

Thus, we see that the DG equations are asymptotically invariant in the sense of Definition 4.1 if we consider the case of an identically zero potential V and omit the R_j functionals with $j = 2,3,4,5$. Then, solution of (40) gives the following asymptotic result $(m|x|^2 \ll 2\hbar t)$:

$$\psi(x,t) \cong \kappa\exp\left(-iD'c_1\left(1 - \frac{\alpha}{2n}\right)\frac{mn}{\hbar}\ln(t)\right)g(x,t) \tag{41}$$

with κ a complex constant.

For reasons of consistency, we must show that a solution ψ of (19), (20) obtained as in (25) from a solution ψ' of (23) is consistent with our asymptotic symmetry result (41). This can be done by noting that, for the case when the potential V is identically zero, ψ' is a solution of the free Schrödinger equation with the mass m replaced by the "effective mass" $m\Lambda$ as in (26). Using the small-argument expansion (35) and related expansions for the arctangent and logarithmic functions that occur in $\operatorname{Arg}\psi'$ and $\ln(|\psi'|)$, respectively, we obtain from (25):

$$\begin{aligned}\psi(x,t) \cong{}& \left(\frac{m\Lambda}{2\pi\hbar t}\right)^{\frac{n}{2}}\frac{\Gamma\left(\frac{n}{2}-\frac{\alpha}{4}\right)}{\Gamma\left(\frac{n}{2}\right)}\exp\left(\frac{2miD}{\hbar}\ln\left(\frac{\Gamma\left(\frac{n}{2}-\frac{\alpha}{4}\right)}{\Gamma\left(\frac{n}{2}\right)}\left(\frac{m\Lambda}{2\pi\hbar t}\right)^{\frac{n}{2}-\frac{\alpha}{4}}\right)\right)\\ &\times\exp\left(-\frac{i\pi}{\Lambda}\left(\frac{n}{4}-\frac{\alpha}{8}\right) + i\left(1-\frac{\alpha}{2n}\right)\frac{m|x|^2}{2\hbar t}\right).\end{aligned} \tag{42}$$

We will compare this expression with the asymptotic symmetry result (41), which can be written in the form:

$$\begin{aligned}\psi_{\text{asym}}(x,t) \cong{}& \kappa\left(\frac{m\Lambda}{2\pi\hbar t}\right)^{\frac{n}{2}-\frac{\alpha}{4}}\frac{\Gamma\left(\frac{n}{2}-\frac{\alpha}{4}\right)}{\Gamma\left(\frac{n}{2}\right)}\exp\left(-i\left[D'c_1\left(1-\frac{\alpha}{2n}\right)\frac{mn}{\hbar}\ln(t)+\right.\right.\\ &\left.\left.+\pi\left(\frac{n}{4}-\frac{\alpha}{8}\right)\right]\right)\exp\left(i\left(1-\frac{\alpha}{2n}\right)\frac{m|x|^2}{2\hbar t}\right).\end{aligned} \tag{43}$$

The expressions (42), (43) will be compared in the spacetime domain $m|x|^2 \ll 2\hbar t$ by setting

$$\kappa = \beta \exp(-i\delta) \qquad \text{with} \quad \beta \quad \text{and} \quad \delta \quad \text{real}. \tag{44}$$

One then obtains

$$\beta = \Lambda^{\frac{n}{2}-\frac{\alpha}{4}} \tag{45}$$

and the coefficients of $\ln t$ in the exponents of (42) and (43) agree because of the equality $D = D'c_1$, which is part of the conditions (21) of DG required for linearizability. Equating the constant parts of the phases of (42) and (43) gives:

$$\delta = n\left(1 - \frac{\alpha}{2n}\right)\left[\frac{\pi}{4}\left(\Lambda^{-1} - 1\right) - \frac{mD}{\hbar}\ln\left(\frac{m\Lambda}{2\pi\hbar}\right)\right] - \frac{2mD}{\hbar}\ln\left(\frac{\Gamma\left(\frac{n}{2} - \frac{\alpha}{4}\right)}{\Gamma\left(\frac{n}{2}\right)}\right). \tag{46}$$

Thus, the solutions of the DG equations (19), (20) obtained by their gauge equivalence to solutions of the linear Schrödinger equation (23) with $V = 0$ is consistent with the asymptotic symmetry result (41) provided that the constant κ is chosen to satisfy (44)–(46).

Equations of Kostin and of Bialynicki-Birula and Mycielski

We consider NSEs of the form:

$$i\hbar\psi_t = -\frac{\hbar^2}{2m}\Delta\psi + \xi_1 \ln(\overline{\psi}\psi)\psi - \frac{\xi_2}{2i}\ln\left(\frac{\psi}{\overline{\psi}}\right)\psi, \tag{47}$$

where ξ_1 and ξ_2 are real constants. The first logarithmic term $\ln\left(|\psi|^2\right)\psi$ was originally proposed by Bialynicki-Birula and Mycielski [16] and the second logarithmic term, which involves the phase of ψ: $(2i)^{-1}\ln\left(\frac{\psi}{\overline{\psi}}\right)$, was first proposed by Kostin [17] in connection with studies of dissipation effects in quantum mechanics. It is appropriate to discuss equation (47) in this section because Doebner and Goldin have shown [7] that their equation (19), (20) extends to include the nonlinear terms in (47) when the parameters γ and Λ in the gauge transformation (22) are time-dependent. Symmetry analyses of equation (47) with $\xi_2 = 0$ have been investigated (cf. [32, 3]) and existence results for this equation (with $\xi_2 = 0$) when an appropriate class of linear potentials is also included have been summarized by Cazenave [15].

Looking for solutions of (47) in the form (31), (33); we obtain the following equation for ϕ:

$$i\hbar\phi_t = \xi_1 \ln\left(|\phi|^2|g|^2\right)\phi - \frac{\xi_2}{2i}\ln\left(\frac{\phi}{\overline{\phi}}\frac{g}{\overline{g}}\right)\phi. \tag{48}$$

Then, assuming that $m|x|^2 \ll 2\hbar t$ and using the expansions (35), we obtain (36) and

$$\frac{g}{\overline{g}} \cong (-1)^{\frac{n}{2}-\frac{\alpha}{4}}\left(1 + O\left(\frac{m|x|^2}{2\hbar t}\right)\right). \tag{49}$$

Equation (48) becomes, to leading order in the quantity $\dfrac{m|x|^2}{2\hbar t}$,

$$\phi_t \cong -\frac{i}{\hbar}\xi_1 \ln\left(\left(\frac{m}{2\pi\hbar t}\right)^{n-\frac{\alpha}{2}} |\phi|^2\right)\phi + \frac{\xi_2}{2\hbar}\ln\left(\frac{\phi}{\overline{\phi}}\right)\phi + \frac{i\xi_2}{2\hbar}\left(\frac{n}{2}-\frac{\alpha}{4}\right)\pi\phi, \tag{50}$$

where the last term on the right-hand side comes from the logarithm in (48) and vanishes $\dfrac{n}{2}-\dfrac{\alpha}{4}$ is an even integer. From the form of eq.(50), we find that its solution must have the form $\phi(t) = \beta\exp(i\delta(t))$ with β a real constant and $\delta(t)$ a real-valued function. Substitution of this expression into (50) yields the following equation for $\delta(t)$:

$$\delta_t = -\frac{\xi_1}{\hbar}\ln\left(\beta^2\left(\frac{m}{2\pi\hbar t}\right)^{n-\frac{\alpha}{2}}\left(\frac{\Gamma\left(\frac{n}{2}-\alpha 4\right)}{\Gamma\left(\frac{n}{2}\right)}\right)^2\right) + \frac{\xi_2\delta}{\hbar} + \frac{\xi_2}{\hbar}\left(\frac{n}{4}-\frac{\alpha}{8}\right)\pi. \tag{51}$$

The solution of this equation has different forms according as ξ_2 is zero or nonzero.

Case A. $\xi_2 = 0$.

$$\begin{aligned}\phi(t) &= \beta\exp\left(-\frac{i\xi_1 t}{\hbar}\left[\ln\left(\beta^2\left(\frac{\Gamma\left(\frac{n}{2}-\frac{\alpha}{4}\right)}{\Gamma\left(\frac{n}{2}\right)}\right)^2\right) + \left(n-\frac{\alpha}{2}\right)\left(\ln\left(\frac{m}{2\pi\hbar}\right)+1\right)\right]\right)\\ &\quad\times\exp\left(\frac{i\xi_1}{\hbar}\left(n-\frac{\alpha}{2}\right)t\ln t + i\zeta\right), \qquad \zeta = \text{real const.}\end{aligned} \tag{52}$$

Case B. $\xi_2 \neq 0$ and $\dfrac{n}{2}-\dfrac{\alpha}{4} \neq$ an even integer

In this case, equation (51) can be written in the form:

$$\begin{aligned}\frac{d}{dt}\left(\exp\left(-\frac{\xi_2 t}{\hbar}\right)\delta(t)\right) &= \frac{\xi_1}{\hbar}\left(n-\frac{\alpha}{2}\right)\ln t\exp\left(-\frac{\xi_2 t}{\hbar}\right) + \frac{\xi_2\pi}{\hbar}\left(\frac{n}{4}-\frac{\alpha}{8}\right)\exp\left(-\frac{\xi_2 t}{\hbar}\right)\\ &\quad-\frac{\xi_1}{\hbar}\ln\left(\beta^2\left(\frac{m}{2\pi\hbar}\right)^{n-\frac{\alpha}{2}}\left(\frac{\Gamma\left(\frac{n}{2}-\frac{\alpha}{4}\right)}{\Gamma\left(\frac{n}{2}\right)}\right)^2\right)\exp\left(-\frac{\xi_2 t}{\hbar}\right).\end{aligned}$$

Integrating this equation from $t^* > 0$ to $+\infty$, and using integration by parts for the $\ln t$ term, we obtain

$$\begin{aligned}\delta(t^*) &= \frac{\xi_1}{\xi_2}\ln\left(\beta^2\left(\frac{m}{2\pi\hbar}\right)^{n-\frac{\alpha}{2}}\left(\frac{\Gamma\left(\frac{n}{2}-\frac{\alpha}{4}\right)}{\Gamma\left(\frac{n}{2}\right)}\right)^2\right) - \frac{\xi_1}{\xi_2}\left(n-\frac{\alpha}{2}\right)\ln t^* - \left(\frac{n}{4}-\frac{\alpha}{8}\right)\pi\\ &\quad-\frac{\xi_1}{\xi_2}\left(n-\frac{\alpha}{2}\right)\exp\left(\frac{\xi_2 t^*}{\hbar}\right)E_1\left(\frac{\xi_2 t^*}{\hbar}\right)\end{aligned}$$

and

$$\begin{aligned}\phi(t) &= \beta\exp(i\delta(t)) = \beta\exp\left(i\frac{\xi_1}{\xi_2}\ln\left(\beta^2\left(\frac{m}{2\pi\hbar}\right)^{n-\frac{\alpha}{2}}\left(\frac{\Gamma\left(\frac{n}{2}-\frac{\alpha}{4}\right)}{\Gamma\left(\frac{n}{2}\right)}\right)^2\right)\right)\\ &\times\exp\left(-i\frac{\xi_1}{\xi_2}\left(n-\frac{\alpha}{2}\right)\ln t - i\frac{\xi_1}{\xi_2}\left(n-\frac{\alpha}{2}\right)\exp\left(\frac{\xi_2 t}{\hbar}\right)E_1\left(\frac{\xi_2 t}{\hbar}\right) - i\left(\frac{n}{4}-\frac{\alpha}{8}\right)\pi\right)\end{aligned} \tag{53}$$

in terms of the exponential integrals $E_1(x) = -\operatorname{Ei}(-x) = \int_x^\infty \exp(-t)t^{-1}dt$ (where the integral is understood as a principal value integral when $x < 0$). In Case B, the terms involving the quantity $\left(\frac{n}{4} - \frac{\alpha}{8}\right)\pi$ are absent when $\frac{n}{2} - \frac{\alpha}{4}$ is an even integer. The final asymptotic solutions to (47) are obtained to leading order in the quantity $\frac{m|x|^2}{2\hbar t}$ by substituting (52) and (53) into the equation $\psi(x,t) = \phi(t)g(x,t)$, where $g(x,t)$ is given by (33).

5 Concluding remarks

We have shown that nonlocal Lorentz symmetry, previously known to be valid for free-particle Schrödinger equations [4, 13], is also valid in a modified form for the Galilei-invariant linear wave equations of Hurley [5], which describe the time evolution of free quantum particles with arbitrary spin. In Section 3 we have discussed similar results for linear Schrödinger equations with linear and harmonic oscillator potentials with arbitrary time-dependent coefficients. For nonlinear equations, we have shown that a subset of the nonlinear Schrödinger equations introduced by Doebner and Goldin [6, 7] in connection with studies of dissipative effects in quantum mechanics as well as some of the related equations discussed by Auberson and Sabatier [26, 8] have exact forms of these symmetries. Moreover, we have also shown that several classes of nonlinear Schrödinger equations – those with power-type nonlinearities as well as the full set of equations proposed by Doebner and Goldin – have asymptotic nonlocal symmetry in the sense described in the present paper.

The question of the nonlocal symmetry of nontrivial (i.e., nonfree) multiparticle Schrödinger equations is open. In general, we expect that the concept of nonlocal symmetry may be helpful in the analysis of such equations.

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References

[1] Niederer U., The Maximal Kinematical Invariance Group of the Free Schrödinger Equation, *Helv. Phys. Acta*, 1972, V.45, N 5, 802–810.

[2] Hagen C.R., Scale and Conformal Transformations in Galilean-Covariant Field Theory, *Phys. Rev. D*, 1972, V.5, N 2, 377–388.

[3] Fushchych W.I., Shtelen W.M. and Serov N.I., Symmetry Analysis and Exact Solutions of Equations of Nonlinear Mathematical Physics, Kluwer, Dordrecht, 1993.

[4] Fushchych W.I. and Segeda Yu.N., On a New Invariance Algebra of the Free Schrödinger Equation, *Dokl. Akad. Nauk SSSR*, 1977, V.232, N 4, 800–801; *Sov. Phys. Dokl.*, 1977, V.22, N 2, 76–77.

[5] Hurley W.J., Nonrelativistic Quantum Mechanics for Particles with Arbitrary Spin, *Phys. Rev. D*, 1971, V.3, N 10, 2339–2347.

[6] Doebner H.-D. and Goldin G.A., Properties of Nonlinear Schrödinger Equations Associated with Diffeomorphism Group Representations, *J. Phys. A: Math. Gen.*, 1994, V.27, N 5. 1771–1780.

[7] Doebner H.-D. and Goldin G.A., Introducing Nonlinear Gauge Transformations in a Family on Nonlinear Schrödinger Equations, *Phys. Rev. A*, 1996, V.54, N 5, 3764–3771.

[8] Auberson G. and Sabatier P.C., On a Class of Homogeneous Nonlinear Schrödinger Equations, *J. Math. Phys.*, 1994, V.35, N 8, 4028–4040.

[9] Niederer U., The Maximal Kinematical Invariance Group of the Harmonic Oscillator, *Helv. Phys. Acta*, 1973, V.46, N 2, 191–200.

[10] Fushchych W.I., On Additional Invariance of Relativistic Equations of Motion, *Teoret. Mat. Fiz.*, 1971, V.7, N 1, 3–12.

[11] Bluman G. and Kumei S., Symmetries and Differential Equations, Applied Mathematical Sciences, V.81, Springer, New York, 1989.

[12] Fushchych W.I. and Nikitin A.G., Symmetries of Maxwell's Equations, D. Reidel, Dordrecht, 1987; Symmetry of the Equations of Quantum Mechanics, Allerton Press, New York, 1994.

[13] Shtelen V.M., On Solutions of Schrödinger Equations Invariant with Respect to the Lorentz Algebra, in Algebraic-Theoretic Analysis of the Equations of Mathematical Physics, Editor W.I. Fushchych, Institute of Mathematics, Ukrainian Academy of Sciences, Kyiv, 1990, 109–112.

[14] Amrein W.O., Jauch J.M. and Sinha K.B., Scattering Theory in Quantum Mechanics, W.A. Benjamin, New York, 1977.

[15] Cazenave T., An Introduction to Nonlinear Schrödinger Equations, Textos de Metodos Matematicos 22, Rio de Janeiro, 1989.

[16] Bialynicki-Birula I. and Mycielski J., Nonlinear Wave Mechanics, *Ann. Phys.*, 1976, V.100, N 1–2, 62–93.

[17] Kostin M.D., On the Schrödinger-Langevin Equation, *J. Chem. Phys.*, 1972, V.57, N 9, 3589–3591; Friction and Dissipative Phenomena in Quantum Mechanics, *J. Statist. Phys.*, 1975, V.12, N 2, 145–151.

[18] Zachary W.W., On Shtelen's Solution of the Free Linear Schrödinger Equation, *J. Nonlin. Math. Phys.*, 1997, V.4, N 3–4, 377–382.

[19] Levy–Leblond J.M., Nonrelativistic Particles and Wave Equations, *Comm. Math. Phys.*, 1967, V.6, N 4, 286–311.

[20] Hagen C.R., The Bargmann-Wigner Method in Galilean Relativity, *Comm. Math. Phys.*, 1970, V.18, N 2, 97–108.

[21] Nieto M.M. and Truax D.R., Displacement-Operator Squeezed States. I. Time-Dependent Systems having Isomorphic Symmetry Algebras, *J. Math. Phys.*, 1997, V.38, N 1, 84–97; II. Examples of Time-Dependent Systems having Isomorphic Symmetry Algebras, *Ibid.*, 1997, V.38, N 1, 98–114.

[22] Truax D.R., Symmetry of Time-Dependent Schrödinger Equations. I. A Classification of Time-Dependent Potentials by Their Maximal Kinematical Algebras, *J. Math. Phys.*, 1981, V.22, N 9, 1959–1964.

[23] Bluman G., On the Transformation of Diffusion Processes into the Wiener Process, *SIAM J. Appl. Math.*, 1980, V.39, N 2, 238–247.

[24] Bluman G. and Shtelen V.M., New Classes of Schrödinger Equations Equivalent to the Free Particle Equation Through Non-local Transformation, *J. Phys. A: Math. Gen.*, 1996, V.29, N 15, 4473–4480.

[25] Feynman R.P. and Hibbs A.R., Quantum Mechanics and Path Integrals, McGraw-Hm, New York, 1965.

[26] Sabatier P.C., Multidimensional Nonlinear Schrödinger Equations with Exponentially Confined Solutions, *Inverse Problems*, 1990, V.6, N 5, L47–L53.

[27] Auberson G., A Class of Homogeneous Evolution Equations with Stable, Localized Solutions in any Dimension, *J. Math. Phys.*, 1997, V.38, N 9, 4576–4593.

[28] Nattermann P., Symmetry, Local Linearization, and Gauge Classification of the Doebner-Goldin Equation, *Reports Math. Phys.*, 1995, V.36, N 2-3, 387–402.

[29] Fushchych W., Chopyk V., Nattermann P. and Scherer W., Symmetries and Reductions of Nonlinear Schrödinger Equations of Doebner-Goldin Type, *Reports Math. Phys.*, 1995, V.35, N 1, 129–138.

[30] Nattermann P. and Zhdanov R., On Integrable Doebner-Goldin Equations, *J. Phys. A: Math. Gen.*, 1996, V.29, N 11, 2869–2886.

[31] Gagnon L. and Wintemitz P., Lie symmetries of a Generalised Non-linear Schrödinger Equation: I. The Symmetry Group and its Subgroups, *J. Phys. A: Math. Gen.*, 1988, V.21, N 7, 1493–1511; Exact Solutions of the Cubic and Quintic Nonlinear Schrödinger Equation for a Cylindrical Geometry, *Phys. Rev. A*, 1989, V.39, N 1, 296–306.

[32] Fushchych W.I. and Chopyk V.I., Symmetry and Non-Lie Reduction of the Nonlinear Schrödinger Equation, *Ukr. Mat. Zh.*, 1993, V.45, N 4, 539–551; *Ukrain. Math. J.*, 1993, V.45, N 4, 581–597.