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Solving Simultaneously Dirac and Ricatti Equations

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Abstract

We analyse the behaviour of the Dirac equation in $d = 1 + 1$ with Lorentz scalar potential. As the system is known to provide a physical realization of supersymmetric quantum mechanics, we take advantage of the factorization method in order to enlarge the restricted class of solvable problems. To be precise, it suffices to integrate a Ricatti equation to construct one-parameter families of solvable potentials. To illustrate the procedure in a simple but relevant context, we resort to a model which has proved useful in showing the phenomenon of fermion number fractionalization.

1 Introduction.

When solving the Dirac equation in $d = 1+1$ with Lorentz scalar potential, the underlying supersymmetric structure is crucial. As the system provides a physical realization of supersymmetric quantum mechanics (susy qm henceforth), the problem reduces itself to a pair of Schrödinger-like hamiltonians related by means of supersymmetry. In doing so, both operators share identical energy spectra unless the Dirac equation allows the existence of a zero-mode (i.e. a normalizable solution with null energy). Accordingly, we would like to show how the methods used in susy qm to obtain analytical solutions of the Schrödinger equation can be extended to the Dirac case. As a relevant example of the aforementioned models one can consider the problem of the Dirac hamiltonian with a scalar potential $\phi(x)$ which characterizes a position dependent band gap. In such a case the Dirac equation serves to describe the behaviour of a x-dependent valence and conduction-band edge of semiconductors in the Brillouin zone [1]. On the other hand, the modified Korteweg-de Vries (mKdV) equation exhibits a striking relation to the Dirac equation with a time-dependent scalar potential. If $\phi(x, t)$ represents a solution of the mKdV equation, the spectrum of the (time-dependent) Dirac operator with the scalar potential $\phi(x, t)$ is time independent. In other words, the Dirac operators at different times are unitarily equivalent if the potential itself evolves according to the mKdV equation [2].

Particular interest acquire however the models in which the Lorentz potential can be looked upon as the kink-like static solution derived from a classical field theory. When going to the quantum level, such solutions give rise in general to new sectors beyond those that are seen in the standard perturbative regime. These classical solutions have received the names of solitary waves, energy lumps, kinks or solitons. In addition, they appear in any number of dimensions: the ϕ^4 kink for $d = 1+1$, the Nielsen-Olesen vortex if $d = 2+1$ or the celebrated 't-Hooft-Polyakov monopole in $d = 3+1$. (The brave reader can enjoy the superb description of the subject contained in the book of Rajaraman [3]). Though more massive than the elementary excitations, the soliton-like configurations become stable since an infinite energy barrier separates them from the ordinary sector. The stability is reinforced by the existence of a topological conserved charge which does not arise by Noether's theorem from a well-behaved symmetry of the lagrangian, but takes into account the large distance behaviour of the field configuration (hence the topological character of the solutions themselves).

On the other hand, dramatic effects appear when a Fermi field is coupled to the soliton. Among other things, such models have proved quite useful in the context of the fermion number fractionalization which has been observed in certain class of polymers like polyacetylene [3]. Describing the soliton by a non-dynamical c-number field, the behaviour of the fermions is governed in this first quantized approach by a Dirac operator which includes the background provided by the scalar field. If one considers interactions enjoying charge conjugation symmetry, the states of positive and negative energies match between them. In addition, the spectrum can exhibit also self-conjugate zero-energy solutions which as a last resort assume the responsibility for the fractionalization of the fermion number. These results are nicely described in topological terms: no matter what the details of the model, the existence of zero-modes is rigorously predicted by general arguments concerning differential operators. It may be worth spelling that susy qm represents in its own right an alternative tool in the analysis of this connection between physics and topology (see further on).

The organization of the article is as follows. Section 2 takes care of the Dirac equation in $d = 1 + 1$ with static Lorentz scalar potential. There we carefully describe the way in which the Ricatti equation enables us to construct one-parameter families of isospectral Lorentz scalar potentials for the Dirac case. To illustrate the procedure in a simple but relevant context, we resort in Section 3 to the problem consisting in the analysis of the Dirac equation on the background provided by the kink-like configuration of the ϕ^4 model. (The first appendix includes a capsule introduction to susy qm while the second one outlines the properties of the eigenfunctions associated with the Pösch-Teller potential).

2 Supersymmetric structure of the Dirac equation.

Although there are several applications of susy qm in the context of the Dirac equation, we discuss only the underlying supersymmetric structure of such equation in $d = 1 + 1$ with static Lorentz scalar potential represented by $f(x)$. In doing so, we return to the ideas exposed by Cooper *et al.* some years ago [4]. The purpose of this section is to show how we can take advantage of susy qm in order to enlarge in a systematic way the restricted class of solvable models. To be precise, it is the general solution of a Ricatti equation that provides us with a one-parameter family of potentials which serve to extend the first results derived from $f(x)$. To begin with, let us give some words about the notation used in the next two sections. Space-time coordinates are represented by x^μ ($\mu = 0, 1; x^0 = t, x^1 = x$). As regards the metric tensor we have $g_{\mu\nu} = g^{\mu\nu}$, where $g_{00} = -g_{11} = 1$ and $g_{\mu\nu} = 0$ otherwise. Moreover ∂_μ stands for space-time derivatives $\partial/\partial x^\mu$. When considering a fermion field $\psi(x, t)$ moving in the static Lorentz scalar potential $f(x)$, the behaviour of the system is governed by the Dirac equation

$$[i\gamma^\mu \partial_\mu - f(x)]\, \psi(x, t) = 0. \tag{1}$$

Letting

$$\psi(x, t) = \psi(x) \exp(-i\omega t)$$

the proper Dirac equation reduces to

$$\gamma^0 \omega\psi(x) + i\gamma^1 \psi'(x) - f(x)\psi(x) = 0,$$

where the prime denotes as usual derivative with respect to the spatial coordinate. Upon choosing now

$$\gamma^0 = \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \gamma^1 = i\sigma^3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \qquad \psi(x) = \begin{pmatrix} \psi_-(x) \\ \psi_+(x) \end{pmatrix},$$

we get the pair of coupled equations

$$A\psi_-(x) = \omega\psi_+(x), \qquad A^\dagger \psi_+(x) = \omega\psi_-(x),$$

where

$$A = \frac{d}{dx} + f(x), \qquad A^\dagger = -\frac{d}{dx} + f(x).$$

It remains to decouple the equations, i.e.

$$A^\dagger A\psi_-(x) = \omega^2 \psi_-(x), \qquad AA^\dagger \psi_+(x) = \omega^2 \psi_+(x)$$

which are of the form of susy qm (see appendix A for a brief introduction to the subject), namely

$$-\psi_-{}''(x) + \left[f^2(x) - f'(x)\right] \psi_-(x) = \omega^2 \psi_-(x), \tag{2}$$

$$-\psi_+{}''(x) + \left[f^2(x) + f'(x)\right] \psi_+(x) = \omega^2 \psi_+(x), \tag{3}$$

where the Lorentz scalar potential $f(x)$ is just the superpotential function. To sum up, we substitute the Dirac equation on behalf of the two Schrödinger-like equations of a genuine susy qm model. In such a case, the components $\psi_-(x)$ and $\psi_+(x)$ represent the eigenfunctions of the hamiltonians $H_- = A^\dagger A$ and $H_+ = AA^\dagger$. It is customarily assumed that whenever the Schrödinger equations for potentials $V_\pm = f^2(x) \pm f'(x)$ are solvable, then there always exists a static Lorentz scalar potential $f(x)$ for which the corresponding Dirac equation is also exactly solvable. As expected, the magic of supersymmetry implies that both hamiltonians H_- and H_+ become solvable simultaneously since the action of the supercharges establish the right connection between the eigenfunctions $\psi_-(x)$ and $\psi_+(x)$. The spectrum of energies is of course degenerate except for the hypothetical zero-mode.

At this point we can recall the way in which the topological behaviour of the superpotential $f(x)$ dictates the existence (absence) of zero-modes. It is customarily assumed that supersymmetry, a new kind of symmetry which puts together both bosonic and fermionic degrees of freedom, represents one of the most ambitious attempts recently made to achieve a trustworthy description of nature. For better or worse, we have no experimental evidence for such a novel symmetry being exactly realized in physics so that one of the most pursued goals in the last two decades has been to describe a suitable mechanism by means of which supersymmetry shows up itself spontaneously broken. As susy qm constitutes an excellent laboratory to analyse the hypothetical spontaneous breaking phenomenon, Witten introduced an order parameter (henceforth the *Witten index* Δ) which in general serves to shed light on this subject. As a matter of fact the aforementioned *index* Δ reads [5]

$$\Delta(\beta) = \mathrm{Tr}\left[\exp(-\beta H_-) - \exp(-\beta H_+)\right],$$

where β stands for a regularization parameter necessary when considering susy qm with scattering states. In doing so, $\Delta(\beta)$ should represent a measurement of the difference of $\psi_-(x)$ and $\psi_+(x)$ modes all of them with zero-energy since for positive energies duplication occurs between the two parts of the model. In order not to clutter the article, we simply point out that when taking into account the subtleties associated with the counting of states in the continuous part of the spectrum the *Witten index* $\Delta(\beta)$ satisfies the equation [6]

$$\frac{d\Delta(\beta)}{d\beta} = \frac{1}{\sqrt{4\pi\beta}}\left[f_+ \exp\left(-\beta f_+^2\right) - f_- \exp\left(-\beta f_-^2\right)\right], \tag{4}$$

where $f_\pm$ represent the asymptotic values of the superpotential $f(x)$ as $x \to \pm\infty$. Although sometimes cumbersome final indices are obtained if any of the two limits f_- or f_+ becomes null, it is the case that whenever we observe a change of sign when going from f_- to f_+, the integration of (4) gives

$$\Delta(\beta) = \pm\left[\frac{1}{2}\Phi\left(f_+\sqrt{\beta}\right) + \frac{1}{2}\Phi\left(f_-\sqrt{\beta}\right)\right]$$

being Φ the Fresnel probability function. Now it suffices the limit $\beta \to \infty$ to reveal the topological roots of the model: the *Witten index* amounts to ± 1 so that the asymptotic behaviour of $f(x)$ dictates the existence (absence) of the non-degenerate zero-mode (see Appendix A). From a qualitative point of view, this result can also be seen as follows: a normalizable zero-energy eigenfunction exists for the superpotentials just exhibiting the right behaviour at infinity. In summary, the *odd-like* superpotentials yields a zero-mode while the *even-like* ones are not capable of producing such a state.

However, the above discussion is only part of the whole story. Once we assume the existence of a solvable quantum mechanical problem for the pair of one-dimensional Schrödinger-like equations written in (2) and (3), different procedures can be profitably used for generating one-parameter families of models which share exactly the same set of eigenvalues. The reasoning becomes much more accesible when resorting to susy qm in the spirit of the factorization schemes exposed by Mielnik [7]. To fix the ideas, let us take $f_+ > 0$ and $f_- < 0$ so that a well-behaved zero-mode exists associated with H_-. In doing so, the energy spectrum of H_+ is positive whereas H_- starts from $E = 0$. Being $\psi_{-0}(x)$ precisely the zero-energy eigenfunction of H_-, the static Lorentz scalar potential $f(x)$ appears as the logarithmic derivative of such a state, i.e. $f(x) = -\psi'_{-0}(x)/\psi_{-0}(x)$. At this stage, we ask ourselves whether the factorization $H_+ = AA^\dagger$ is unique or can be more general than usually realized. Writing now a new factorization like

$$H_+ = BB^\dagger$$

for

$$B = \frac{d}{dx} + F(x), \qquad B^\dagger = -\frac{d}{dx} + F(x)$$

the function $F(x)$ must satisfy the Ricatti-like equation

$$F'(x) + F^2(x) = V_+(x). \tag{5}$$

Whenever we know a particular solution of (5) (the potential $f(x)$ itself!), the general solution can be obtained by putting [8]

$$F(x) = f(x) + \phi(x)$$

so that

$$\phi'(x) + \phi^2(x) + 2f(x)\phi(x) = 0. \tag{6}$$

Now it proves convenient the introduction of $\rho(x) = 1/\phi(x)$ to transform (6) into a first-order differential equation like

$$-\rho'(x) + 2f(x)\rho(x) + 1 = 0$$

whose general solution corresponds to

$$\rho(x) = \exp\left(\int_{-\infty}^{x} 2f(t)\,dt\right)\left[\lambda + \int_{-\infty}^{x} \exp -\left(\int_{-\infty}^{t} 2f(z)\,dz\right) dt\right],$$

where in principle λ is a real number. Bearing in mind the relation between $f(x)$ and the zero-mode $\psi'_{-0}(x)$, we get for the function $F(x)$ the useful expression

$$F(x) = f(x) + \frac{\psi^2_{-0}(x)}{\left(\lambda + \int_{-\infty}^{x} \psi^2_{-0}(t)\,dt\right)},$$

whenever the choice for λ avoids any singularity along the real axis. To sum up, we go from the *small potential* $f(x)$ to the *large potential* represented by $F(x)$ just by solving in the standard way a Ricatti differential equation. In doing so, we can take advantage again of the commutation formula of susy qm to get a family of hamiltonians $\tilde{H}$, i.e.

$$\tilde{H} = B^\dagger B$$

whose close relation with H_- can be explained as follows. Except for the zero-energy eigenstate, the potential $\tilde{V}(x)$ included in $\tilde{H}$, namely

$$\tilde{V}(x) = V_+(x) - 2\frac{d}{dx}\left[f(x) + \frac{\psi^2_{-0}(x)}{\left(\lambda + \int_{-\infty}^{x} \psi^2_{-0}(t)\,dt\right)}\right]$$

becomes isospectral with respect to the $V_+(x)$ itself. As anticipated in the second section, the zero-modes are found from first-order differential operators so that in our case it suffices to consider

$$\left(\frac{d}{dx} + F(x)\right) \tilde{\psi}_0(x) = 0$$

to obtain

$$\tilde{\psi}_0(x) = \frac{\sqrt{\lambda(\lambda+1)}\psi_{-0}(x)}{\lambda + \int_{-\infty}^{x} \psi_{-0}^2(t)\, dt}$$

whenever $\lambda < -1$ and $\lambda > 0$ in order to guarantee the right normalization of the state. In this regards, the procedure may be understood as a *renormalization of the zero-mode*, leading to a one-parameter of hamiltonians $\tilde{H}$ isospectral with the original H_-. Returning to the Dirac equation itself, the situation is both surprising and unexpectedly subtle. First of all, we start from a static Lorentz scalar potential $f(x)$ which yields solvable Schrödinger-like potentials $V_\pm(x) = f^2(x) \pm f'(x)$. Afterwards, we enlarge in a systematic way the restricted class of such solvable models just by going from the *small potential* $f(x)$ to the *large potential* $F(x)$ through the right combination of the commutation formula of susy qm and the general theory of Ricatti equations. In addition, the down components of the spinor remain unchanged whereas the transformation $\psi_-(x) \to \tilde{\psi}(x)$ must be carried out as far as the upper components are concerned. In any case, we avoid the difficulties associated with the handling of the potential $\tilde{V}(x)$ since the $\tilde{\psi}(x)$ themselves can be obtained from $\psi_+(x)$ by the action of the operator $B^\dagger$.

3 Illustration: kinks in the presence of Fermi fields.

To illustrate the above formalism in a simple but physically relevant scenario, we resort now to the Dirac equation in the background provided by the kink-like configuration of the ϕ^4 model. Such system has proved quite useful in the context of the so-called fractionalization of the fermion number which can be observed in polymers like the polyacetylene. By any standard, the search for classical solutions represents nowadays one of the most prolific areas in quantum field theory. The interest is mainly motivated by the belief that such classical configurations, which of course make stationary the action, may shed light on the properties of the system at issue. In this section we restrict ourselves to non-dissipative configurations with finite energy because they constitute the natural candidates to describe new sectors of the model beyond the perturbative regime. On the other hand, dramatic effects like the fractionalization of the fermion number appear when a Dirac field is introduced to the kink itself. At this stage, it should be interesting to review the pioneering model of Jackiw and Rebbi [9] in the light of the formalism exposed in the previous section. For reference, we recall that the behaviour of the system is governed by the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\, \partial_\mu\phi\partial^\mu\phi - \frac{1}{2}\left(\phi^2 - 1\right)^2 + i\bar{\psi}\gamma^\mu\partial_\mu\psi - 2\phi\bar{\psi}\psi, \tag{7}$$

where as usual $\phi(x)$ and $\psi(x)$ stand for the boson and the fermion fields respectively. Notice that for the sake of simplicity the coupling constants have been absorbed in the fields themselves. Unless otherwise noted, it suffices here to discuss a first quantized approach where the fermion moves in the non-dynamical c-number field provided by the static kink $\phi_k(x)$. In doing so, the Dirac equation derived from (7) reads

$$i\gamma^\mu \partial_\mu \psi(x,t) - 2\phi_k(x)\psi(x,t) = 0$$

which is of the form written in (1). In other words, the kink-like profile represents in this case the static Lorentz scalar potential. At this stage, it is customarily assumed the existence of the well-known kink of the ϕ^4 model, namely [3] $\phi_k(x) = \tanh x$ a topological configuration which smoothly interpolates between the constant configurations $\phi_- = -1$ and $\phi_+ = 1$. On comparing now with the general formalism of the second section, we face a standard Dirac problem with $f(x) = 2\tanh x$ whose properties (spectrum and eigenfunctions) are well studied (see Appendix B). For reference, we simply point out that in this case we have

$$V_-(x) = 4 - \frac{6}{\cosh^2 x}, \qquad V_+(x) = 4 - \frac{2}{\cosh^2 x}$$

while the zero-mode $\psi_{-0}(x)$ reads

$$\psi_{-0}(x) = \frac{\sqrt{3}}{2}\frac{1}{\cosh^2 x}.$$

Now we can write that

$$F(x) = 2\tanh x + \frac{3\cosh^{-4} x}{4\lambda + 3\tanh x - \tanh^3 x + 2}.$$

As regards the one-parameter family of potentials $\tilde{V}(x)$ we have

$$\tilde{V}(x) = 4 - \frac{6}{\cosh^2 x} - 2\frac{d}{dx}\left[\frac{3\cosh^{-4} x}{4\lambda + 3\tanh x - \tanh^3 x + 2}\right]$$

while the *renormalization of the zero-mode* translates into

$$\tilde{\psi}_0(x) = \frac{2\sqrt{3}\sqrt{\lambda(\lambda+1)}\cosh^{-2} x}{4\lambda + 3\tanh x - \tanh^3 x + 2}.$$

The procedure closes itself when writing the spinor solutions of the *large potential* $F(x)$ in terms of the components $\psi_+(x)$ and $\tilde{\psi}(x)$ according to the general method.

4 Conclusions.

In this article we have considered in detail the procedure for obtaining exact energy eigenvalues and eigenfunctions for the Dirac equation involving a Lorentz scalar potential in $d = 1 + 1$. The generic model is useful in different branches of physics, ranging from condensed matter to nonlinear equations. For instance, it serves to study the fermion number fractionalization observed in certain class of polymers like polyacetylene, the position dependent band gap in semiconductors or the theory of solitons of the modified Korteweg-de Vries equation. Taking into account the way in which the system itself provides us with a realization of supersymmetric quantum mechanics, we face ultimately a pair of Schrödinger-like hamiltonians. As expected, the two aforementioned operators share the energy spectra except for the hypothetical existence of a zero-mode, i.e. a normalizable solution with $E = 0$.

In addition, the so-called topological techniques have proved quite useful in modern physics. When we need to analyze the spectrum of a well-behaved linear differential operator, the zero-modes are the most crucial ones for applications. To determine the properties of these modes we can resort to the results contained in the *index theorems*: no matter what the local details of the model at issue, the existence of zero-modes depends on the global behaviour of the system (hence the topological character of the procedure itself). Taking advantage of the structure associated with susy qm, it suffices to consider a Ricatti equation in order to enlarge the restricted class of problems whose eigenvalues and eigenfunctions can be presented in closed form. To sum up, whenever we have a pair of solvable Schrödinger-like equations then there also exists a one-parameter family of Lorentz scalar potentials for which the Dirac equation is exactly solvable.

Appendix A

In this appendix we present a brief introduction of one-dimensional susy qm, as tailored to our needs. In its most simple formulation the model deals with a time-independent hamiltonian matrix H_s of the form

$$H_s = \begin{pmatrix} H_- & 0 \\ 0 & H_+ \end{pmatrix}$$

which contains both one-particle hamiltonians H_- and H_+. This operator H_s is part of a graded algebra (superalgebra in the jargon of physics) whose closure is achieved by means of the right mixing of commutation and anti-commutation relations. At this point, it is customarily assumed the existence of the so-called superpotential function $W(x)$ which gives rise to the couple of first-order differential operators A, $A^\dagger$ given by

$$A = \frac{d}{dx} + W(x), \qquad A^\dagger = -\frac{d}{dx} + W(x).$$

Now we can write the supercharges Q, $Q^\dagger$, i.e.

$$Q = \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix}, \qquad Q^\dagger = \begin{pmatrix} 0 & A^\dagger \\ 0 & 0 \end{pmatrix}$$

which together with the hamiltonian H_s constitute the superalgebra of susy qm, namely

$$H_s = \{Q, Q^\dagger\}, \qquad [H_s, Q] = \left[H_s, Q^\dagger\right] = 0, \qquad \{Q, Q\} = \{Q^\dagger, Q^\dagger\} = 0.$$

In more physical terms susy qm can be understood as a matrix hamiltonian H_s acting on two-component wave-functions $\psi(x)$ like

$$\psi(x) = \begin{pmatrix} \psi_-(x) \\ \psi_+(x) \end{pmatrix},$$

where it should be emphasized that H_+ is obtained from H_- just by reversing the order of the operators A and $A^\dagger$. In other words

$$H_- = A^\dagger A, \qquad H_+ = A A^\dagger$$

so that the two one-particle hamiltonians H_- and H_+ read

$$H_\pm = -\frac{d^2}{dx^2} + V_\pm(x)$$

for

$$V_\pm(x) = W^2(x) \pm W'(x),$$

where the prime denotes derivative with respect to the spatial coordinate. It is often the case to refer to the pair of potentials $V_-(x)$ and $V_+(x)$ as supersymmetric partners in accordance with the standard susy qm language. To finish this brief exposition, let us outline the main properties of our model as far as the standard realization of susy qm is concerned. First of all, the energy eigenvalues of both H_- and H_+ are positive semi-definite since the eigenkets $\psi_-(x)$ and $\psi_+(x)$ verify in fact the relations

$$\langle\psi_-|H_-|\psi_+\rangle = \|A|\psi_-\rangle\|^2, \qquad \langle\psi_+|H_+|\psi_-\rangle = \|A^\dagger|\psi_-\rangle\|^2,$$

where $\||\psi_\pm\rangle\|$ represents the norm of the eigenstate $|\psi_\pm\rangle$. As regards the hypothetical zero-energy solutions $\psi_{-0}(x)$ or $\psi_{+0}(x)$ we have that

$$A\psi_{-0}(x) = 0, \qquad A^\dagger\psi_{+0}(x) = 0.$$

In doing so, they can be obtained by solving first-order differential equations so that we write the formal solutions

$$\psi_{\pm 0}(x) \sim \exp\left(\pm\int^x W(y)\,dy\right). \tag{8}$$

As any well respected normalizable zero-energy eigenfunction requires boundary conditions of the type $\psi_{\pm 0} \to 0$ as $x \to \pm\infty$, the meaning of (8) is clear: the zero-mode, if any, must be non-degenerate. The above argument illuminates the framework in which supersymmetry itself appears either unbroken or spontaneously broken. To be more specific, if susy remains exact the energy of the non-degenerate ground-state is zero while the spontaneous breaking phenomenon translates into twofold ground-state. On the other hand, all other energy eigenvalues appear duplicated. When considering the eigenket $\psi_-(x)$ with energy E_-, i.e.

$$H_-\psi_-(x) = E_-\psi_-(x)$$

the action of the operator A on the $\psi_-(x)$ itself provides an eigenfunction of the partner hamiltonian H_+. In other words

$$H_+\left[A\psi_-(x)\right] = AA^\dagger A\psi_-(x) = E_-\left[A\psi_-(x)\right].$$

Just by reversing the order of the operators, we start now from the eigenkets $\psi_+(x)$, namely

$$H_+\psi_+(x) = E_+\psi_+(x)$$

thus obtaining as expected eigenfunctions of H_- through the action of $A^\dagger$, i.e.

$$H_-\left[A^\dagger\psi_+(x)\right] = A^\dagger AA^\dagger\psi_+(x) = E_+\left[A^\dagger\psi_+(x)\right].$$

To sum up, except for the hypothetical zero-modes all other energy eigenvalues appear duplicated when the whole model is studied. Further on, are the first-order differential operators A, $A^\dagger$ that establish the connection between the two sectors of the system. However, the above structure exhibits important advantages if we study the problem from a practical point of view. If we resort for instance to a computational technique to estimate the energy spectrum, the original hamiltonian can be substituted on behalf of the isospectral one to improve the properties of convergence. The same argument can be applied of course when considering variational methods.

Appendix B

For reference, we outline in this appendix the mean features concerning the Pösch-Teller potentials which arise in the analysis of the Dirac equation on non-trivial backgrounds like the one associated with the kink of the ϕ^4 model. The problem itself can be elegantly solved in a Lie-algebra framework according to the technique described by Frank and Wolf [10], whose notation will be used in the following. To start from scratch, let us consider the Pösch-Teller hamiltonian

$$H_{PT} = -\frac{1}{2}\frac{d^2}{dx^2} + \frac{c}{\cosh^2 x} + \frac{s}{\sinh^2 x}, \tag{9}$$

where we choose a system of units in which $\hbar = m = 1$. Now it proves convenient the introduction of the parameters k_i $(i = 1, 2)$ given by

$$c = -\frac{1}{2}\left[(2k_1 - 1)^2 - \frac{1}{4}\right], \qquad s = \frac{1}{2}\left[(2k_2 - 1)^2 - \frac{1}{4}\right]$$

which in turn enable us to write the energies E_k of the bound states associated with the Pösch-Teller potential of (9), namely

$$E_k = -\frac{(2k-1)^2}{2}, \tag{10}$$

where $k = k_{\min}, k_{\min -1}, \ldots > 1/2$ with $k_{\min} = k_1 - k_2$. Letting k to be of the form

$$k = \frac{1 + i\kappa}{2}\kappa \geq 0$$

the expression of E_k in (10) also takes care of the continuous part of the spectrum. It remains to identify the behaviour of the wave-functions themselves which, as expected, depends on the values of the singularity parameter s in (9). At first glance the situation is as follows:

i) For the bound states we have

$$\Psi_k^{(k_1,k_2)} \sim (\cosh x)^{-2k_1+3/2}(\sinh x)^{2k_2-1/2}F\left[\alpha,\beta,\gamma;z\right],$$

where $F\left[\alpha,\beta,\gamma;z\right]$ stand for the ${}_2F_1$ Gauss hypergeometric function with the identifications

$$\alpha = -k_1 + k_2 + k, \qquad \beta = -k_1 + k_2 - k + 1, \qquad \gamma = 2k_2, \qquad z = -\sinh^2 x.$$

ii) When considering the scattering states we find

$$\Psi_k^{(k_1,k_2)} \sim (\cosh x)^{2k_1-1/2}(\sinh x)^{2k_2-1/2}F\left[\alpha,\beta,\gamma;z\right]$$

now with

$$\alpha = k_1 + k_2 - k, \qquad \beta = k_1 + k_2 + k - 1, \qquad \gamma = 2k_2, \qquad z = -\sinh^2 x.$$

With regards to singular potentials like the ones written in (9), the careful analysis of the problem can be carried out in accordance with the subtle mathematical tools exposed by Reed and Simon [11]. As far as our model concerns, it might seem plausible to infer the existence of three well different regions depending on the values of the parameter s, namely:

i) If $s < -1/8$, the model itself becomes physically unrealistic since the spectrum is not bounded from below.

ii) Whenever $-1/8 < s < 3/8$, the singularity is not strong enough to make the wave-functions vanish at the origin. However, it proves necessary to define *one-parameter family of self-adjoint extensions* so that a bounded from below spectrum appears once a proper domain is specified. In more physical terms, the wave-functions pass across the singularity point and the model extends itself along the real axis.

iii) Finally, if $s > 3/8$ the singularity acts as an impenetrable barrier thus dividing the space into two independent regions, i.e. $x > 0$ and $x < 0$, forcing the wave-functions to vanish at the origin.

The above results shed light on the family of susy qm models governed by superpotentials like

$$W(x) = \ell \tanh x, \qquad \ell = 1, 2, \ldots \tag{11}$$

which give rise to

$$H_- = -\frac{d^2}{dx^2} + \ell^2 - \frac{\ell(\ell+1)}{\cosh^2 x}, \qquad H_+ = -\frac{d^2}{dx^2} + \ell^2 - \frac{\ell(\ell-1)}{\cosh^2 x}.$$

In accordance with the previous results we have for H_-

$$k_1 = \frac{\ell}{2} + \frac{3}{4}, \qquad k_1 = -\frac{\ell}{2} + \frac{1}{4}$$

together with

$$k_2 = \frac{3}{4}, \qquad k_2 = \frac{1}{4}$$

so that the discrete energy spectrum reads

$$E_{-j} = \ell^2 - (\ell - j)^2, \qquad j = 0, 1, \ldots, \ell - 1$$

while the scattering states start at $E = \ell^2$. With regards to H_+ we find

$$k_1 = \frac{\ell}{2} + \frac{1}{4}, \qquad k_1 = -\frac{\ell}{2} + \frac{3}{4}$$

and again

$$k_2 = \frac{3}{4}, \qquad k_2 = \frac{1}{4}$$

with energies

$$E_{+j} = \ell^2 - (\ell - j)^2, \qquad j = 1, \ldots, \ell - 1.$$

The difference between both discrete spectra reduces itself to the zero-mode of H_- which adopts the form

$$\psi_{-0}(x) \sim P_\ell^\ell(\tanh x),$$

where the $P_\ell^\ell(\tanh x)$ represent associated Legendre polynomials. Bearing in mind that

$$P_\ell^\ell(\tanh x) \sim \cosh^{-\ell} x$$

the logarithmic derivative of the zero-mode itself provides us with the superpotential functions written in (11), i.e.

$$W(x) = -\frac{P'^\ell_\ell(\tanh x)}{P_\ell^\ell(\tanh x)}.$$

In this case, being $s = 0$, we get rid of the difficulties associated with the singular behaviour of the potential. Among other things, the system extends along the real line and the expected pattern of even and odd wave-functions (notice that we are dealing with even potential) appears in the end.

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