Viewing the Efficiency of Chaos Control

Philippe CHANFREAU † *and Hannu LYYJYNEN* ‡

† *Department of Mathematics, Åbo Akademi, FIN-20500 Turku, Finland*

‡ *Department of Applied Mathematics, Luleå University of Technology,*
  S-97187 Luleå, Sweden

Abstract

This paper aims to cast some new light on controlling chaos using the OGY- and the Zero-Spectral-Radius methods. In deriving those methods we use a generalized procedure differing from the usual ones. This procedure allows us to conveniently treat maps to be controlled bringing the orbit to both various saddles and to sources with both real and complex eigenvalues. We demonstrate the procedure and the subsequent control on a variety of maps. We evaluate the control by examining the basins of attraction of the relevant controlled systems graphically and in some cases analytically.

1   Introduction

One of the basic properties of a chaotic dynamic system is topological transitivity. We can take advantage of that property. In practice it can be used to get from any point in the attractor to the vicinity of a given point. The popular OGY-method is typically used when the target point is an unstable fixed point or a point in an unstable periodic cycle. Once we have arrived close enough to the target point we use linear control with respect to a parameter in order to stay in that neighborhood. The OGY-method and its various versions aim at bringing the orbit to the stable set of the target point and keeping it there. In that way the system itself will attract the trajectory to the target. The method is described in [14] and among many other well formulated descriptions and derivations we want to mention [7, 8, 9]. Among others, [15] deals with special techniques for stabilising unstable periodic cycles of higher order. Several other variants of local control through parameter perturbation and their comparison to the OGY-method are presented in [1, 2, 10]. An investigation of the relations between these methods and the "pole placement" methods can be found for instance in [13]. An attempt to establish a general methodology is made in [6, 11, 12]. In order to get close to a certain point in a chaotic attractor rapidly from a relatively distant point, various versions of the "Targeting"-method can be used. The original idea of that method is described in [16, 17, 18]. A large

overview of early chaos control methods was made in [4]. And recently the same authors have assembled most of recent work on the subject in [5].

In this paper methods of controlling by parameter perturbation in a map are used to stabilize the system to an unstable fixed point. A complementary set of examples can be seen in [3]. We will use the OGY-method and the Zero-Spectral-Radius method. The examples were chosen in order to get a broad variation in the properties of the unstable fixed points or periodic cycles. They vary from saddles to sources. Some eigenvalues of the corresponding Jacobian matrices are real and some are complex. We investigate two- and three-dimensional maps.

In some cases control is applied several times on the same map, each time perturbing a different parameter.

Our main goal is to evaluate the efficiency of the above methods on the variety of maps they are tested. The efficiency can be viewed through the plots (with level curves) of the basins of attraction of the controlled systems. In the case of the three-dimensional maps the basins of attraction are viewed in perpendicular cross-sections through the fixed point. Explicit expressions for the controlled systems are derived in several cases. In some of the cases the basin of attraction of the controlled system is so small or so narrow that the local control method in question is to be employed with special precaution or to be preceded by a "targeting" procedure.

This paper is organized in the following manner:

Section 2. The derivation of formulae for linear control by parameter perturbation. Both the Zero-Spectral-Radius method (ZSR) and the OGY-method are derived through matrix manipulations.

Section 3. A study of the control of a 2-dimensional discrete map:

The Hénon map is stabilised to a saddle fixed point in several cases. Control is implemented with respect to several available parameters and using the two methods mentioned.

Section 4. A study of the control of three-dimensional discrete maps:

In subsection 4.1 a Hénon-like map is controlled to a fixed point with one unstable real eigenvalue and two stable complex ones. Both the OGY- and the ZSR-methods are used.

In subsection 4.2 a system of three coupled logistic maps is controlled with the OGY-method. The map is stabilised to a fixed point with one stable and two unstable directions.

2   General theory and control formulae

Let us consider the map

$$\overline{x} \to F(\overline{x}, p), \tag{2.1}$$

where $\overline{x}$ is an n-dimensional vector and p is a real parameter. Without loss of generality we assume the fixed point $\overline{x}^*$ to be the origin for the parameter value $p = 0$ (if we are interested in other periodic orbits we here use the corresponding iterate of f instead of f itself). The map can then be written in the form

$$\overline{x} \to J\overline{x} + p\overline{w} + F_J, \tag{2.2}$$

where J is the Jacobian matrix and $\overline{w} = \frac{\partial F}{\partial p}$, both calculated in the fixed point (here: the origin) and with $p = 0$. F_J is of higher order.

To control the system we choose an expression for p being a linear function of the components of $\overline{x} : p = \overline{\alpha}_J^T \overline{x}$.

Thus the function representing the controlled system can also be written in the form

$$\overline{x} \to J_c \overline{x} + F_J, \tag{2.3}$$

where $J_c = J + \overline{w}\,\overline{\alpha}_J^T$.

The next step is a change of coordinates. Suppose the new coordinates $\overline{u}$ are defined by $\overline{x} = Q\overline{u}$, where Q is an $n \times n$ non-degenerate matrix. In the new coordinates the map (2.1) can be written in the following form:

$$\overline{u} \to M\overline{u} + p\overline{b} + F_M, \tag{2.4}$$

where $M = Q^{-1}JQ$, $\overline{b} = Q^{-1}\overline{w}$ and F_M is of higher order.

The controlled system can be written in the form

$$\overline{u} \to M_c \overline{u} + F_J, \tag{2.5}$$

where $M_c = M + \overline{b}\,\overline{\alpha}_M^T$, $\overline{\alpha}_M^T = \overline{\alpha}_J^T Q$.

The case where M becomes a diagonal matrix or a matrix in Jordan normal form is of special interest.

We shall look at ways of choosing the coefficients in order to make the control work. If the coefficients in $\overline{\alpha}_M$ (equivalently $\overline{\alpha}_J$) are chosen so that the absolute value of the eigenvalues are less than one the fixed point becomes stable for the controlled system. This means that we can control the system in the vicinity of the fixed point (actually the controlled system will converge towards the fixed point).

We shall use two methods: the Zero-Spectral-Radius method (ZSR) and the OGY-method. In the ZSR-method all eigenvalues of the Jacobian matrix for the controlled system are required to be zero. In the OGY-method only the unstable eigenvalues, that is the ones with absolute value greater than one, are required to be zero and the stable ones, those with absolute value less than one are left unchanged. Let's have a look at the first case.

2.1 The Zero-Spectral-Radius method (ZSR)

We can derive general expressions for the coefficients α_i (the components of $\overline{\alpha}_M$) making all the eigenvalues of M_c equal to zero.

Theorem 2.1. *Let the matrix M be a diagonal matrix*

$$\begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix},$$

where $\lambda_i \neq \lambda_j$ for $i \neq j$. Furthermore we suppose that all $b_i \neq 0$. Then the eigenvalues of M_c are all equal to zero if and only if

$$\alpha_i = \frac{-\lambda_i^n}{\prod\limits_{j=1, j\neq i}^{n} (\lambda_i - \lambda_j) b_i}, \qquad i = 1, \ldots, n. \tag{2.6}$$

(λ_i can be complex).

Proof. The characteristic polynomial $P(\lambda) = \det(M_c - \lambda I) \equiv (-1)^n \lambda^n$, since all its roots are equal to zero. We will develop $M_c = M + \bar{b}\bar{\alpha}_M^t$. We get

$$
M_c = \begin{pmatrix}
\lambda_1 + b_1\alpha_1 & b_1\alpha_2 & \cdots & b_1\alpha_n \\
b_2\alpha_1 & \lambda_2 + b_2\alpha_2 & \cdots & b_2\alpha_n \\
\cdots & \cdots & \cdots & \cdots \\
b_n\alpha_1 & b_n\alpha_2 & \cdots & \lambda_n + b_n\alpha_n
\end{pmatrix},
$$

and $P(\lambda) = \det(M_c - \lambda I)$ equals

$$
\det \begin{pmatrix}
\lambda_1 + b_1\alpha_1 - \lambda & b_1\alpha_2 & \cdots & b_1\alpha_n \\
b_2\alpha_1 & \lambda_2 + b_2\alpha_2 - \lambda & \cdots & b_2\alpha_n \\
\cdots & \cdots & \cdots & \cdots \\
b_n\alpha_1 & b_n\alpha_2 & \cdots & \lambda_n + b_n\alpha_n - \lambda
\end{pmatrix}.
$$

Now develop $P(\lambda_1)$ by elementary column operations, which results in

$$
\alpha_1 \det \begin{pmatrix}
b_1 & b_1\alpha_2 & \cdots & b_1\alpha_n \\
b_2 & \lambda_2 - \lambda_1 + b_2\alpha_2 & \cdots & b_2\alpha_n \\
\cdots & \cdots & \cdots & \cdots \\
b_n & b_n\alpha_2 & \cdots & \lambda_n - \lambda_1 + b_n\alpha_n
\end{pmatrix}
$$

$$
= \alpha_1 \det \begin{pmatrix}
b_1 & 0 & \cdots & 0 \\
b_2 & \lambda_2 - \lambda_1 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots \\
b_n & 0 & \cdots & \lambda_n - \lambda_1
\end{pmatrix}
$$

$$
= b_1\alpha_1(-1)^{n-1}(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)\cdots(\lambda_1 - \lambda_n) = (-1)^n \lambda_1^n.
$$

Solving for α_1 we get (2.6) for $i = 1$. For α_i, $i = 1, \ldots, n$ we get the analogous expressions as necessary conditions.

If the characteristic polynomial has coefficients c_i, $i = 0, \ldots, n$, that is $P(\lambda) = c_n\lambda^n + \cdots + c_1\lambda + c_0$, then we can immediately see from the determinant expansion that the leading coefficient must be $(-1)^n$ then giving a system of n equations $c_{n-1}\lambda_i^{n-1} + \cdots + c_1\lambda_i + c_0 = 0$, $i = 1, \ldots, n$ which has only the trivial solution because the λ_i:s are all supposed to be different from each other. (The system can be written in the form $C(c_{n-1}, \ldots, c_1, c_0)^T = \bar{0}^T$ and it is well known that in this case the determinant of C is different from zero). Thus the conditions are also sufficient. The proof is finished.

We shall now look for the concrete formulae in some special cases used in this paper.

Suppose J is two-dimensional and has two real eigenvalues $\lambda_1 \neq \lambda_2$. Then we get the following formulae for the coefficients:

$$
\alpha_1 = \frac{-\lambda_1^2}{b_1(\lambda_1 - \lambda_2)}, \qquad \alpha_2 = \frac{-\lambda_2^2}{b_2(\lambda_2 - \lambda_1)}. \tag{2.7}
$$

Suppose J is two-dimensional and has two complex eigenvalues $\lambda \pm i\mu$, where $\mu \neq 0$. We remark that

$$
S^{-1} \begin{pmatrix} \lambda + \mu i & 0 \\ 0 & \lambda - \mu i \end{pmatrix} S = \begin{pmatrix} \lambda & \mu \\ -\mu & \lambda \end{pmatrix},
$$

where

$$S = \begin{pmatrix} i & 1 \\ 1 & i \end{pmatrix}.$$

Thus the expressions for α_1 and α_2 in this case can be obtained from the case with two different real eigenvalues if we put $\lambda_1 = \lambda + \mu i$, $\lambda_2 = \lambda - \mu i$ and use $\overline{b} = S\tilde{b}$ and $\tilde{\alpha}^T = \overline{\alpha}^T S$, where $\overline{b}$ gives b_1 and b_2 for the real eigenvalue case and $\tilde{b}$ gives b_1 and b_2 for the complex eigenvalue case and analogously for $\overline{\alpha}$ and $\tilde{\alpha}$.

Consequently if M is of the form

$$\begin{pmatrix} \lambda & \mu \\ -\mu & \lambda \end{pmatrix},$$

where $\mu \neq 0$ then we get the following expressions for the coefficients:

$$\alpha_1 = (b_2(\mu^2 - \lambda^2) - 2b_1\mu\lambda)/A, \qquad \alpha_2 = (b_1(\lambda^2 - \mu^2) - 2b_2\mu\lambda)/A, \tag{2.8}$$

where $A = \mu(b_1^2 + b_2^2)$.

Suppose now J is three-dimensional and has one real eigenvalue λ_u and two complex eigenvalues $\lambda \pm i\mu$, where $\mu \neq 0$. Suppose M is of the form

$$\begin{pmatrix} \lambda_u & 0 & 0 \\ 0 & \lambda & \mu \\ 0 & -\mu & \lambda \end{pmatrix},$$

where $\mu \neq 0$. Then we get the coefficients:

$$\alpha_1 = -\lambda_u^3/(Db_1), \qquad \alpha_2 = -(Bb_2 + Cb_3)AD, \qquad \alpha_3 = (Cb_2 - Bb_3)/AD, \tag{2.9}$$

where

$$A = \mu(b_2^2 + b_3^2), \qquad B = 2\lambda^3\mu + \lambda_u\mu^3 + 2\mu^3\lambda - 3\lambda_u\lambda_2\mu,$$
$$C = \lambda^4 - \lambda_u\lambda^3 + 3\lambda_u\mu^2\lambda - \mu^4, \qquad D = \lambda^2 + \mu^2 - 2\lambda_u\lambda + \lambda_u^2.$$

2.2   The OGY formulae

The same techniques and the formulae in the previous subsection can also be used for the OGY-control. We give the result in some cases.

Suppose for a two-dimensional system the fixed point is a saddle and M has the form

$$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix},$$

where the absolute value of λ_1 is less than one and the absolute value of λ_2 is greater than one. Then we get the coefficients:

$$\alpha_1 = 0, \qquad \alpha_2 = -\lambda_2/b_2. \tag{2.10}$$

This is easily seen to coincide with the known form of the OGY-formula. We have $\overline{\alpha}_J^T = \overline{\alpha}_M^T Q^{-1}$. And since the columns of Q consist of the eigenvectors of J the rows of

Q^{-1} consist of the contravariant basis vectors $\overline{f}_1$ and $\overline{f}_2$. This leads to $\overline{\alpha}_J^T = -\lambda_2 \overline{f}_2^T / b_2$. But because $\overline{b} = Q^{-1}\overline{w}$ we get $b_2 = \overline{f}_2 \cdot \overline{w}$ resulting in the familiar

$$\overline{\alpha}_J^T = -\frac{\lambda_2 \overline{f}_2^T}{\overline{f}_2^T \overline{w}}.$$

Suppose for a three-dimensional system the fixed point is a saddle with two-dimensional unstable manifold and M has the form

$$\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix},$$

where the absolute values of λ_1 and λ_2 are greater than one and the absolute value of λ_3 is less than one.

Then we get the coefficients:

$$\alpha_1 = \frac{-\lambda_1^2}{b_1(\lambda_1 - \lambda_2)}, \qquad \alpha_2 = \frac{-\lambda_2^2}{b_2(\lambda_2 - \lambda_1)}, \qquad \alpha_3 = 0 \tag{2.11}$$

similar to formula (2.7).

3   Comparing ways of controlling a two-dimensional map

We have used our method to control some two-dimensional maps with only linear and quadratic terms. The fixed points were saddles and even sources. In this section, however, we will concentrate on showing different ways of controlling the familiar Hénon map with a saddle. The cases with sources are investigated in [3] and shortly treated in the discussion.

3.1   The Hénon map

Let us now consider the map

$$\begin{pmatrix} X \\ Y \end{pmatrix} \rightarrow \begin{pmatrix} a - bY - cX^2 \\ dX + e \end{pmatrix}. \tag{3.1}$$

For $c = d = 1$, $e = 0$ this is the famous Hénon map. The parameters c, d and e are introduced in order to apply control in several ways to the standard Hénon map. Control is applied by perturbing one of the parameters while the other ones are held fixed. For instance applying control with respect to the parameter a is carried out by permitting the parameter to vary about a nominal value a^*, so that $a = a^* + p$. Applying control with respect to any of the parameters b, c, d and e is carried out analogously by adding the perturbation p to the nominal value of the parameter in turn to be used. In addition we apply control by adding multiples of p to both coordinates.

In the following, $c^* = d^* = 1$ and $e^* = 0$ will be used. We will assume that the values of a^* and b^* are such that the map has a chaotic attractor and that within the attractor there is a saddle fixed point (x^*, y^*), with one stable direction with the stable eigenvalue λ_s and one unstable direction with the unstable eigenvalue λ_u. Notice that, given the above values of the parameters, $x^* = y^*$.

One example of such a case is the common version of the Hénon map with $a^* = 1.4$ and $b^* = -0.3$. It is treated in many papers and textbooks. In the numerical example in Section 3.1.7 we will consider a special case with values $a^* = 1.05$ and $b^* = -0.5$. Iterating the map yields a familiar multi-folded horseshoe-shaped attractor shown in Fig. 1a). One reason for choosing this particular version of the Hénon map is that the stable eigenvalue of the fixed point is larger (0.2665) than in the "common" case (0.1559). This fact leads to bigger differences in the results when we apply the two control methods, ZSR and OGY.

We control the system by the ZSR and OGY methods. In order to use the formulae in Section 2 we transform the system in two steps:

1) Moving the fixed point to the origin. Separating the linear and non-linear parts.

2) Taking the eigenvectors as new basis vectors.

3.1.1   Moving the fixed point and separating matrix terms

We introduce the new variables $(x, y) = (X - x^*, Y - y^*)$. The map (3.1) can now be rewritten as

$$\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} a - b(y + y^*) - c(x + x^*)^2 - x^* \\ d(x + x^*) + e - y^* \end{pmatrix}. \tag{3.2}$$

We apply control with respect to each of the parameters a, b, c, d and e one by one as described in the beginning of Section 3.1. Expanding and separating the matrix terms (keeping in mind that $x^* = y^*$, $c^* = 1$, $d^* = 1$ and $e^* = 0$), we obtain the formula (2.2) in each case with the following expression for J:

$$J = \begin{pmatrix} -2x^* & -b^* \\ 1 & 0 \end{pmatrix}.$$

The expressions for $\overline{w}$ will vary:

$$a)\ \overline{w} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad b)\ \overline{w} = \begin{pmatrix} -y^* \\ 0 \end{pmatrix}, \tag{3.3}$$

$$c)\ \overline{w} = \begin{pmatrix} -(x^*)^2 \\ 0 \end{pmatrix}, \qquad d)\ \overline{w} = \begin{pmatrix} 0 \\ x^* \end{pmatrix} \quad \text{or} \quad e)\ \overline{w} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{3.4}$$

depending on which parameter to be perturbed.

In the following we consider especially the first two cases $a)$ and $b)$. On the other hand we have applied control perturbing both a and e at the same time in the following cases (meaning $(a, e) \rightarrow (a, e)^T + p\overline{w}$, and control with respect to p):

$$f)\ \overline{w} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \qquad g)\ \overline{w} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad \text{and} \qquad h)\ \overline{w} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}. \tag{3.5}$$

Out of these, we especially consider the case $f)$.

The expressions for F_J is the same in cases $a)$, $e)$, $f)$, $g)$ and $h)$:

$$\overline{F_J} = \begin{pmatrix} -x^2 \\ 0 \end{pmatrix}.$$

The expression for F_J in case $b)$ is

$$\overline{F_J} = \begin{pmatrix} -py - x^2 \\ 0 \end{pmatrix}.$$

3.1.2   Changing basis vectors

The eigenvalues of J are $\lambda = -x^* \pm \sqrt{(x^*)^2 - b^*}$ which results in $\lambda_1\lambda_2 = b^*$ and $\lambda_1 + \lambda_2 = -2x^*$. As mentioned in the beginning of this section we are particularly interested in the fixed point being a saddle. The eigenvalues can then be noted λ_s and λ_u and the eigenvectors are

$$\overline{e}_s = \begin{pmatrix} \lambda_s \\ 1 \end{pmatrix}, \qquad \text{and} \qquad \overline{e}_u = \begin{pmatrix} \lambda_u \\ 1 \end{pmatrix}.$$

The basis is changed so that the eigenvectors of the Jacobian matrix become the new basis vectors. The coordinates in the new eigenvector basis are called (u, v), with u being the coordinate in the stable direction and v the coordinate in the unstable direction. The matrix Q is the transition matrix

$$Q = \begin{pmatrix} \lambda_s & \lambda_u \\ 1 & 1 \end{pmatrix}$$

leading to a diagonal matrix $M = Q^{-1}JQ$. Since $\overline{u} = Q^{-1}\overline{x}$, we get

$$u = \frac{1}{\lambda_s - \lambda_u}(x - \lambda_u y), \qquad v = \frac{1}{\lambda_s - \lambda_u}(-x + \lambda_s y). \tag{3.6}$$

Furthermore, since

$$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = Q^{-1}\overline{w},$$

we get for control with respect to a

$$b_1 = \frac{1}{\lambda_s - \lambda_u}, \qquad b_2 = \frac{-1}{\lambda_s - \lambda_u}, \tag{3.7}$$

we get for control with respect to b

$$b_1 = \frac{-y^*}{\lambda_s - \lambda_u}, \qquad b_2 = \frac{y^*}{\lambda_s - \lambda_u}, \tag{3.8}$$

and for control in the f)-case:

$$b_1 = \frac{1 + \lambda_u}{\lambda_s - \lambda_u}, \qquad b_2 = \frac{-1 - \lambda_s}{\lambda_s - \lambda_u}. \tag{3.9}$$

3.1.3   Using Zero-Spectral-Radius control

Although the fixed point in our Hénon attractor typically is a saddle point it does not matter in the case of the ZSR-method if there is a stable eigenvalue or not. So we will here use the more general notation (λ_1, λ_2) instead of (λ_s, λ_u). We can now obtain expressions for α_1 and α_2 in accordance with (2.7) leading to explicit expressions for the parameter perturbation $p = \alpha_1 u + \alpha_2 v$ for the Hénon map (using $\lambda_1\lambda_2 = b$ and $\lambda_1 + \lambda_2 = -2x^*$)

In the case of control with respect to a we get

$$\alpha_1 = -\lambda_1^2, \qquad \alpha_2 = -\lambda_2^2 \qquad \text{and} \qquad p = 2x^*x + b^*y. \tag{3.10}$$

In the case of control with respect to b we get

$$\alpha_1 = \frac{\lambda_1^2}{y^*}, \qquad \alpha_2 = \frac{\lambda_2^2}{y^*} \qquad \text{and} \qquad p = -2x - \frac{b^*}{y^*}y. \tag{3.11}$$

In the case $f)$ we get

$$\alpha_1 = \frac{-\lambda_1^2}{1+\lambda_2}, \qquad \alpha_2 = \frac{-\lambda_2^2}{1+\lambda_1}. \tag{3.12}$$

In the case $f)$ the analytical expression for p cannot be brought to a simple form, so it is left out. Thus in the following only the $a)$- and $b)$-cases are treated analytically.

3.1.4   Deriving the controlled system for the ZSR-method

Control parameters are determined from the linear part of the system, but nevertheless the non-linear part is still present and together with the parameter perturbation a new system, the "controlled system" is created. In general, the controlled system will differ more from the original one the further one gets from the fixed point. We insert the expressions for p into equation (2.2) and get the following controlled systems.

- In the case of control with respect to the parameter a:

$$\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} -x^2 \\ x \end{pmatrix}. \tag{3.13}$$

- In the case of control with respect to the parameter b:

$$\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} -x^2 - \dfrac{(\lambda_1+\lambda_2)}{y^*}xy + \dfrac{\lambda_1\lambda_2}{y^*}y^2 \\ x \end{pmatrix}, \tag{3.14}$$

which is the same as

$$\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} -x^2 + 2xy + \dfrac{b^*}{y^*}y^2 \\ x \end{pmatrix}. \tag{3.15}$$

In the first case one can clearly see that the controlled system is only dependent on x and convergence to the fixed point (the origin) will occur under the same circumstances as for the one-dimensional map $x \rightarrow -x^2$. So there is convergence for (3.13) to the fixed point $(0,0)^T$, that is $(X,Y)^T = (x^*,y^*)^T$, from starting points with $X \in]x^* - 1, x^* + 1[$. For $X = x^* \pm 1$ there is convergence to the point $(x^* - 1, x^* - 1)^T$ and divergence otherwise.

3.1.5 Using OGY-control

Since the OGY-method is based on the very property of a fixed point being a saddle with at least one stable direction we will here use the notations (λ_s, λ_u) instead of (λ_1, λ_2).

According to (2.10) we get $p = \alpha_1 u + \alpha_2 v = 0u - \frac{\lambda_u}{b_2}v$. Thus

$$p = \frac{-\lambda_u v}{b_2}. \tag{3.16}$$

Using (3.6) and (3.7)–(3.9) we can get the particular expressions for each version of parameter perturbation.

3.1.6 Deriving the controlled system for the OGY-control

Inserting the expressions for p into equation (2.2) we get the following expressions for the controlled system applying OGY-control.

- In the case of control with respect to a:

$$\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} \lambda_s x - x^2 \\ x \end{pmatrix}. \tag{3.17}$$

- In the case of control with respect to b:

$$\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} \lambda_s x - x^2 - \frac{\lambda_u}{y^*}xy + \frac{\lambda_s \lambda_u}{y^*}y^2 \\ x \end{pmatrix}. \tag{3.18}$$

Here again in the first case the controlled system is only dependent on x and convergence to the fixed point (the origin) will occur under the same circumstances as for the one-dimensional map $x \rightarrow \lambda_s x - x^2$. So there is convergence for (3.17) to the fixed point $(0,0)^T$, that is $(X,Y)^T = (x^*, y^*)^T$, from starting points with $X \in]x^* + \lambda_s - 1, x^* + 1[$. For $X = x^* + \lambda_s - 1$ and $X = x^* + 1$ there is convergence to the point $(x^* + \lambda_s - 1, x^* + \lambda_s - 1)^T$ and divergence otherwise.

3.1.7 A numerical example

The Hénon map with parameter values $a = 1.05$ and $b = -0.5$, (and $c = d = 1$, $e = 0$) will be used. The corresponding chaotic attractor is shown in Fig. 1a). The unstable fixed point is $(0.8048, 0.8048)^T$ and it is marked with a circle (o) in the plot. The eigenvalues for the Jacobian matrix are $\lambda_s = 0.2665$ and $\lambda_u = $ -1.8760. The corresponding eigenvectors are $\overline{e}_s = (0.2665, 1)^T$ and $\overline{e}_u = (-1.8760, 1)^T$. The two methods, OGY and ZSR, yield almost congruent basins of attraction using the same parameter (parameters b, c, d or e). On the other hand depending on the parameter to be perturbed the basins of attraction are very different in shape. In the cases g) and h) the basins are similar in shape and geometry regardless of the method (mostly almost vertical contour lines). The f)-case application of the two methods yields a vastly differing inner geometry, as can be seen in the Fig. 1a) and 1c). In all cases the ZSR-method displays larger regions of the fastest convergence than the OGY-method perturbing the same parameter.

a)   b)

c)   d)

Figure 1: *a)* OGY-control in the case *f.* *b)* OGY-control with respect to the parameter *b.*
c) ZSR-control in the case *f.* *d)* OGY- and ZSR-control with respect to the parameter *a.*

In Fig. *a)*, *b)* and *c)* the map is iterated 5 times. The central dotted regions contain
the points from which the map converges and reaches within a distance of less than 0.001
from the fixed point (o). The peripheral dotted regions represent points from which the
map certainly diverges.

Fig. *a)* displays the result of OGY-control in the *f)*-case. The level curves are: 0.001,
0.01 and 0.1. Fig. *b)* shows the result of OGY-control with respect to the parameter *b*.
The level curves are: 0.003 and 0.06. Fig. *c)* displays the result of ZSR-control in the
f)-case. The level curves are: 0.00001, 0.0001, 0.001 and 0.01. Finally in Fig. *d)* the
results of control using both OGY- and ZSR-methods with respect to the parameter *a* are
displayed. It was stated in subsections 3.1.4 and 3.1.6 that the controlled systems only
depend on x. The graphs show the number of iterates needed to get within a distance of
0.001 from the fixed point from various starting points. The solid graph represents the
ZSR-method. The dotted lines $X = x^* \pm 1$ mark the boundaries of the basin of attraction.
The dotted graph represents the OGY-method. The dotted lines $X = x^* + \lambda_s - 1$ and
$X = x^* + 1$ mark the boundaries of the basin of attraction.

In this section we have not so far paid particular interest in limiting the size of the
parameter perturbation p. We have looked for the entire basin of attraction regardless of
how much an unlimited value for p might distort the system. This calls for a discussion.

Where is the region of the smallest initial p:s? Let us have a look at the OGY-method first. The parameter perturbation p is generally expressed in formula (3.16). Clearly the smallest values for p are required close to the straight line determined by $v = 0$ (formula (3.6)), that is along the direction of the stable eigenvector $(\lambda_s, 1)^T$ or $(1, 3.75)^T$. This applies in all of our cases and is no big surprise giving the basic idea of the OGY-method. Investigating the expressions for p (formulae (3.10)–(3.11) and the other cases c)–h) not explicitly given in this text) in the ZSR-method yield correspondingly straight lines as the centre lines of stripes constituting the regions of the smallest initial p:s. These lines have directions $(1, -2x^*/b^*)^T$ or $(1, 3.22)^T$ in the cases a), b) and c). The cases d) and e) have a centre line with direction $(1, 3.84)^T$. In many cases the direction of the line fits well in the image of the basins of attraction. For instance, in Fig. 1b) this stripe of small p:s follows narrow parts of fast convergence in the basin of attraction. Nevertheless, the further one gets along the line from the fixed point the more powerful the non-linear part of the controlled system becomes yielding uncertain results.

4   Controlling three-dimensional maps

4.1   Controlling a 3-dimensional Hénon-like map

Let us consider a three-dimensional map somewhat similar to the Hénon map:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \rightarrow \begin{pmatrix} ax + by + cz - x^2 + p \\ x \\ y \end{pmatrix}, \tag{4.1}$$

where normally p $= 0$.

We will investigate the effect of controlling the map with the OGY-method and the Zero-Spectral-Radius method. We will consider a case where there is a chaotic attractor including an unstable fixed point (the origin) with one unstable direction, with the eigenvalue λ_u, and a stable manifold with two complex eigenvalues λ_2 and λ_3. We will apply control using the parameter p.

4.1.1   The control formulae

In order to write the OGY-control formula we refer to Section 2 formula (2.10). Analogously with that formula we have $\alpha_1 = -\lambda_u/b_1$, $\alpha_2 = 0$ and $\alpha_3 = 0$ yielding $p = -\lambda_u u_1/b_1$, where u_1 is the first coordinate of the new basis vector $\overline{u}$.

As to the ZSR-control formula we refer to Section 2 formula (2.9). The derivation of the coefficients is quite tedious but is conveniently carried out with the aid of a computer.

4.1.2   The controlled systems

The controlled system of the map applying the control methods can be derived as follows. Consider the Jacobian matrix of the controlled system:

$$\begin{pmatrix} A & B & C \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

where $A = a + \alpha_1$, $B = b + \alpha_2$, $C = c + \alpha_3$ and $p = \alpha_1 x + \alpha_2 y + \alpha_3 z$. The characteristic equation is

$$\lambda^3 - A\lambda^2 - B\lambda - C = 0. \tag{4.2}$$

In the case of OGY-control the characteristic equation can also be written in the following form since the one unstable eigenvalue is required to be zero: $\lambda(\lambda - \lambda_2)(\lambda - \lambda_3) = 0$. Developing and comparing the two equations yield: $A = \lambda_2 + \lambda_3$, $B = -\lambda_2\lambda_3$, and $C = 0$. So the map has a controlled system of the following form when λ_u is required to be zero in accordance with the OGY-method:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \rightarrow \begin{pmatrix} (\lambda_2 + \lambda_3)x - \lambda_2\lambda_3 y - x^2 \\ x \\ y \end{pmatrix}. \tag{4.3}$$

This implies that the controlled system does not depend on z.

In the case of ZSR-control the characteristic equation can also be written in the following form since all eigenvalues are required to be zero: $\lambda^3 = 0$.

Comparing with the equation 4.2 yields: $A = B = C = 0$. So the linear part of the first equation vanishes and the map has a controlled system of the following form:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \rightarrow \begin{pmatrix} -x^2 \\ x \\ y \end{pmatrix}. \tag{4.4}$$

As for the Hénon map the map 4.4 will depend only on x and convergence will occur under the same circumstances as the one-dimensional map $x \rightarrow -x^2$. That is the map will converge to the fixed point $(0, 0, 0)^T$ from any point with $x \in]-1, 1[$. For points with $x = \pm 1$ there is convergence to the point $(-1, -1, -1)^T$ and divergence otherwise.

4.1.3   A numerical example

We will consider the particular case:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \rightarrow \begin{pmatrix} -1.65x - 0.3y - 0.2z - x^2 + p \\ x \\ y \end{pmatrix}. \tag{4.5}$$

The corresponding attractor can be seen in Fig. 2a).

As can easily be seen the map has a fixed point in the origin $(0, 0, 0)$ situated in the chaotic attractor. The eigenvalues and eigenvectors of the Jacobian matrix in this point are

$$\lambda_u = -1.5395,$$
$$\overline{e}_u = (0.7906, -0.5135, 0.3336)^T,$$
$$\lambda_{2,3} = -0.0552 \pm 0.3562i,$$
$$\overline{e}_{2,3} = (-0.1211 \mp 0.0076i, -0.0306 \pm 0.3352i - 0.9059 \mp 0.2265i)^T.$$

That is, there are one real unstable eigenvalue λ_u and two complex stable eigenvalues λ_2 and λ_3.

a)   b)

y
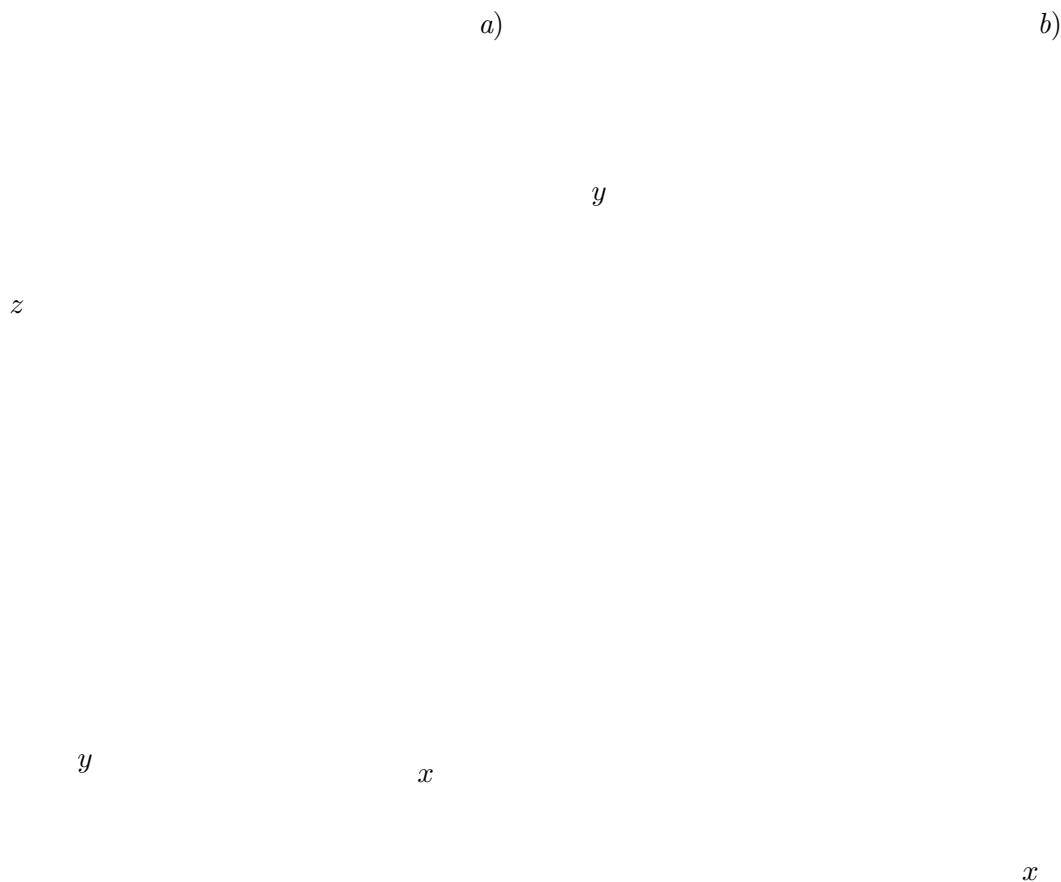
z

y   x

x

Figure 2: *a)* The attractor. *b)* A cross-section of the basin of attraction through the fixed point at $z = 0$ using OGY-control.

We shall now have a look at the basin of attraction and the rate of convergence of the control methods depending on the starting point. Three cross-sections were made through the fixed point: those were the planes $x = 0$, $y = 0$ and $z = 0$. The controlled system was iterated five times starting from each point on each plane. Contour lines show how close the system point has reached the fixed point after five iterations. We choose to display the result only from the cross-section $z = 0$.

Controlling with the OGY-method. Applying the method in order to control the map to the fixed point $(0, 0, 0)^T$ works well within certain limits of distance from that point. First, the planes $x = 0$ and $y = 0$ only display parallel contour lines which is explained by the controlled system not depending on z. The contour lines of the plane $z = 0$ are shown in Fig. 2b).

The fixed point is marked with a cross $(+)$. The vaguely boomerang shaped contour lines represent the levels (counted from inside out): 0.005, 0.010, 0.020, 0.050, and 1.000. The last level represents more or less the boundary of the basin of attraction for the OGY-controlled system. The 0.020 level makes an extra turn at the left side.

Controlling with the Zero-Spectral-Radius method. The $y = 0$ and $z = 0$ planes display parallel lines and $x = 0$ does not display any contour lines at all. This is explained by looking at the extremely simple expression for the controlled system.

From all starting points in the $x = 0$ plane the system goes exactly to the origin in the second iteration at the latest.

4.2 Coupled logistic map with OGY-control

In this subsection we examine the efficiency of the OGY-control for another three-dimensional map. The map is stabilized towards a saddle fixed point with two unstable eigendirections. Another version of this map with a saddle point with one unstable eigendirection is examined in [3]. We consider the system consisting of three coupled logistic maps

$$
\begin{pmatrix} x \\ y \\ z \end{pmatrix} \rightarrow \begin{pmatrix} 1-2p & p & p \\ p & 1-2p & p \\ p & p & 1-2p \end{pmatrix} \begin{pmatrix} r_1 & x & (1-x) \\ r_2 & y & (1-y) \\ r_3 & z & (1-z) \end{pmatrix},
$$

where r_1, r_2, r_3 and p are parameters. The coupling parameter p is chosen to be the control parameter. As the state space of the system we consider the unit cube $[0,1] \times [0,1] \times [0,1]$.

With parameter values $r_1 = 3.9$, $r_2 = 3.95$, $r_3 = 1.0$ and $p = 0.0736$, the point

$$
\overline{x}^{*T} = (0.7291, 0.7323, 0.2889)
$$

is a fixed point and there are one stable and two unstable eigendirections.

The stable eigenvalue and the unstable ones with the corresponding eigenvectors are

$$
\begin{aligned}
\lambda_s &= 0.35600 & \overline{e}_s &= (0.021204, 0.020760, 1.3728)^T \\
\lambda_{u1} &= -1.4094 & \overline{e}_{u1} &= (0.63971, -0.54033, 0.0063062)^T \\
\lambda_{u2} &= -1.6750 & \overline{e}_{u2} &= (0.57127, 0.65842, 0.080606)^T.
\end{aligned}
$$

The vector $\overline{w}$ becomes

$$
\overline{w} = \frac{\partial f}{\partial p}(\overline{x}^*, p^*) = (-0.56084, -0.57308, 1.1339).
$$

Here we calculate the control coefficients according to (2.11) using again in Section 2.2 $\overline{x} - \overline{x}^*$ and $p - p^* = \delta p$ instead of $\overline{x}$ and p.

The explicit expression for the control is obtained as

$$
\delta p = 104.33x - 107.46y + 0.013503z + 2.6172. \tag{4.6}
$$

The controlled map has the Jacobian matrix

$$
J_c = \begin{pmatrix} -60.036 & 60.130 & 0.023502 \\ -59.921 & 60.016 & 0.023337 \\ 118.17 & -121.98 & 0.37538 \end{pmatrix}
$$

with $(0.11687, 0.1163, 1)$ as eigendirection corresponding to the eigenvalue zero and $(1, 0.97904, 64.737)$ as eigendirection for the non-zero eigenvalue.

The cross-sections through the fixed points of the small region of convergence are depicted in Fig. 3. The convergence levels are (calculated using 10 iterates) $(0.048, 0.125, 0.5, 1, 5, 10, 50, 100) \cdot 10^{-9}$.
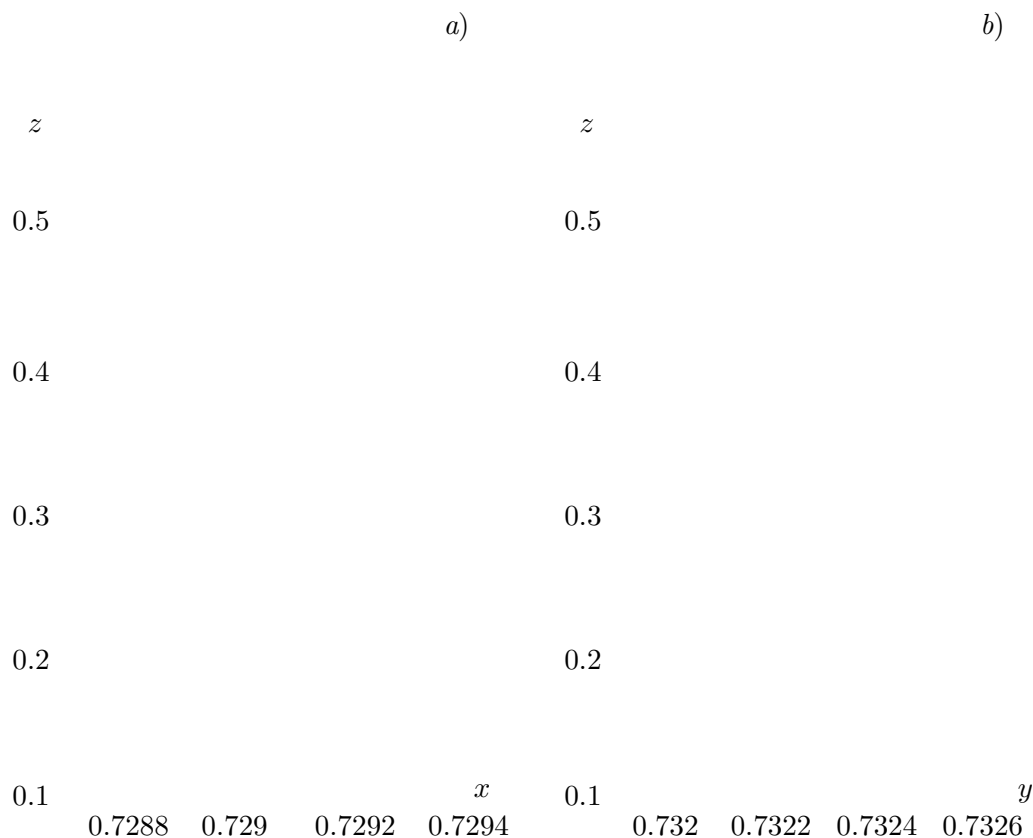
a)   b)

Figure 3: One stable, two unstable eigendirections. Convergence levels for a) $y = 0.7323$ and b) $x = 0.7291$.

5 Discussion

The message of this paper is two-fold:

1. The display of basins of attraction with level-curves for the controlled systems for a variety of maps.

2. A new procedure for deriving control formulae for the OGY-method and related methods.

We have chosen a variety of maps covering some unusual cases with real or complex eigenvalues. For this purpose we have brought forward a procedure, which in our opinion simplifies the derivations of control formulae through matrix diagonalising. Moreover the procedure yields convenient expressions (Section 2). The OGY-method and the ZSR-method thereby appear as mere special cases of a general way of thinking, regardless of the number of dimensions, the proportion of unstable eigendirections or the quality of the eigenvalues. There are possibilities to bring forward other similar methods than the mentioned ones. A comparison with the original form of the OGY formula is given in subsection 2.2. Our procedure is implemented in all sections but it is displayed more in detail in Section 3. This section 3 contains a broad study of many possible ways to control the well-known Hénon map. The goal of the control is to bring the orbit to a saddle fixed

point, which works successfully in all cases. But the geometry of the basin of attraction varies considerably depending on how the control is applied.

Our "three-dimensional Hénon" map has two complex eigenvalues in the origin and serves as an example of applying our procedure in such a case.

In the case of the three-dimensional coupled logistic map all the eigenvalues are real, one of them stable and two unstable. The control coefficients become very large and the basin of attraction to the fixed point very small. The eigenvector corresponding to the eigenvalue zero (in the controlled system) dictates the direction of fastest convergence.

Further examples with convergence level graphs can be found in [3]:

- Another version of the coupled logistic map with one unstable and two stable eigendirections and controlled with the OGY-method.

- Two-dimensional maps are stabilised to sources with either real or complex eigenvalues. Obviously, the OGY-method is not applicable. The control with the ZSR-method worked well although the basins of attraction were rather narrow due to the fact that the two eigenvalues were near each other in size. This was the case also in Section 4.2.

- There is also an example on OGY-controlling the Poincaré map of a Duffing system either to a saddle fixed point or to a two-periodic saddle point. An analysis of the rate of convergence is included.

As to the local speed of convergence: the ZSR is always the more efficient method due to the fact that the eigenvalues of all directions are placed at zero. The speed of the OGY method depends on the value of the stable eigenvalues. If their absolute value is close to one, the convergence is slow. If it is close to zero the behaviour is similar to the one of the ZSR-method. On the other hand, further away from the fixed point the importance of the non-linearity of the map grows. There the properties of the linearisation almost vanish. Therefore, in general, both methods display almost congruent outer boundaries of their basins of attraction. The non-linear influence also create fairly distant areas with fast convergence requiring only limited control. Much closer areas may have slow convergence or even divergence.

We have concentrated on applying linear control to one parameter at a time. In general we would archive better results by perturbing several parameters or by applying non-linear control. Another procedure would be to combine the linear methods with targeting.

Acknowledgements

References

[1] Aston P.J. and Bird C.M., Analysis of the Control of Chaos – Rate of Convergence, *Int. J. of Bifurcation and Chaos*, 1995, V.5, N 4, 1157–1165.

[2] Aston P.J. and Bird C.M., Analysis of the Control of Chaos – Extending the Basin of Attraction, *Chaos, Solitons & Fractals*, 1997, V.8, N 9, 1413–1429.

[3] Chanfreau Ph. and Lyyjynen H., Viewing the Efficiency of Chaos Control on a Variety of Maps, Luleå University of Technology, Research Report 1998–20.

[4] Chen G. and Dong X., From Chaos to Order – Perspectives and Methodologies in Controlling Chaotic Nonlinear Dynamical Systems, *Int. J. of Bifurcation and Chaos*, 1993, V.3, N 6, 1363–1409.

[5] Chen G. and Dong X., From Chaos to Order – Methodologies, Perspectives and Applications, World Scientific series of Nonlinear Science, Series A, Vol.24, 1998.

[6] Ding M.,Yang W., In V., Ditto W.L., Spano M.L. and Gluckman B., Controlling Chaos in High Dimensions: Theory and Experiment, *Phys. Rev. E*, 1996, V.53, N 5, 4334–4344.

[7] Ditto W.L., Rauseo S.N. and Spano M.L., Experimental Control of Chaos, *Phys. Rev. Lett.*, 1990, V.65, N 26, 3211–3213.

[8] Ditto W.L., Spano M.L. and Lindner J.F., Techniques for the Control of Chaos, *Physica D*, 1995, V.86, 198–211.

[9] Dressler U. and Nitsche G., Controlling Chaos Using Time Delay Coordinates, *Phys. Rev. Lett.*, 1992, V.68, N 1, 1–4.

[10] Galias Z., New Method for Stabilization of Unstable Periodic Orbits in Chaotic Systems, *Int. J. of Bifurcation and Chaos*, 1995, V.5, N 1, 281–295.

[11] Gluckman B.J., Spano M.L., Yang W., Ding M., In V. and Ditto W., Tracking Unstable Periodic Orbits in Nonstationary High-Dimensional Chaotic Systems: Method and Experiment, *Phys. Rev. E*, 1997, V.55, N 5, 4935–4942.

[12] Hong Z., Jie Y., Jiao W. and Yinghai W., General Method of Controlling Chaos, *Phys. Rev. E*, 1996, V.53, N 1, 299–306.

[13] Obradovic D. and Lenz H., When is OGY Control More than Just Pole Placement? *Int. J. of Bifurcation and Chaos*, 1997, V.7, N 3, 691–699.

[14] Ott E., Grebogi C. and Yorke J.A., Controlling Chaos, *Phys. Rev. Lett.*, 1990, V.64, N 11, 1196–1199.

[15] Paskota M., Mees A.I. and Teo K.L., Stabilizing Higher Periodic Orbits, *Int. J. of Bifurcation and Chaos*, 1994, V.4, N 2, 457–460.

[16] Shinbrot T., Ott E., Grebogi C. and Yorke J., Using Chaos to Direct Trajectories to Targets, *Phys. Rev. Lett.*, 1990, V.65, N 26, 3215–3218.

[17] Shinbrot T., Ditto W., Grebogi C., Ott E., Spano M. and Yorke J., Using the Sensitive Dependence of Chaos ("the Butterfly Effect") to Direct Trajectories in an Experimental Chaotic System, *Phys. Rev. Lett.*, 1992, V.68, N 19, 2863–2866.

[18] Shinbrot T., Chaos: Unpredictable Yet Controllable? *Nonlinear Science Today*, 1993, V.3, N 2, 1–8.