New Mathematical Models for Particle Flow Dynamics

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Abstract

A new class of integro-partial differential equation models is derived for the prediction of granular flow dynamics. These models are obtained using a novel limiting averaging method (inspired by techniques employed in the derivation of infinite-dimensional dynamical systems models) on the Newtonian equations of motion of a many-particle system incorporating widely used inelastic particle-particle force formulas. By using Taylor series expansions, these models can be approximated by a system of partial differential equations of the Navier-Stokes type. The exact or approximate governing equations obtained are far from simple, but they are less complicated than most of the continuum models now being used to predict particle flow behavior. Solutions of the new models for granular flows down inclined planes and in vibrating beds are compared with known experimental and analytical results and good agreement is obtained.

1   Introduction

The last two decades have witnessed an intensification of research in granular flow dynamics, in large measure spurred by a burgeoning array of engineering and industrial applications of particle technology. There are several features that make granular flow research attractive to engineers, mathematicians and scientists, among which are the following: A need still exists to formulate the underlying principles of particle interactions in a completely satisfactory manner; there are as yet few if any definitive mathematical models that can reliably predict a wide range of granular flows; particle flow phenomena such as arching, surface waves and convection are still not entirely understood from a

mathematical or engineering perspective; there is a panoply of extremely complex nonlinear dynamical behaviors exhibited in granular flow regimes that has not yet been fully analyzed and has severely tested or exceeded the capabilities of current experimental and computer technologies for accurate characterization; and the techniques and devices for optimizing certain features of particle flows are for the most part only understood on an ad hoc basis. In this paper we use some averaging and limiting ideas associated with infinite-dimensional dynamical systems theory to derive a new class of continuum mathematical models for granular flow that may be capable of predicting the dynamical characteristics of particle flows in a large variety of circumstances, and thereby help to make some progress in solving the many outstanding problems in this field. Our purpose is not to compete with the host of interesting models that try to incorporate as much of the physics of granular flows as possible, including (vibrational) energy equations. Rather, we aim to produce mathematical models that are relatively tractable and ignore just enough of the physics to still provide useful predictions of granular flow dynamics for a wide range of applications.

Although there have been several partial successes in recent years, the state-of-the-art in mathematical modeling of granular flow phenomena pales in comparison to that of fluid mechanics where there is a universally accepted model – the Navier-Stokes equations – whose reliability has been tested and confirmed for over a century, and is considered in many quarters to be capable of apprehending even what may be the most elusive of all physical processes – fluid turbulence. Approaches based on continuum mechanics (transport theory) and kinetic theory (statistical mechanics) have been those most often used for obtaining mathematical models for particle flows in the form of systems of partial differential equations. Notable examples derived using these methods which have enjoyed some success in predicting granular flow dynamics may be found in An & Pierce [1], Anderson & Jackson [2], Farrel, Lun & Savage [7], Gardiner & Schaeffer [8], Goldshtein & Shapiro [10], Jenike & Shield [12], Jenkins & Savage [13], Jenkins & Richman [14], Johnson, Nott & Jackson [15], Lun [18], Lun & Savage [19], Numan & Keller [21], Pasquarell [22], Pitman [24], Rajagopal [26], Richman [27], Tsimring & Aranson [38], Savage [29, 30], Savage & Jeffrey [31], Schaeffer [32], Schaeffer, Shearer & Pitman [33] and Shen & Ackermann [34] (see also Fan & Zhu [6] and Walton [39]). Several of these models have proven to be rather effective in characterizing certain granular flows, for example in chutes and hoppers with simple geometries, but they tend to be fairly complicated systems of nonlinear partial differential equations that are difficult to analyze and solve except by approximate numerical methods, and the information they provide has barely made a dent in the host of practical problems associated with industrial uses of particle technology. There have also been a number of simple, idealized models formulated by neglecting a variety of physical factors, but these tend to miss many of the features of granular flow of interest in applications. Much work still remains in finding a really effective balance between mathematical tractability and adherence to the underlying principles of physics in the models for a large class of granular flow phenomena. It is hoped that the models introduced here will provide a useful step in the direction of achieving such a balance.

Granular flows have also been extensively studied using methods inspired by molecular dynamics research. The basic idea of the molecular dynamics approach is to use realistic models for interparticle forces, developed from both theory and empirical investigations, in a Newtonian dynamics context with a large number of particles (hundreds or thousands)

to determine the evolution of a particle flow configuration. Analytical means are of little use in solving the very high dimensional dynamical systems encountered in such an approach, but some very sophisticated simulations, employing a variety of numerical solution techniques, have been devised for studying granular flows, such as those of Goldhirsch et al. [9], Lan & Rosato [16, 17], McNamara & Luding [20], Pöeschel & Herrmann [25], Rosato et al. [28], Swinney et al. [35] and Walton [39]. Alternative approaches based on cellular automata models and kinetic models of random walks in discrete lattices have also proven to be quite useful; see, for example, Baxter & Behringer [3] and Caram & Hong [5]. These and other simulations have proven to be so remarkably accurate in manifesting most of the complex aspects of particle flow behavior, that one is inescapably drawn to the conclusion that the formulation of a more concise and tractable mathematical representation of such simulations should greatly enhance our ability to analyze particle flow phenomena.

It was this idea of finding more succinct ways of mathematically characterizing granular flow simulations for extremely large numbers of particles that served as the inspiration for the new models derived in this paper by computing limiting forms of the relevant Newtonian dynamical systems. To be more precise, we obtain systems of nonlinear partial differential equations – infinite-dimensional dynamical systems – for velocity fields of granular flows by using an averaging method together with the computation of a limit as the number of particles tends to infinity, followed by a Taylor series approximation. The approach employed is akin to the methods used to obtain limiting partial differential equations for systems of ordinary differential equations (as the size of systems tend toward infinity) in the theory of infinite-dimensional dynamical systems; for example, as when the Korteweg-de Vries equation is obtained as the "limit" of an infinite string of coupled nonlinear oscillators (cf. Tabor [36] and Temam [37]). Our method leads to an infinite class of mathematical models of widely varying levels of complexity, depending on the form of the particle-particle force laws chosen and the order of the Taylor series expansions employed. Several of these models appear to enjoy certain advantages over existing models in terms of simplicity and ease of analysis, and they have the potential for providing a better developed mathematical understanding of granular flow phenomena.

This paper is organized as follows: The particle-particle models, based on the Hertz-Mindlin theory and some empirical observation, that we shall employ for the granular flows under consideration are described in Section 2. Then, in Section 3 we develop the Newtonian differential equations of motion for the particle flow dynamics using the particle-particle force formulas introduced in Section 2, and we describe a decomposition of the forces into interparticle forces, body forces and transmitted forces. In Section 4 we delineate a limiting procedure on the Newtonian equations of motion of the granular flow, ignoring boundary contributions, that produces a system of integro-partial differential equations that models the velocity field of the particle flow. This class of mathematical models for particle flow dynamics is infinite and depends on the form and parameters of the particle-particle force formulas. We also show how our integro-differential equation models can yield an infinitude of partial differential equation approximations to the governing equations when the dynamical variables are approximated by Taylor series. A choice of interparticle parameters and order of Taylor series expansion that leads to a particularly simple system of partial differential equations for the particle velocities in a granular flow is also described in this section. We compute, in Section 5, an exact solution of the simple model of Section 6 for a fully developed flow through a vertical pipe with a

uniform circular cross-section, and in so doing give our first illustration of how to append relevant boundary conditions to our system of equations. The model of Section 4 and the introduction of appropriate auxiliary conditions necessary to model fully developed, two-dimensional granular flow down an inclined plane is treated in Section 6, and we compare our results with those obtained from other analytical and experimental studies of inclined plane flows. In Section 7 we develop the boundary conditions for flows in a vibrating bed and study numerically our model subject to these boundary conditions. We discuss our results and compare them with experimental studies. Finally, we conclude in Section 8 with a discussion of the consequences of the work in this paper and possible directions for future research involving the new class of models.

2   Interparticle forces

In this section we shall describe the particle-particle force models that are the foundation upon which we construct our derivation of the governing equations of motion for the granular flows under consideration. We assume that the flow system is comprised of a large number N of identical inelastic spherical particles distributed throughout some region in $\mathbf{R}^3$ at points $\mathbf{x}^{(i)}$, $1 \leq i \leq N$. The common radius of all the particles is a very small positive number that we denote by r, and the point $\mathbf{x}^{(i)}$ corresponding to the ith particle is located at the center of the particle for all $1 \leq i \leq N$.

We may select any particle, say the ith one, and suppose that the jth particle at $\mathbf{x}^{(j)}$ is near $\mathbf{x}^{(i)}$. For convenience, we define

$$\mathbf{r}_i^j := \mathbf{x}^{(j)} - \mathbf{x}^{(i)} \tag{1}$$

and $\mathbf{v}_i^j$ to be the velocity of the jth particle relative to the ith particle; namely,

$$\mathbf{v}_i^j := \mathbf{v}\left(\mathbf{x}^{(j)}\right) - \mathbf{v}\left(\mathbf{x}^{(i)}\right). \tag{2}$$

Taking our cue from Hertz-Mindlin theory as supported by numerous experimental observations and granular flow simulations (see [6, 12, 28, 29] and [39]), we shall assume that the model for the force $\mathbf{P}_i^j$ exerted on the ith particle by the jth particle is described as follows: $\mathbf{P}_i^j$ is the sum of a (inelastic) *normal force* $\mathcal{N}_i^j$ and a *tangential force* $\mathcal{T}_i^j$ due to friction

$$\mathbf{P}_i^j = \mathcal{N}_i^j + \mathcal{T}_i^j, \tag{3}$$

where

$$\mathcal{N}_i^j := \left[-\chi\left(\left\|\mathbf{r}_i^j\right\|^2\right)\left\|\mathbf{r}_i^j\right\|^\alpha + \eta\left(\left\|\mathbf{r}_i^j\right\|^2\right)\left\langle\mathbf{v}_i^j,\mathbf{r}_i^j\right\rangle\left\|\mathbf{r}_i^j\right\|^\beta \right]\hat{\mathbf{r}}_i^j \tag{4}$$

and

$$\mathcal{T}_i^j := \psi\left(\left\|\mathbf{r}_i^j\right\|^2\right)\left\|\mathbf{r}_i^j\right\|^\gamma\left\|\vartheta(\mathbf{v}_i^j)\right\|^\delta \widehat{\vartheta(\mathbf{v}_i^j)}. \tag{5}$$

Here $\langle\cdot,\cdot\rangle$ is the standard inner product with induced norm $\|\cdot\|$ in $\mathbf{R}^3$ and α, β, γ and δ are positive exponents chosen according to the particular properties of the material particles;

among the most often used values are $\alpha = 1$ or $3/2$ (Hertzian), $\beta = 1$, $\gamma = 0, 1/3, 2/3$ or $3/2$ and $\delta = 1$ or 2. The functions χ, ψ and η are smooth ($= C^\infty$) on $[0, \infty)$ and have the following properties:

$$\chi(\tau), \psi(\tau), \eta(\tau) \geq 0 \qquad \text{for all} \quad \tau \geq 0; \tag{6}$$

$$\chi'(\tau), \psi'(\tau), \eta'(\tau) \leq 0 \qquad \text{for all} \quad \tau \geq 0 \quad (' = d/d\tau); \tag{7}$$

$$\chi(\tau) = \psi(\tau) = \eta(\tau) = 0 \qquad \text{when} \quad \tau > 4r^2; \tag{8}$$

$$\chi'(\tau) = \psi'(\tau) = \eta'(\tau) = 0 \qquad \text{for} \quad 0 \leq \tau \leq q^2 < 4r^2; \tag{9}$$

$$\eta(0) < \chi(0); \tag{10}$$

and

$$\eta(\tau) \leq \chi(\tau) \qquad \text{for all} \quad \tau \geq 0. \tag{11}$$

Graphs of these functions are shown in Figure 1. As we are going to ignore the rotational motion of the particles in our treatment, we shall assume that the tangential force is very small compared to the normal force and, more specifically, that $\psi(\tau) \ll \eta(\tau)$ for all τ such that $\psi(\tau) > 0$.

Figure 1: Force functions: a) $y = \chi(x)$, b) $y = \eta(x)$, c) $y = \psi(x)$

The role of the function η is to represent an energy loss due to inelasticity in the restoring mode when the particles are separating after a collision. A caret over a vector $\mathbf{u}$ indicates the unit vector in the direction of $\mathbf{u}$; i.e., $\hat{\mathbf{u}} := \mathbf{u}/\|\mathbf{u}\|$. The vector $\vartheta\left(\mathbf{v}_i^j\right)$ is the component of the relative velocity at the point of contact of a pair of particles obtained

by projecting $\mathbf{v}_i^j$ onto the tangent plane of the ith particle at the point of contact. This vector can be written in the form

$$\vartheta\left(\mathbf{v}_i^j\right) := \mathbf{v}_i^j - \left\langle \mathbf{v}_i^j, \hat{\mathbf{r}}_i^j \right\rangle \hat{\mathbf{r}}_i^j. \tag{12}$$

The normal and tangential interparticle forces are depicted in Figure 2.

Figure 2: Particle-particle forces: a) normal force, b) tangential force

Summing over all particles in the granular flow system, we find that the total force exerted by all the particles on the ith particle is

$$\mathbf{P}_i := \sum_{j=1, j\neq i}^{N} \mathbf{P}_i^j = \sum_{j=1, j\neq i}^{N} \left(\mathcal{N}_i^j + \mathcal{T}_i^j \right). \tag{13}$$

Note that our assumed particle-particle force models account for the geometry of the particles only with regard to the region where the force vanishes (its *support)* and the manner in which the tangential frictional component of force is defined. We observe that (13) can also be obtained from a specific force density field surrounding the ith particle with the force supplied by each grain equal to this specific density multiplied by the volume $\frac{4}{3}\pi r^3$. We shall return to this point in the sequel when we compute limiting forms of the particle dynamical system.

3   Newtonian equations of motion

The motion of the particles in the granular flow field may be described by a system of $3N$ second-order, ordinary differential equations expressing Newton's second law of motion;

viz.

$$m\ddot{\mathbf{x}}^{(i)} = \mathbf{F}_i := \mathbf{P}_i + \mathbf{T}_i + \mathbf{E}_i + \mathbf{B}_i \qquad (1 \le i \le N), \tag{14}$$

where $\dot{} = d/dt$, m is the mass of each of the N identical particles, $\mathbf{P}_i$ is the force exerted on the ith particle by all particles in direct contact with it as described in the preceding section, $\mathbf{T}_i$ is the *transmitted force* on the ith particle exerted by connected arrays of particles in contact with one another that touch a particle in direct contact with the ith particle, $\mathbf{E}_i$ is the *external* or *body force* on the ith particle which is usually just the gravitational force (but may sometimes also include electromagnetic and other forces) and $\mathbf{B}_i$ is the *boundary force* exerted on the ith particle by fixed or motile boundaries in direct contact with it that delimit the region in space in which the particles can move. Observe that the variables on which each of the components of force depend can be described as follows:

$$\mathbf{P}_i = \mathbf{P}_i\left(\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(N)}, \dot{\mathbf{x}}^{(1)}, \ldots, \dot{\mathbf{x}}^{(N)}\right),$$

$$\mathbf{T}_i = \mathbf{T}_i\left(\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(N)}, \dot{\mathbf{x}}^{(1)}, \ldots, \dot{\mathbf{x}}^{(N)}\right),$$

$$\frac{\mathbf{E}_i}{m} \quad \text{is usually constant,}$$

$$\mathbf{B}_i = \mathbf{B}_i\left(\mathbf{x}^{(i)}, \dot{\mathbf{x}}^{(i)}, t\right),$$

where the dependence on t in $\mathbf{B}_i$ occurs when the material boundary of the flow region moves with time such as in the case of particles moving in a vibrating container.

We can write the Newtonian equations of motion in a more concise form by introducing the following vector notation: Define the vector $\mathbf{X}$ in $\mathbf{R}^{3N}$ to be

$$\mathbf{X} := \left(\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \ldots, \mathbf{x}^{(N)}\right).$$

Then (14) can be rewritten in vector form as

$$\ddot{\mathbf{X}} = \mathbf{\Phi}(\mathbf{X}, \dot{\mathbf{X}}, t) := \mathbf{\Phi}_p(\mathbf{X}, \dot{\mathbf{X}}) + \mathbf{\Phi}_T(\mathbf{X}, \dot{\mathbf{X}}, t) + \mathbf{\Phi}_e + \mathbf{\Phi}_b(\mathbf{X}, \dot{\mathbf{X}}, t), \tag{15}$$

where

$$\mathbf{\Phi}_p := m^{-1}\left(\mathbf{P}_1(\mathbf{X}, \dot{\mathbf{X}}), \ldots, \mathbf{P}_N(\mathbf{X}, \dot{\mathbf{X}})\right)$$

is the *interparticle force per unit mass*,

$$\mathbf{\Phi}_T := m^{-1}(\mathbf{T}_1(\mathbf{X}, \dot{\mathbf{X}}, t), \ldots, \mathbf{T}_N(\mathbf{X}, \dot{\mathbf{X}}, t))$$

is the *transmitted force per unit mass*,

$$\mathbf{\Phi}_e := m^{-1}\left(\mathbf{E}_1, \ldots, \mathbf{E}_N\right)$$

is the *external force per unit mass* and

$$\mathbf{\Phi}_b := m^{-1}\left(\mathbf{B}_1(\mathbf{x}^{(1)}, \dot{\mathbf{x}}^{(1)}, t), \ldots, \mathbf{B}_N(\mathbf{x}^{(N)}, \dot{\mathbf{x}}^{(N)}, t)\right)$$

is the *boundary force per unit mass.* In theory, if initial values of $\mathbf{X}$ and $\dot{\mathbf{X}}$ are specified, then (15) uniquely determines the ensuing motion of all the particles, at least for small values of $|t|$ (see [36] and [37]). However, for extremely large values of N the work required to integrate (15) – analytically, when in the rare cases that this is possible, or numerically otherwise – tends to be prohibitive. Thus it is desirable to find an infinite-dimensional limit in some sense for (15) as $N \to \infty$, presumably in the form of a partial differential equation, that may prove to be more amenable to analysis. This is precisely what we shall do in the next section.

4   Limiting models

We shall demonstrate how new models for granular flow phenomena can be obtained by applying a certain type of dynamical limit procedure to the Newtonian equations (15). The reader will no doubt notice at least a vague similarity between our method and the continuum limit used in the Fermi-Ulam-Pasta model to obtain the Korteweg-de Vries equation (cf. [36]). To begin with, we restrict our attention to points in the interior of the granular flow region that are not directly affected by interaction with the boundary. Consequently, for the time being we ignore the boundary force contribution in (14) or (15); the boundary effects shall be considered in the sequel when we study specific boundary-value problems.

Referring to (14), we assume that the body forces are exclusively gravitational and that the Cartesian coordinate system has been chosen so that the gravitational force acting on each particle has the form

$$\mathbf{E}_i = -mg\hat{\mathbf{e}}, \tag{16}$$

where g is the acceleration of gravity and $\hat{\mathbf{e}}$ is a unit vector in the opposite direction to the gravitational field. Now we select a point in the interior of the granular flow field corresponding to the ith particle (at time t) which is moving along a trajectory determined by the vector field

$$\dot{\mathbf{x}}^{(i)} = \mathbf{v}\left(\mathbf{x}^{(i)}, t\right) \tag{17}$$

and the location of this particle at time $t = 0$.

The interparticle forces on the ith particle at the point $\mathbf{x}^{(i)} = \mathbf{x}$ are given by (13). Since we are going to take a limit as the number of grains goes to infinity, we need to average or distribute these forces in a way that insures the existence of such a limit and is conducive to its computation. This can be done by smearing the particles into a continuum (assumed to be locally uniform) and considering the interparticle force field to be obtained from a specific force density field. To be more precise, we assume that each particle is surrounded by a specific force density field of the same form $c\mathbf{P}_*$. Whence, the interparticle force on the ith particle can be written

$$\mathbf{P}_i := c \sum_{j \neq i} \mathbf{P}_*\left(\mathbf{x}^{(j)}; \mathbf{v}_i^j\right) \Delta V_j, \tag{18}$$

where ΔV_j is the volume increment occupied by the jth particle and $c > 0$ is a multiplicative factor with units $volume^{-1}$ (associated with the geometry of the particles). This can be rewritten in the form

$$\mathbf{P}_i = (N-1)^{-1} cc_0 \sum_{j \neq i} \mathbf{P}_* \left(\mathbf{x}^{(j)}; \mathbf{v}_i^j \right), \tag{19}$$

where c_0 is a positive constant equal to the volume of the (compact) support of $\mathbf{P}_i$ and we have assumed that all the particles occupy volume increments of the same size. In (19) we plainly see the averaging aspect of this approach. Taking the limit as $N \to \infty$ in (19) [or equivalently as $\Delta V_j \to 0$ in (18)] using standard results from integration theory, we obtain

$$\lim_{N \to \infty} \mathbf{P}_i = c \int_{\mathbf{R}^3} \mathbf{P}_* \, dy_1 dy_2 dy_3 = c \int_{\mathbf{R}^3} \mathbf{P}_* \, d\mathbf{y}, \tag{20}$$

where it follows from (3), (4), (5) and (12) that

$$\begin{aligned}
\mathbf{P}_* :={}& \mathbf{P}_* \left(\mathbf{y}; \mathbf{v}(\mathbf{x}), \mathbf{v}(\mathbf{x} + \mathbf{y}) \right) \\
={}& \left[-\chi \left(\|\mathbf{y}\|^2 \right) \|\mathbf{y}\|^{\alpha-1} + \eta \left(\|\mathbf{y}\|^2 \right) \langle \mathbf{v}(\mathbf{x} + \mathbf{y}) - \mathbf{v}(\mathbf{x}), \mathbf{y} \rangle \|\mathbf{y}\|^{\beta-1} \right] \mathbf{y} \\
& + \psi \left(\|\mathbf{y}\|^2 \right) \|\mathbf{y}\|^{\gamma} \left\| \mathbf{v}(\mathbf{x} + \mathbf{y}) - \mathbf{v}(\mathbf{x}) - \langle \mathbf{v}(\mathbf{x} + \mathbf{y}) - \mathbf{v}(\mathbf{x}), \mathbf{y} \rangle \frac{\mathbf{y}}{\|\mathbf{y}\|^2} \right\|^{\delta-1} \\
& \times \left[\mathbf{v}(\mathbf{x} + \mathbf{y}) - \mathbf{v}(\mathbf{x}) - \langle \mathbf{v}(\mathbf{x} + \mathbf{y}) - \mathbf{v}(\mathbf{x}), \mathbf{y} \rangle \frac{\mathbf{y}}{\|\mathbf{y}\|^2} \right],
\end{aligned} \tag{21}$$

where $\mathbf{y} = (y_1, y_2, y_3)$ represents the position vector measured from the reference point $\mathbf{x}$ that has been introduced to simplify the notation for $\mathbf{r}_i^j$. As for the transmitted force in the Newtonian equation (14), we make the standard assumption that in the continuum limit it can be represented by a gradient field, $\operatorname{grad} p$, where the function $p = p(\mathbf{x}, t)$ is naturally called the *pressure*. The particle at $\mathbf{x}^{(i)}$ is represented by a *density field*, $\rho = \rho(\mathbf{x}, t)$, with compact support. Hence the external force is

$$\left(\int_{W_i} \rho g \, dV \right) \hat{\mathbf{e}},$$

where W_i is a spherical (control) region centered at $\mathbf{x}^{(i)}$ with (Lebesgue) measure ΔV_i, and this converges to $\rho g \hat{\mathbf{e}}$ as $N \to \infty$ ($\Leftrightarrow \Delta V_i \to 0$). In the same spirit, the right-hand side of (14) is replaced by

$$\frac{d}{dt} \int_{W_i} \rho \mathbf{v} \, dV,$$

which upon applying the usual continuum limit converges to the density times the total (material) derivative of the velocity:

$$\rho \frac{D\mathbf{v}}{Dt} := \rho \left(\frac{\partial \mathbf{v}}{\partial t} + \sum_{k=1}^{3} \frac{\partial \mathbf{v}}{\partial x_k} v_k \right).$$

Upon combining all of the above computations, we obtain the following system of nonlinear integro-partial differential equations for the momentum balance of the particle flow in the interior of the region under consideration:

$$\frac{D\mathbf{v}}{Dt} = \frac{\partial \mathbf{v}}{\partial t} + v_k \frac{\partial \mathbf{v}}{\partial x_k} = -g\hat{\mathbf{e}} + \frac{1}{\rho}\operatorname{grad} p + \kappa \int_{\mathbf{R}^3} \mathbf{P}_*(\mathbf{y}; \mathbf{v}(\mathbf{x}), \mathbf{v}(\mathbf{x}+\mathbf{y}))\, dy, \tag{22}$$

where $\kappa := c/\rho$ and we have employed the Einstein summation convention. If ρ is constant and $\operatorname{grad} p$ is known a priori, then (22) together with appropriate initial and boundary data suffices to determine the velocity field. When the granular flow is compressible and $\operatorname{grad} p$ is known a priori, we have to add the continuity equation

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{v}) = 0 \tag{23}$$

to (22), and then by imposing additional auxiliary data the velocity field and density may be determined. Of course, in general, both p and ρ are unknown variables, in which case (22) and (23) are insufficient to determine p, ρ and $\mathbf{v}$. One more equation must be added, and this may be accomplished by appending an energy equation to (22) and (23). The easiest way to do this is to obtain an equation of state of the granular flow medium that provides a relationship between the pressure and the density. If we make the same assumption as above (in particular, that the particles are uniformly and isotropically distributed locally), then by applying the same type of limit as $N \to \infty$ to the equations representing the kinetic energy of the Newtonian system (14), we obtain the equation of state of an ideal gas; namely

$$p = A \rho^\omega, \tag{24}$$

where $A > 0$ and $\omega > 1$ are constants that are obtained from the properties of the granular flow medium. The same result can be derived by applying the standard thermodynamic limit of statistical mechanics in conjunction with the virial theorem.

For certain purposes, including comparison with other continuum models for particle flows, it is useful to replace (22) with an approximate partial differential equation. Although, it should be pointed out that, mathematically speaking, (22) enjoys certain inherent advantages over such partial differential equation models. In particular, as we shall demonstrate in a forthcoming paper, solutions of the system with (22) exhibit considerably more regularity than the pure differential equation models that we shall discuss in the sequel.

In order to approximate (22) by a system of mth order partial differential equations, we may use the following Taylor series expansion of order m:

$$\mathbf{v}(\mathbf{x}+\mathbf{y}) - \mathbf{v}(\mathbf{x}) \simeq \sum_{k=1}^{m} \frac{1}{k!} \frac{\partial^k \mathbf{v}}{\partial \mathbf{x}^k}(\mathbf{x})\mathbf{y}^k. \tag{25}$$

The positive integer m is at our disposal, and it is plausible to assume that the larger we choose m, the more accurate the approximation. Substituting (25) in (22), we obtain the system of mth order, nonlinear partial differential equations as an approximate model for the momentum balance of the granular flow field given by

$$\frac{\partial \mathbf{v}}{\partial t} + v_k \frac{\partial \mathbf{v}}{\partial x_k} = -g\hat{\mathbf{e}} + \frac{1}{\rho}\operatorname{grad} p + \mathbf{\Gamma}\left(\mathbf{v}, \frac{\partial \mathbf{v}}{\partial \mathbf{x}}, \dots, \frac{\partial^m \mathbf{v}}{\partial \mathbf{x}^m}\right), \tag{26}$$

where $\boldsymbol{\Gamma}$, a function that does not depend explicitly on $\mathbf{x}$, is defined by

$$
\begin{aligned}
\boldsymbol{\Gamma} := \kappa \Bigg\{ &- \int_{\mathbf{R}^3} \chi\left(\|\mathbf{y}\|^2\right) \|\mathbf{y}\|^{\alpha-1} \, \mathbf{y} d\mathbf{y} + \sum_{k=1}^m \frac{1}{k!} \int_{\mathbf{R}^3} \left\langle \frac{\partial^k \mathbf{v}}{\partial \mathbf{x}^k} \mathbf{y}^k, \mathbf{y} \right\rangle \|\mathbf{y}\|^{\beta-1} \, \eta\left(\|\mathbf{y}\|^2\right) \mathbf{y} d\mathbf{y} \\
&+ \sum_{k=1}^m \frac{1}{k!} \int_{\mathbf{R}^3} \psi\left(\|\mathbf{y}\|^2\right) \|\mathbf{y}\|^{\gamma} \left\| \sum_{k=1}^m \frac{1}{k!} \left[\frac{\partial^k \mathbf{v}}{\partial \mathbf{x}^k} \mathbf{y}^k - \left\langle \frac{\partial^k \mathbf{v}}{\partial \mathbf{x}^k} \mathbf{y}^k, \mathbf{y} \right\rangle \frac{\mathbf{y}}{\|\mathbf{y}\|^2} \right] \right\|^{\delta-1} \\
&\times \left[\frac{\partial^k \mathbf{v}}{\partial \mathbf{x}^k} \mathbf{y}^k - \left\langle \frac{\partial^k \mathbf{v}}{\partial \mathbf{x}^k} \mathbf{y}^k, \mathbf{y} \right\rangle \frac{\mathbf{y}}{\|\mathbf{y}\|^2} \right] d\mathbf{y} \Bigg\}.
\end{aligned}
$$

Hence we have infinitely many possible partial differential equation models for granular flow corresponding to the choices of the functions χ, η and ψ, of parameters α, β, γ and δ, and the order m of the Taylor series approximation. This leads to a very natural question: What order of Taylor series approximation in (26) should be used for a given application? As we shall show in the sequel, $m = 2$ works rather well for tube, inclined plane and vibrating bed flows. However, it will probably be necessary to consider several choices in other applications and determine an acceptable order of approximation on a case-by-case basis, where an educated guess is made based upon known properties of the flow.

A simple flow model

Depending on the choice of parameters and the order, the equation (26) can range from relatively simple to quite complicated. In this section we make a choice of parameters and order that leads to a rather simple yet ostensibly realistic model for the velocity field of a granular flow. Specifically, we choose $\alpha = \beta = 1$, $\gamma = 0$, $\delta = 1$ and $m = 2$. Then (26) takes the form

$$
\begin{aligned}
\frac{\partial \mathbf{v}}{\partial t} + v_k \frac{\partial \mathbf{v}}{\partial x_k} = &-g\hat{\mathbf{e}} + \frac{1}{\rho} \operatorname{grad} p + \kappa \int_{\mathbf{R}^3} \Bigg\{ -\chi\left(\|\mathbf{y}\|^2\right) \mathbf{y} + \sum_{k=1}^2 \frac{1}{k!} \left[\psi\left(\|\mathbf{y}\|^2\right) \frac{\partial^k \mathbf{v}}{\partial \mathbf{x}^k} \mathbf{y}^k \right. \\
&+ \left\langle \frac{\partial^k \mathbf{v}}{\partial \mathbf{x}^k} \mathbf{y}^k, \mathbf{y} \right\rangle \left(\eta\left(\|\mathbf{y}\|^2\right) - \psi\left(\|\mathbf{y}\|^2\right) \|\mathbf{y}\|^{-2}\right) \mathbf{y} \Bigg] \Bigg\} d\mathbf{y},
\end{aligned}
$$

which upon integration using spherical coordinates simplifies to

$$
\frac{\partial \mathbf{v}}{\partial t} + v_k \frac{\partial \mathbf{v}}{\partial x_k} = -g\hat{\mathbf{e}} + \frac{1}{\rho} \operatorname{grad} p + \nu \Delta \mathbf{v} + \lambda \operatorname{grad}(\operatorname{div} \mathbf{v}), \tag{27}
$$

where

$$
\nu := \frac{2\pi\kappa}{15} \int_0^\infty \left[\eta(s^2)s^2 - 4\psi(s^2) \right] s^4 ds
$$

and

$$
\lambda := \frac{4\pi\kappa}{15} \int_0^\infty \left[\eta(s^2)s^2 - \psi(s^2) \right] s^4 ds
$$

depend only on the density and the particle-particle force functions described in Section 2 and may be assumed to be constants in many applications.

Observe that (27) is essentially just the momentum part of the Navier-Stokes equations (cf. [23, 36] and [37]). There are at least two interesting inferences that may be drawn from this result: Firstly, it provides a partial confirmation of the validity of our integro-partial differential equation model as a predictive tool for granular flow. Secondly, it lends support to the contention that the Navier-Stokes equations are a good model for granular flow behavior obtained from simulation of the Newtonian equations of motion.

5   Flow through a tube: an exact solution

In this section we obtain an exact solution of the approximate model (27), (23), (24) subject to appropriate boundary conditions, for the case of fully developed (steady-state) granular flow, under the action of gravity, through a vertical circular cylindrical pipe illustrated in Figure 3. We assume that the density and pressure are constant, hence it suffices to solve (26) subject to some boundary conditions.

Figure 3: Flow through a tube

Under the circumstances, it is convenient to recast (27) in terms of standard cylindrical coordinates (r, θ, z) with corresponding velocity components (u, v, w), where u is the radial, v the azimuthal and w is the vertical(axial) component of the flow velocity. The system

assumes the following form with respect to cylindrical coordinates:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + \frac{v}{r}\frac{\partial u}{\partial \theta} + w\frac{\partial u}{\partial z} - \frac{v^2}{r}$$

$$= \nu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2}\right] + \lambda\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial(ru)}{\partial r} + \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial z}\right],$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial r} + \frac{v}{r}\frac{\partial v}{\partial \theta} + w\frac{\partial v}{\partial z} + \frac{uv}{r}$$

$$= \nu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} + \frac{\partial^2 v}{\partial z^2}\right] + \frac{\lambda}{r}\frac{\partial}{\partial \theta}\left[\frac{1}{r}\frac{\partial(ru)}{\partial r} + \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial z}\right], \tag{28}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial r} + \frac{v}{r}\frac{\partial w}{\partial \theta} + w\frac{\partial w}{\partial z}$$

$$= -g + \nu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} + \frac{\partial^2 w}{\partial z^2}\right] + \lambda\frac{\partial}{\partial z}\left[\frac{1}{r}\frac{\partial(ru)}{\partial r} + \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial z}\right],$$

where g is the acceleration of gravity. We assume that the pipe has radius $R > 0$ and that its length is so great that the domain of the granular flow can be represented in idealized form as

$$\Omega := \{(r, \theta, z) : 0 \leq r < R\}.$$

Now we deal with the task of appending appropriate auxiliary data to (28) on Ω. As we are seeking a steady-state solution, we assume that the velocity is independent of the time t. There remains the question of realistic auxiliary data on the boundary $\partial\Omega$. Of course, $u \leq 0$ on $\partial\Omega$ is required by the geometry of the pipe (assuming it is rigid and impenetrable). Over time, one may reasonably expect the radial and azimuthal fluctuations in velocity along the inside surface of the pipe to cease, so we shall assume that both u and v vanish on $\partial\Omega$. As for the axial velocity along $\partial\Omega$: the motion of a particle in contact with $\partial\Omega$ is that of free fall with a resisting force due to friction. This suggests that there is a constant limiting (or terminal) velocity along the wall of the pipe (that is achieved in the long-term flow configuration), so it is reasonable to assume that w is a negative constant along $\partial\Omega$. In summary, we take the auxiliary data for (28) in Ω to be

$$\frac{\partial u}{\partial t} = \frac{\partial v}{\partial t} = \frac{\partial w}{\partial t} \equiv 0 \qquad \text{in} \quad \Omega,$$

$$u = v = 0 \qquad \text{and} \qquad w = -w_\infty \qquad \text{on} \quad \partial\Omega, \tag{29}$$

where w_∞ is a positive constant.

In view of the boundary conditions, it makes sense to seek a solution of (28)–(29) with $u = v = 0$ and $w = \varphi(r)$. Then the first two equations of (27) are trivially satisfied and the third equation yields

$$\frac{\nu}{r}\frac{d}{dr}\left(r\frac{d\varphi}{dr}\right) = g. \tag{30}$$

Integrating (30), we obtain

$$\varphi = \frac{g}{4\nu}r^2 + c_1\log r + c_2,$$

where c_1 and c_2 are constants of integration. The solution should be regular at $r = 0$, so we must set $c_1 = 0$. Then (29) leads to the following solution:

$$u = v = 0 \qquad \text{and} \qquad w = \frac{g}{4\nu}\left(r^2 - R^2\right) - w_\infty. \tag{31}$$

It is easy to check that (31) satisfies (28)–(29).

6 Inclined plane flow model

The fully developed flow of particles down a two-dimensional inclined plane will be studied in this section using the governing equations (27), (23), (24). As inclined plane flow has been extensively investigated (see, for example [2] and [10]), we shall have an opportunity to compare the predictions based upon the simple approximate model with the results obtained by other researchers, thereby further testing the effectiveness of our approach.

Figure 4 depicts the flow geometry for a plane inclined at an angle of θ to the horizontal.

Figure 4: Inclined plane flow

It is convenient to use a Cartesian coordinate system with x measured down along the surface of the inclined plane and y-axis normal to the plane and pointing into the flowing layer of granular material. Here u represents the component of the flow velocity along the x-axis and h the depth of the flowing layer. We assume that the density is constant and that the pressure gradient exactly balances the gravitational force normal to the inclined plane throughout the flowing layer. Therefore, it suffices to solve (27) along with the

necessary boundary conditions. Of course, the figure embodies the usual assumption that the granular flow is essentially two-dimensional.

A clockwise rotation of θ of the coordinate system and a balancing of the gravitational and reaction forces normal to the inclined plane yields the following pair of equations for the granular flow:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = g\sin\theta + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + \lambda\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x\partial y}\right), \tag{32}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) + \lambda\left(\frac{\partial^2 u}{\partial x\partial y} + \frac{\partial^2 v}{\partial y^2}\right), \tag{33}$$

where v is the y-component of the velocity of the granular flow. Since the flow is taken to be fully developed (steady-state), it is reasonable to assume that both u and v are independent of the time t. It is also sensible to presuppose that the y-component of the velocity vanishes identically and that u is a function of y only. With these assumptions (33) reduces to the trivial equation $0 = 0$ and we are left only with the simple ordinary differential equation

$$\frac{d^2 u}{dy^2} = -\frac{g}{\nu}\sin\theta \tag{34}$$

representing (32).

Appropriate auxiliary data for (34) are the free-boundary condition along the free surface representing the interface between the flowing particles and the air that defines the depth of the flowing layer h as the smallest number satisfying

$$\frac{du}{dy}(h) = 0 \qquad (h > 0), \tag{35}$$

and a slip condition along the inclined plane granular material interface

$$\frac{du}{dy}(0) = ku_\infty, \tag{36}$$

where u_∞ is a limiting velocity along the surface of the plane. The component u_∞ may be the result of a partial balance between the frictional properties of the plane and the particles and the gravitational component of the force in the x-direction or a combination of gravitational and frictional effects and some constant mass flow rate supplied to the system. Here k is some nonnegative constant connected with the nature of the shearing stress in the flowing layer adjacent to the plane that is related to the frictional characteristics of the plane and particles and the dynamical state of the system.

Integrating (34) twice using (36), we obtain the solution

$$u = u(y) = -\left(\frac{g\sin\theta}{2\nu}\right)y^2 + u_\infty(ky + 1). \tag{37}$$

Whence we determine the depth of the flowing layer by substitution of (35) in (37); namely,

$$h = \frac{k\nu u_\infty}{g\sin\theta}, \tag{38}$$

for $\theta > 0$. A typical velocity profile is shown in Figure 5.

Figure 5: Velocity profiles for inclined plane flow

The extremely simple nature of the solution (37) obtained from the governing equation (27) notwithstanding, it compares rather well with observations from experimental studies and the predictions from more complicated flow models (cf. [2] and [15]). For example, the form of the velocity profiles illustrated in Fig. 5 is qualitatively similar to those measured in experiments and derived from more comprehensive constitutive equations. Moreover, unlike some fairly popular models, (38) shows that our approach predicts a decrease in the depth of the flowing layer with increasing inclination angle of the plane – a property that is consistent with experimental observations.

7   Vibrating bed model

In this section we shall apply our model (27), (23), (24) to the study of granular flows in a two-dimensional vibrating bed. Namely, we consider the motion of a very large number of particles in a rectangular container in the plane with fixed vertical side walls and a horizontal bottom that is oscillating periodically in the vertical (x_2) direction. The only body force is a gravitational force in the negative (x_2) direction and the interstitial and surrounding medium is air which we assume has no effect on the granular flow.

At $t = 0$ the particles are contained in the following region:

$$K_0 := \{(x_1, x_2) \in \mathbf{R}^2 : |x_1| < \sigma, \ x_2 > 0\}, \tag{39}$$

where $\sigma > 0$ is half of the width of the container. Then, the container is subject to a vertical oscillation of the form $a\sin(\omega t)$ that is illustrated in Figure 6.

Figure 6: Granular material in a vibrating container

We shall use boundary conditions at the walls similar to those employed in the previous section. Namely, we assume that the normal component of the particle velocity near the wall is equal to the normal component of the wall velocity. As for the tangential direction, we use the equation (36) in the following form

$$\frac{\partial v_T}{\partial n} = -k v_T, \tag{40}$$

where v_T denotes the relative tangential component of velocity between the particle and the wall, $\partial/\partial n$ is the partial derivative in the outer normal direction. The equation (40) as well as (36) represents a type of balance law between the interparticle and particle-wall friction forces which has been used by other reseachers. The (constant) coefficient $k > 0$ is a measure of the boundary friction that we shall call the *wall friction coefficient.* It would be most natural to use the particle size as the characteristic length for the non-dimensionalization of k, but it tends to zero in the continuum model limit. Therefore we use for this purpose the space step of the numerical integration scheme.

In summary then, we take the following as the governing equations plus the initial and

boundary conditions for the granular flow in the planar vibrating bed:

$$\frac{\partial v_1}{\partial t} + v_1 \frac{\partial v_1}{\partial x_1} + v_2 \frac{\partial v_1}{\partial x_2} = \nu \left(\frac{\partial^2 v_1}{\partial x_1^2} + \frac{\partial^2 v_1}{\partial x_2^2} \right) + \lambda \left(\frac{\partial^2 v_1}{\partial x_1^2} + \frac{\partial^2 v_2}{\partial x_1 \partial x_2} \right),$$

$$\frac{\partial v_2}{\partial t} + v_1 \frac{\partial v_2}{\partial x_1} + v_2 \frac{\partial v_2}{\partial x_2} = \nu \left(\frac{\partial^2 v_2}{\partial x_1^2} + \frac{\partial^2 v_2}{\partial x_2^2} \right) + \lambda \left(\frac{\partial^2 v_1}{\partial x_1 \partial x_2} + \frac{\partial^2 v_2}{\partial x_2^2} \right)$$

(41)

in $\Sigma := \{(x,t) : x \in \Omega_t, t > 0\}$;

$$v_1(x_1, x_2, 0) = v_1^0(x_1, x_2), \qquad v_2(x_1, x_2, 0) = v_2^0(x_1, x_2) \qquad \text{at} \quad t = 0 \tag{42}$$

for all $x \in \Omega_0 = \{x \in \mathbf{R}^2 : |x_1| < \sigma, 0 < x_2 < h\}$, where the functions $v_1^0(x_1, x_2)$ and $v_2^0(x_1, x_2)$ determine the initial velocity distribution;

$$\frac{\partial v_1}{\partial x_2} = -kv_1, \qquad v_2 = a\omega \cos(\omega t) \tag{43}$$

for all particles on the bottom, $\{(x_1, x_2) : |x_1| < \sigma, x_2 = a\sin(\omega t)\}$, of the bed when $t > 0$;

$$v_1 = 0, \qquad \frac{\partial v_2}{\partial x_1} = -kv_2 \tag{44}$$

for all particles on the left wall, $\{(x_1, x_2) : x_1 = -\sigma, x_2 > a\sin(\omega t)\}$, of the container when $t > 0$;

$$v_1 = 0, \qquad \frac{\partial v_2}{\partial x_1} = -kv_2 \tag{45}$$

for all particles on the right wall, $\{(x_1, x_2) : x_1 = \sigma, x_2 > a\sin(\omega t)\}$, of the container when $t > 0$; and

$$\frac{\partial \mathbf{v}}{\partial n} = 0 \tag{46}$$

for all particles on the free-boundary, consisting of all points in $\partial \Omega_t \backslash \partial K_t$, when $t > 0$.

Now we shall consider some numerical solutions to the model (41) subject to the initial and the boundary conditions (42)–(46). To simplify our analysis, we shall ignore the effects of surface waves and free-boundary components at the bottom of the container.

In order to avoid difficulties with vibrating bed boundary conditions like $v_2(x, a\sin(\omega t)) = a\omega \cos(\omega t)$ at the bottom of the bed, we shall write our equations in the vibrating system of coordinates:

$$x = x^*, \qquad y = y^* + a\sin(\omega t), \qquad t = t^*. \tag{47}$$

The operators of partial differentiation in this frame are

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial x^*}, \qquad \frac{\partial}{\partial y} = \frac{\partial}{\partial y^*}, \qquad \frac{\partial}{\partial t} = \frac{\partial}{\partial t^*} - a\omega \cos(\omega t^*) \frac{\partial}{\partial y^*}. \tag{48}$$

We have to include also into the system (41) the inertial force term proportional to $a\omega^2 \sin(\omega t)$ and directed along the y-axis. The governing system of equations in the "starred" system takes the following form (we have dropped the index '$*$')

$$v_{1,t} = \nu(v_{1,xx} + v_{1,yy}) + a\omega \cos(\omega t)v_{1,y} - \alpha(v_1 v_{1,x} + v_2 v_{1,y}) + \lambda(v_{1,xx} + v_{2,xy}),$$

$$v_{2,t} = a\omega^2 sin(\omega t) + \nu(v_{2,xx} + v_{2,yy}) + a\omega \, cos(\omega t)v_{2,y} - \alpha(v_1 v_{2,x} + v_2 v_{2,y}) \tag{49}$$

$$+ \lambda(v_{1,xy} + v_{2,yy}),$$

where $0 \leq x \leq 2$, $0 \leq y \leq 2$, and the boundary conditions can be written as

$$v_1(0, y, t) = 0, \qquad v_1(2, y, t) = 0, \qquad v_2(x, 0, t) = 0, \qquad v_2(x, 2, t) = 0,$$

$$\frac{\partial v_1}{\partial y}(x, 0, t) + k v_1(x, 0, t) = 0, \qquad \frac{\partial v_1}{\partial y}(x, 2, t) = 0, \qquad (50)$$

$$\frac{\partial v_2}{\partial x}(0, y, t) + k v_2(0, y, t) = 0, \qquad \frac{\partial v_2}{\partial x}(2, y, t) + k v_2(2, y, t) = 0.$$

We use an explicit finite difference scheme for solving the system (49)–(50) with the spatial step $\Delta x = \Delta y = 0.001$ and the time step $\Delta t = 10^{-5}$. Estimates show that such a small time step is needed to satisfy the stability condition for the explicit finite-difference scheme.

We investigated the system (49)–(50) using both multi-vortex and random initial conditions. For certain ranges of the parameters (corresponding to the particle-particle forces) the motion of the system starting with a multi-vortex configuration changes rapidly into a pair of vortices that persists for a long time (relative to the period of the forced oscillations). The centers of this "stable" vortex pair oscillate with small amplitude synchronistically with the forced oscillations. In some cases this vortex pair evolved into a single vortex over a very large time period. Increase of the constant ν results in a corresponding increase of the particle-particle friction and leads to damping of the vorticity (see Figure 7).

When the motion starts from a random initial velocity distribution we also observed the "stable" vortex type of motion and bifurcation between different types of relatively stable patterns (Figure 8). Some of the values used for the control parameters were $\omega = 2$, $a = 1$, $\nu = 0.3$, $\lambda = 1.0$ in the case of the vortex type of motion, and $\omega = 3$, $a = 1$, $\nu = 1.0$, $\lambda = 0.5$ in the case of a mixing motion. These types of particle dynamics are in agreement with experimental observations and computer simulation results [17, 25]. We note also that Hayakawa and Hong [11] obtained similar results from numerical solutions of their models but they assumed no-slip boundary conditions at the walls which are not physically realistic for vibrating bed granular flows.

Similar types of the flow behavior were obtained by Bourzutschky and Miller [4] for their Navier-Stokes models. By using negative slip boundary conditions in numerical experiments corresponding to granular flows with a high mobility boundary layer, they obtained experimentally observable vortex type solutions. Unlike us, they did not obtain convective flow behavior coinciding with experimentally observed results for possible values of the wall friction coefficient. A possible explanation for this discrepancy between their findings and ours may be the fact that we included the gravity force directly in our model and they did not.

As mentioned above, we have suppressed the free-boundary conditions that occur in an actual vibrating bed in our numerical experiments. This has been done to simplify the numerical solution of the problem, since incorporation of the free-boundaries significantly complicates the problem and for us is still in the developmental stage. Preliminary results indicate that the addition of free-boundaries will still result in the appearance of "stable" convective vortices, and we plan to demonstrate this in a forthcoming paper. Apparently, the frictional effects of the walls is the primary mechanism in the generation of convective rolls. For now then, our analysis of the vortices must be considered to be of a local rather than a global nature.

Figure 7: Motion of particles starting with four-vortex initial configuration (numerical solutions)

Figure 8: Motion of particles starting with random initial velocity distribution (numerical solutions)

8 Concluding Remarks

Starting with well-established representations for particle-particle normal and tangential frictional forces (based on sound theoretical principles and a large body of experimental observations), we derived a new class of integro-partial differential equations to describe the velocity field in the granular flow of rough, inelastic particles. These granular flow models were obtained by taking a dynamical limit as $N \to \infty$ of the Newtonian system of differential equations of motion of an N-particle array using integral averages of an assumed uniform distribution of particles comprising the flow field. Then by employing Taylor series expansions of key variables of the flow field, we were able to obtain an infinite collection of approximations of the model equations in the form of a system of three nonlinear partial differential equations for the velocity components of the granular flow. The simplest of these approximate models, obtained by retaining only the first two terms in the Taylor expansions, is a system of equations that is significantly less complicated than most of the continuum models currently being used to investigate particle flow dynamics.

Our models, and especially the simplest of the approximations, certainly do not incorporate as much of the physics involved in granular flows as do the more comprehensive partial differential equation models, yet they appear to be quite promising instrumentalities for the prediction of particle flow behavior. A good indication of this is the results of our application of the simplest model to granular flow through a vertical tube, fully developed flow down an inclined plane and flow in a vibrating bed which produced relatively simple solutions that compared remarkably well, in a qualitative sense, with experimental observations and the predictions of more complete models. This suggests that, in spite of the simplifying assumptions we used in the derivation, these models may be capable of accurately predicting dynamical properties of a wider range of granular flow configurations than one might imagine. And that they certainly warrant further investigation and testing. Moreover, the new models are far more amenable to analysis (particularly from the viewpoint of infinite-dimensional dynamical systems theory) than the majority of governing equations in the literature. Therefore it is quite possible that one may be able to apprehend important new insights into several elusive granular flow phenomena from a more penetrating mathematical investigation of the properties of the new equations.

In the near future we plan to undertake an intensive analytical and computational study of the models introduced in this paper. For example, we shall use dynamical systems theory to identify and analyze such phenomena as inertial manifolds, bifurcations, strange attractors and regimes of spatio-temporal chaos that will then be correlated to a variety of complex granular flow behaviors. In addition, it should be useful to develop and implement algorithms for the approximate numerical solution of the models, and then compare the results obtained with those from simulations,experimental studies and other governing equations. We shall begin this program of investigation by conducting a more thorough analysis of vibrating bed flows and also studying granular flow in hoppers.

Acknowledgments

of the original version of this paper.

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