Algebraic Spectral Relations for Elliptic Quantum Calogero-Moser Problems

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Abstract

Explicit algebraic relations between the quantum integrals of the elliptic Calogero–Moser quantum problems related to the root systems $\mathbf{A_2}$ and $\mathbf{B_2}$ are found.

1 Introduction

The notion of algebraic integrability for the quantum problems arised as a multidimensional generalisations of the "finite-gap" property of the one-dimensional Schrödinger operators (see [1, 2, 3]).

Recall that the Schrödinger equation

$$L\psi = -\Delta\psi + u(x)\psi = E\psi, \qquad x \in R^n,$$

is called *integrable* if there exist n commuting differential operators $L_1 = L, L_2, \ldots, L_n$ with the constant algebraically independent highest symbols $P_1(\xi) = \xi^2, P_2(\xi), \ldots, P_n(\xi)$, and *algebraically integrable* if there exists at least one more differential operator L_{n+1}, which commutes with the operators L_i, $i = 1, \ldots, n$, and whose highest symbol $P_{n+1}(\xi)$ is constant and takes different values on the solutions of the algebraic system $P_i(\xi) = c_i$, $i = 1, \ldots, n$, for generic c_i.

According to the general result [1] in the algebraically integrable case there exists an algebraic relation between the operators L_i, $i = 1, \ldots, n + 1$:

$$Q(L_1, L_2, \ldots, L_{n+1}) = 0.$$

The corresponding eigenvalues λ_i of the operators L_i obviously satisfy the same relation:

$$Q(\lambda_1, \lambda_2, \ldots, \lambda_{n+1}) = 0.$$

Sometimes it is more suitable to add more generators from the commutative ring of quantum integrals: $L_1, L_2, \ldots, L_{n+k}$; in that case we have more than one relation. We will call these relations *spectral*. They determine a spectral variety of the corresponding Schrödinger operator.

The main result of the present paper is the explicit description of the spectral algebraic relations for the three-particle elliptic Calogero-Moser problem with the Hamiltonian

$$L = -\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial x_3^2} + 4(\wp(x_1 - x_2) + \wp(x_2 - x_3) + \wp(x_3 - x_1)) \tag{1}$$

and for the generalised Calogero-Moser problem related to the root system $\mathbf{B_2}$ (see [4]) with the Hamiltonian

$$L = -\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} + 2(\wp(x_1) + \wp(x_2) + 2\wp(x_1 + x_2) + 2\wp(x_1 - x_2)). \tag{2}$$

Here $\wp$ is the classical Weierstrass elliptic function satisfying the equation

$$\wp'^2 - 4\wp^3 + g_2\wp + g_3 = 0.$$

These operators are known as the simplest multidimentional generalizations of the classical Lame operator (see [2]).

For the problem (1) reduced to the plane $x_1 + x_2 + x_3 = 0$ the explicit equations of the spectral variety have been found before in [5]. The derivation of [5] is indirect and based on the idea of the "isoperiodic deformations" [6]. In the present paper we give a direct derivation of the spectral relations for the problem (1) using explicit formulae for the additional quantum integrals, which have been found in [7]. The derived formulae are in a good agreement with the formulae written in [5].

For the elliptic Calogero-Moser problem related to the root system $\mathbf{B_2}$ the algebraic integrability with the explicit formulae for the additional integrals were obtained in the recent paper [8]. We use these formulae to derive the spectral relations for the operator (2).

We would like to mention that although the procedure of the derivation of the spectral relations provided the quantum integrals are given is effective, the actual calculations are huge and would be very difficult to perform without computer. We have used a special program, which has been created for this purpose.

2   Spectral relations for the three-particle elliptic Calogero-Moser problem

Let's consider the quantum problem with the Hamiltonian (1). This is a particular case of the three-particle elliptic Calogero-Moser problem corresponding to a special value of the parameter in the interaction. The algebraic integrability in this case has been conjectured by Chalykh and Veselov in [2] and proved later in [5, 7] and (in much more general case) in [9]. We should mention that only the paper [7] contains the explicit formulae for the additional integrals, the other proofs are indirect.

The usual integrability of this problem has been established by Calogero, Marchioro and Ragnisco in [10]. The corresponding integrals have the form

$$L_1 = L = -\Delta + 4(\wp_{12} + \wp_{23} + \wp_{31}),$$
$$L_2 = \partial_1 + \partial_2 + \partial_3,$$
$$L_3 = \partial_1\partial_2\partial_3 + 2\wp_{12}\partial_3 + 2\wp_{23}\partial_1 + 2\wp_{31}\partial_2,$$

where we have used the notations $\partial_i = \partial/\partial x_i$, $\wp_{ij} = \wp(x_i - x_j)$.

The following additional integrals have been found in [7]:

$$
\begin{aligned}
I_{12} = {} & (\partial_1 - \partial_3)^2(\partial_2 - \partial_3)^2 - 8\wp_{23}(\partial_1 - \partial_3)^2 - 8\wp_{13}(\partial_2 - \partial_3)^2 \\
& + 4(\wp_{12} - \wp_{13} - \wp_{23})(\partial_1 - \partial_3)(\partial_2 - \partial_3) - 2(\wp'_{12} + \wp'_{13} + 6\wp'_{23})(\partial_1 - \partial_3) \\
& - 2(-\wp'_{12} + 6\wp'_{13} + \wp'_{23})(\partial_2 - \partial_3) - 2\wp''_{12} - 6\wp''_{13} - 6\wp''_{23} + 4(\wp^2_{12} + \wp^2_{13} + \wp^2_{23}) \\
& + 8(\wp_{12}\wp_{13} + \wp_{12}\wp_{23} + 7\wp_{13}\wp_{23}),
\end{aligned}
$$

two other integrals I_{23}, I_{31} can be written simply by permuting the indices. Unfortunately, none of these operators implies algebraic integrability, because the symbols do not take different values on the solutions of the system $P_i(\xi) = c_i$, $i = 1, 2, 3$ (see the definition in the Introduction). But any non-symmetric linear combination of them, e.g. $L_4 = I_{12} + 2I_{23}$ would fit into the definition.

Lemma. *The operators L_1, L_2, L_3, I, where I is equal to I_{12}, I_{23} or I_{13}, satisfy the algebraic relation:*

$$Q(L_1, L_2, L_3, I) = I^3 + A_1 I^2 + A_2 I + A_3 = 0, \tag{3}$$

where

$$
\begin{aligned}
& A_1 = 6g_2 - X^2, \; A_2 = 2XY - 15g_2^2 - 2g_2 X^2, \\
& A_3 = -Y^2 - 2g_2 XY - 108g_3 Y + 16g_3 X^3 + 15g_2^2 X^2 - 100g_2^3, \\
& X = 3/2 L_1 + 1/2 L_2^2, \\
& Y = 1/2 L_1^3 + 27 L_3^2 + 1/4 L_2^6 + L_1 L_2^4 - 5 L_2^3 L_3 + 5/4 L_1^2 L_2^2 - 9 L_1 L_2 L_3.
\end{aligned}
$$

The idea of the proof is the following. Let the relation Q be a polynomial of third degree of I such that I_{12}, I_{23}, I_{13} are its roots:

$$Q = I^3 + A_1 I^2 + A_2 I + A_3.$$

Then $A_1 = -(I_{12} + I_{23} + I_{13})$, $A_2 = I_{12}I_{23} + I_{12}I_{13} + I_{23}I_{13}$, $A_3 = -I_{12}I_{23}I_{13}$. From the explicit formulae for I_{ij} it follows that the operators A_i are symmetric and their highest symbols a_i are constants. So there exist polynomials p_i such that $a_i = p_i(l_1, l_2, l_3)$. Consider $A'_i = A_i - p_i(L_1, L_2, L_3)$, A'_i commute with L. It follows from the Berezin's lemma [11] that if a differential operator commutes with a Schrödinger operator then the coefficients in the highest symbol of this operator are polynomials in x. Since the coefficients in the highest symbols of the operators A'_i are some elliptic functions, they must be constant in x. It is clear that the operators A'_i are also symmetric and $\deg a'_i < \deg a_i$. So we can continue this procedure until we come to zero. Thus we express A_i as the polynomials of L_1, L_2 and L_3. To calculate the explicit expressions we can use the fact that the coefficients in the highest symbols a'_i are constant and therefore may be calculated

at some special point. The most suitable choice is when $x_1 - x_2 = \omega_1$, $x_2 - x_3 = \omega_2$, $x_1 - x_3 = (\omega_1 + \omega_2)$, where $2\omega_1$, $2\omega_2$ are the periods of Weierstrass $\wp$-function. Then $\wp'(x_1 - x_2) = \wp'(x_2 - x_3) = \wp'(x_1 - x_3) = 0$ and $\wp(x_1 - x_2) = e_1$, $\wp(x_2 - x_3) = e_2$, $\wp(x_1 - x_3) = e_3$, where e_1, e_2, e_3 are the roots of the polynomial $4z^3 - g_2 z - g_3 = 0$ (see e.g. [12])

Theorem 1. *The integrals of the three-particle elliptic Calogero-Moser problem (1) L_1, L_2, L_3, $I = I_{12}$, $J = I_{23}$ satisfy the algebraic system:*

$$I^3 + A_1 I^2 + A_2 I + A_3 = 0,$$
$$I^2 + IJ + J^2 + A_1(I + J) + A_2 = 0,$$

where A_1, A_2, A_3 are given by the formulae (3). An additional integral L_4 which guarantees the algebraic integrability of the problem can be chosen as $L_4 = I + 2J$.

Proof. The first relation for I is proved in the lemma. The proof of the second relation is the following. The operators $I_{12} = I$, $I_{23} = J$, I_{13} satisfy the relations: $I + J + I_{13} = -A_1$, $IJ + (I + J)I_{13} = A_2$, so $I_{13} = -(I + J + A_1)$ and therefore we obtain $IJ - (I + J)(I + J + A_1) = A_2$. This completes the proof of Theorem 1.

Remark. Putting $L_2 = 0$ we can reduce the problem (1) to the plane $x_1 + x_2 + x_3 = 0$ and obtain two-dimensional elliptic Calogero–Moser problem related to the root system $\mathbf{A_2}$. The spectral curve of this problem was found in the paper [5]. The corresponding formula from [5] has the form:

$$\nu^3 + (6\lambda\mu^2 - 3(\lambda^2 - 3g_2)^2)\nu - \mu^4 + (10\lambda^3 - 18g_2\lambda + 108g_3)\mu^2 + 2(\lambda^2 - 3g_2)^3 = 0.$$

If in our formula (3) put $L_2 = 0$, substitute $L_1 = 2\lambda$, $L_3 = \sqrt{3}/9\mu$, $I = \nu - 1/3(6g_2 - X^2)$, then for the relation (3) we get:

$$\nu^3 + (6\lambda\mu^2 - 3(\lambda^2 - 3g_2)^2)\nu - \mu^4 + (10\lambda^3 - 18g_2\lambda - 108g_3)\mu^2 + 2(\lambda^2 - 3g_2)^3 = 0.$$

The elliptic curves $y^2 = 4x^3 - g_2 x - g_3$ and $y^2 = 4x^3 - g_2 x + g_3$ are isomorphic: $x \to -x$, $y \to iy$, so the difference in the sign by g_3 is not important.

3   Spectral relations for the elliptic Calogero-Moser problem related to the root system $\mathbf{B_2}$

Consider now the Schrödinger operator (2). Its algebraic integrability conjectured in [2] has been proved in [8].

The formulae for the quantum integrals in this case are (see [8])

$$L_1 = L = -\Delta + 2(\wp(x) + \wp(y) + 2\wp(x + y) + 2\wp(x - y)),$$
$$\begin{aligned}
L_2 = {} & \partial_x^2 \partial_y^2 - 2\wp(y)\partial_x^2 - 2\wp(x)\partial_y^2 - 4(\wp(x + y) - \wp(x - y))\partial_x \partial_y \\
& - 2(\wp'(x + y) + \wp'(x - y))\partial_x - 2(\wp'(x + y) - \wp'(x - y))\partial_y \\
& - 2(\wp''(x + y) + \wp''(x - y)) + 4(\wp^2(x + y) + \wp^2(x - y)) \\
& + 4(\wp(x) + \wp(y))(\wp(x + y) + \wp(x - y)) - 8\wp(x + y)\wp(x - y) - 4\wp(x)\wp(y),
\end{aligned}$$
$$L_3 = I_x + 2I_y,$$

where

$$
\begin{aligned}
I_x ={}& \partial_x^5 - 5\partial_x^3\partial_y^2 - 10(1/2\wp(x) - \wp(y) + \wp(x+y) + \wp(x-y))\partial_x^3 \\
&+ 30(\wp(x+y) - \wp(x-y))\partial_x^2\partial_y + 15\wp(x)\partial_x\partial_y^2 \\
&- 15/2\wp(x)(\partial_x^2 - \partial_y^2) + 30(\wp(x+y) - \wp(x-y))\partial_x\partial_y \\
&+ (10\wp''(x+y) - 10\wp''(x-y) - 30\wp(y)(\wp(x+y) - \wp(x-y)))\partial_y \\
&+ (30\wp(y)(\wp(x) - \wp(x+y) - \wp(x-y)) + 120\wp(x+y)\wp(x-y) \\
&+ 10\wp''(x+y) + 10\wp''(x-y) - 5\wp''(x) - 9/2g_2)\partial_x \\
&- 15(\wp'(x+y) + \wp'(x-y))(\wp(x) + \wp(y)) - 15(\wp'(x)\wp(y) + \wp'(y)\wp(x)) \\
&+ 60(\wp'(x+y) + \wp'(x-y))(\wp(x+y) + \wp(x-y)),
\end{aligned}
$$

operator I_y can be written by exchanging in the previous formula x and y, and we use the notations $\partial_x = \partial/\partial x, \partial_y = \partial/\partial y$.

Theorem 2. *The quantum integrals* $L = 1/2L_1$, $M = L_2$, $I = I_x$, $J = I_y$, $L_3 = I + 2J$ *of the elliptic Calogero-Moser system related to the root system* $\mathbf{B_2}$ *satisfy the following algebraic relations:*

$$
I^4 + B_1 I^2 + B_2 = 0, \qquad I^2 + J^2 + B_1 = 0,
$$

where

$$
\begin{aligned}
B_1 ={}& 32L^5 - 120ML^3 + 120M^2L + g_2(-82L^3 + 114LM) \\
&+ g_3(-270L^2 + 486M) + 102g_2^2L + 486g_3g_2, \\
B_2 ={}& 400M^3L^4 - 1440M^4L^2 + 1296M^5 \\
&+ g_2(-400ML^6 + 840M^2L^4 - 576M^3L^2 + 648M^4) \\
&+ g_3(800L^7 - 8280ML^5 + 22032M^2L^3 - 17496LM^3) \\
&+ g_2^2(800L^6 - 1815ML^4 + 3510M^2L^2 - 3807M^3) \\
&+ g_3g_2(3870L^5 + 324ML^3 - 13122M^2L) \\
&+ g_3^2(18225L^4 - 65610ML^2 + 59049M^2) \\
&+ g_2^3(-2930L^4 + 5418ML^2 - 4536M^2) \\
&+ g_3g_2^2(-21708L^3 + 26244LM) \\
&+ g_3^3g_2(-65610L^2 + 118098M) + g_2^4(2772L^2 - 1539M) \\
&+ 21870g_3g_2^3L + 59049g_2^2g_3^2 - 162g_2^5.
\end{aligned}
$$

The proof is analogous to the previous case. The first relation is a polynomial of second order in I^2 such that I_x^2 and I_y^2 are its roots. Calculations of the coefficients as in the previous case can be done at the special point $(x, y) : x = \omega_1, y = \omega_2$. Then $\wp(x) = e_1$, $\wp(y) = e_2$, $\wp(x+y) = \wp(x-y) = e_3$, $\wp'(x) = \wp'(y) = \wp'(x-y) = \wp'(x-y) = 0$.

As we have mentioned already these calculations have been done with the help of the special computer program created by the authors.

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