

A Note on the Third Family of $N = 2$ Supersymmetric KdV Hierarchies

F. DELDUC[†] and L. GALLOT[‡]

[†] *Laboratoire de Physique*, Groupe de Physique Théorique ENS Lyon,
46 allée d'Italie, 69364 Lyon CEDEX 07, France
E-mail: Francois.Delduc@ens-lyon.fr*

[‡] *Dipartimento di Fisica Teorica Università di Torino,
Via P. Giuria 1, 10125 Torino, Italy
E-mail: gallot@to.infn.it*

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Abstract

We propose a hamiltonian formulation of the $N = 2$ supersymmetric KP type hierarchy recently studied by Krivonos and Sorin. We obtain a quadratic hamiltonian structure which allows for several reductions of the KP type hierarchy. In particular, the third family of $N = 2$ KdV hierarchies is recovered. We also give an easy construction of Wronskian solutions of the KP and KdV type equations.

1 Introduction

The existence of three different $N = 2$ supersymmetric integrable n -KdV hierarchies with the $N = 2$ super \mathcal{W}_n algebra as a hamiltonian structure has been made plausible by the works of many authors [1, 2, 3]. For two of these families, a complete description by means of a classical r -matrix approach using the algebra of chirality preserving pseudo-differential operators (Ψ DOs) has been proposed in [4]. A formulation of the same hierarchies in the Drinfeld-Sokolov approach also exists [5, 6].

The last remaining family of $N = 2$ n -KdV hierarchies is of a somewhat different nature. Actually, the bosonic limit of the two first $N = 2$ n -KdV hierarchies is composed of two decoupled KdV and non-standard KdV hierarchies [4, 7] whereas the bosonic limit of the third one is the $(1, n)$ KdV hierarchy [7] which is irreducible [8, 9, 10]. Recently, Krivonos and Sorin [11] gave a Lax representation for the third family of $N = 2$ KdV hierarchies.

The aim of the present letter is to give the hamiltonian formulation of the $N = 2$ KP type hierarchy which contains as reductions the above mentioned third family of $N = 2$ n -KdV hierarchies and a third type of $N = 2$ constrained KP systems as well. We shall

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proceed as follows. First, we define the KP type equations using the algebra \mathcal{D} of bosonic Ψ DOs with $N = 2$ superfields as their coefficients. A peculiarity of these flows on a Ψ DO \mathcal{L} is that, although they are associated with a classical r -matrix on \mathcal{D} , they do not take the Lax form. Second, we show that, as was expected from [11], the evolution equations of two Ψ DOs constructed from \mathcal{L} , one being chiral and the other antichiral, have the Lax form. This allows to determine an infinite set of $N = 2$ supersymmetric conserved quantities, or hamiltonians, for the KP type flows. Third, we find a quadratic hamiltonian structure for the KP type hierarchy. Since the defining flows do not have the Lax form, this Poisson bracket does not have a standard form associated with a classical r -matrix. Hamiltonians are found to be in involution with respect to this Poisson bracket, which achieves the proof of integrability. Reductions to a finite number of fields and the bosonic limit are also briefly discussed.

In the last part of this letter, we construct Wronskian solutions and τ -functions of the KdV and KP type equations. This construction is a simple extension of the bosonic one. No such construction is known at present for the first two families of $N = 2$ KP and KdV hierarchies.

2 $N = 2$ supersymmetric KdV hierarchies: the third family

$N = 2$ supersymmetry. We shall consider an $N = 2$ superspace with space coordinate x and two Grassmann coordinates $\theta, \bar{\theta}$. We shall use the notation \underline{x} for the triple of coordinates $(x, \theta, \bar{\theta})$. The supersymmetric covariant derivatives are defined by

$$\partial \equiv \frac{\partial}{\partial x}, \quad D = \frac{\partial}{\partial \theta} + \frac{1}{2}\bar{\theta}\partial, \quad \bar{D} = \frac{\partial}{\partial \bar{\theta}} + \frac{1}{2}\theta\partial, \quad (2.1)$$

and satisfy the $N = 2$ supersymmetry algebra

$$D^2 = \bar{D}^2 = 0, \quad \{D, \bar{D}\} = \partial. \quad (2.2)$$

Beside ordinary superfields $H(\underline{x})$ depending arbitrarily on Grassmann coordinates, one can also define chiral superfields $\varphi(\underline{x})$ satisfying $D\varphi = 0$ and antichiral superfields $\bar{\varphi}(\underline{x})$ satisfying $\bar{D}\bar{\varphi} = 0$. We define the integration over the $N = 2$ superspace to be

$$\int d^3\underline{x} H(x, \theta, \bar{\theta}) = \int dx \bar{D}DH(x, \theta, \bar{\theta})|_{\theta=\bar{\theta}=0}. \quad (2.3)$$

Let us consider the algebra \mathcal{D} of pseudo-differential operators \mathcal{L} of the form

$$\mathcal{L} = \sum_{k < M} u_k \partial^k \quad (2.4)$$

with the usual product rule. The coefficients functions u_k are commuting $N = 2$ superfields. The highest power of ∂ with non zero coefficient will be called the order of \mathcal{L} . We define as usual the residue of the pseudo-differential operator \mathcal{L} by $\text{res } \mathcal{L} = u_{-1}$. The residue of a commutator is a total space derivative, $\text{res } [\mathcal{L}, \mathcal{L}'] = (\partial\Omega)$. The trace of \mathcal{L} is the integral over the superspace of the residue

$$\text{tr } \mathcal{L} = \int d^3\underline{x} \text{res } \mathcal{L}, \quad \text{tr } [\mathcal{L}, \mathcal{L}'] = 0. \quad (2.5)$$

\mathcal{D} can be split into two associative subalgebras $\mathcal{D} = \mathcal{D}_+ \oplus \mathcal{D}_-$, where \mathcal{L} is in \mathcal{D}_+ if it is a differential operator and \mathcal{L} is in \mathcal{D}_- if it is a strictly pseudo-differential operator ($M = 0$ in 2.4). We shall note

$$\mathcal{L} = \mathcal{L}_+ + \mathcal{L}_-, \quad \mathcal{L}_\pm \in \mathcal{D}_\pm. \quad (2.6)$$

Moreover, \mathcal{D}_+ and \mathcal{D}_- are isotropic subalgebras with respect to the trace. As a consequence of these facts, the endomorphism R of \mathcal{D} defined by $R(\mathcal{L}) = \frac{1}{2}(\mathcal{L}_+ - \mathcal{L}_-)$ is a skew-symmetric classical r -matrix (the very same as in the bosonic case),

$$\text{tr}(R(\mathcal{L})\mathcal{L}' + \mathcal{L}R(\mathcal{L}')) = 0, \quad (2.7)$$

$$R([R(\mathcal{L}), \mathcal{L}'] + [\mathcal{L}, R(\mathcal{L}')] = [R(\mathcal{L}), R(\mathcal{L}')] + \frac{1}{4}[\mathcal{L}, \mathcal{L}']. \quad (2.8)$$

KP type equations. We shall first write the evolution equations for an $N = 2$ supersymmetric KP type hierarchy. Let us consider a pseudo-differential operator of the type

$$\mathcal{L} = \partial^{n-1} + \sum_{k=1}^{\infty} U_k \partial^{n-1-k} \quad (2.9)$$

containing an infinite set of bosonic $N = 2$ superfields. Following [11], we associate with \mathcal{L} the two pseudo-differential operators¹⁾

$$L = \{D, \mathcal{L}\bar{D}\mathcal{L}^{-1}\}, \quad \bar{L} = \{\bar{D}, \mathcal{L}^{-1}D\mathcal{L}\}. \quad (2.10)$$

By definition, L and \bar{L} are respectively chiral and anti-chiral operators, that is to say $[D, L] = [\bar{D}, \bar{L}] = 0$. L and \bar{L} are pseudo-differential operators in \mathcal{D} , of order one, which becomes clear when they are written in the form

$$L = \partial + [D\mathcal{L}[\bar{D}\mathcal{L}^{-1}]], \quad \bar{L} = \partial + [\bar{D}\mathcal{L}^{-1}[D\mathcal{L}]], \quad (2.11)$$

where the notation $[D\mathcal{L}]$ means that the odd derivative D only acts on the coefficient functions of the operator \mathcal{L} . L and \bar{L} are conjugate operators

$$L = \mathcal{L}\bar{L}\mathcal{L}^{-1} \quad (2.12)$$

and, as a consequence, the following basic commutation relation between powers of L and \bar{L} holds

$$L^p \mathcal{L} = \mathcal{L} \bar{L}^p. \quad (2.13)$$

This suggests to define the following flows on \mathcal{L}

$$\frac{\partial}{\partial t_p} \mathcal{L} = R(L^p) \mathcal{L} - \mathcal{L} R(\bar{L}^p) \quad (2.14)$$

which do not have the Lax form. Using equation (2.13), these flows can be put into the form

$$\partial_p \mathcal{L} = (L^p)_- \mathcal{L} - \mathcal{L} (\bar{L}^p)_-. \quad (2.15)$$

¹⁾Our conventions differ from those of Krivonos and Sorin. In particular, chiralities are exchanged.

The right-hand side is a Ψ DO of order $n-2$, which proves the consistency of the flows (2.15). We used the notation $\partial_p = \partial/\partial t_p$. A crucial point is that, although the flows on \mathcal{L} do not take the Lax form, the evolution equations of L and \bar{L} indeed do

$$\partial_p L = [R(L^p), L], \quad \partial_p \bar{L} = [R(\bar{L}^p), \bar{L}]. \quad (2.16)$$

In order to show this, one can study the evolution equations of the operator $\mathcal{L}\bar{D}\mathcal{L}^{-1}$, which using (2.14) are given by

$$\partial_p \mathcal{L}\bar{D}\mathcal{L}^{-1} = [R(L^p), \mathcal{L}\bar{D}\mathcal{L}^{-1}] + \mathcal{L}[\bar{D}, R(\bar{L}^p)]\mathcal{L}^{-1}. \quad (2.17)$$

The second term on the right-hand side vanishes because of the anti-chirality of the operator \bar{L} . The evolution equations of L are then obtained as

$$\partial_p L = \{D, \partial_p \mathcal{L}\bar{D}\mathcal{L}^{-1}\} = \{D, [R(L^p), \mathcal{L}\bar{D}\mathcal{L}^{-1}]\} = [R(L^p), L], \quad (2.18)$$

where the chirality of the operator L has been used in the last equality. Using the Lax equations (2.16) for L and \bar{L} and the modified Yang-Baxter equation (2.8) for R , one can show that the flows (2.14) commute

$$\begin{aligned} [\partial_p, \partial_q] \mathcal{L} &= (R([R(L^q), L^p]) - R([R(L^p), L^q]) + [R(L^p), R(L^q)]) \mathcal{L} \\ &\quad - \mathcal{L} (R([R(\bar{L}^q), \bar{L}^p]) - R([R(\bar{L}^p), \bar{L}^q]) + [R(\bar{L}^p), R(\bar{L}^q)]) = 0, \end{aligned} \quad (2.19)$$

which suggests that this hierarchy is integrable.

To conclude this paragraph, we would like to mention a geometric formulation of the $N = 2$ KP type flows described before. Actually, one can define the following two sets of derivative operators

$$\nabla_C = D, \quad \bar{\nabla}_C = \mathcal{L}\bar{D}\mathcal{L}^{-1}, \quad \partial_C = L, \quad (2.20)$$

$$\nabla_A = \mathcal{L}^{-1}\nabla_C\mathcal{L} = \mathcal{L}^{-1}D\mathcal{L}, \quad \bar{\nabla}_A = \mathcal{L}^{-1}\bar{\nabla}_C\mathcal{L} = \bar{D}, \quad \partial_A = \mathcal{L}^{-1}\partial_C\mathcal{L} = \bar{L} \quad (2.21)$$

which both satisfy the $N = 2$ supersymmetry algebra

$$\nabla_X^2 = \bar{\nabla}_X^2 = 0, \quad \partial_X = \{\nabla_X, \bar{\nabla}_X\} \quad \text{for } X = C, A. \quad (2.22)$$

Hence, an easy computation shows that the KP type flows (2.14) imply a Lax type evolution equation for all these derivatives

$$\partial_p \nabla_X = [R(\partial_X^p), \nabla_X], \quad \partial_p \bar{\nabla}_X = [R(\partial_X^p), \bar{\nabla}_X] \quad \text{for } X = C, A. \quad (2.23)$$

A discrete symmetry. The flows of the KP type hierarchy are invariant under the following involutive transformation T :

$$\begin{aligned} x_T &= x, & \theta_T &= \bar{\theta}, & \bar{\theta}_T &= \theta, & t_{Tp} &= (-)^{p+1} t_p, \\ \mathcal{L}_T(x_T, \theta_T, \bar{\theta}_T) &= (-)^{n-1} \mathcal{L}^t(x, \theta, \bar{\theta}), \end{aligned} \quad (2.24)$$

where \mathcal{L}^t is the adjoint operator, defined as in the bosonic case [12], and $n-1$ is the order of \mathcal{L} . Indeed the flows (2.14) are equivalent to

$$(-)^{p+1} \partial_p \mathcal{L}_T = R(L_T^p) \mathcal{L}_T - \mathcal{L}_T R(\bar{L}_T^p) \quad (2.25)$$

where $L_T = \{\bar{D}_T, \mathcal{L}_T D_T \mathcal{L}_T^{-1}\}$ and $\bar{L}_T = \{D_T, \mathcal{L}_T^{-1} \bar{D}_T \mathcal{L}_T\}$. Using this result, certain discrete invariance for the nonlinear evolution equations [11] can be extracted directly from the operator \mathcal{L} . We shall see a simple example below.

Conserved quantities. Another consequence of Lax equations (2.16) is that they provide us with an infinite set of conserved quantities for the flows (2.14). Standard arguments coming from the study of KP type hierarchies [12] tell us that conservation laws are associated with $\text{res } L^k$ and $\text{res } \bar{L}^k$ so that the quantities [7, 11]

$$H_k = \frac{1}{k} \int dx \text{res } L^k|_{\theta, \bar{\theta}=0} = \frac{1}{k} \int dx \text{res } \bar{L}^k|_{\theta, \bar{\theta}=0} \quad (2.26)$$

are conserved. Notice here that the integration is over the space coordinate x only, so that it is not clear a priori why these quantities are invariant under $N = 2$ supersymmetry. This fact, as well as the second equality in (2.26), are consequences of the basic relation (2.13) which yields

$$\text{res } \bar{L}^k = \text{res } L^k + \text{res } [\mathcal{L}^{-1}, L^k \mathcal{L}] \quad (2.27)$$

so that, from the properties of the residue [12], the quantities $\text{res } L^k$ and $\text{res } \bar{L}^k$ differ only by a space derivative, which we denote by

$$\text{res } [\mathcal{L}^{-1}, L^k \mathcal{L}] = k \partial \mathcal{H}_k \quad (2.28)$$

where \mathcal{H}_k is an $N = 2$ differential polynomial in the coefficients of \mathcal{L} . We may rewrite equation (2.27) as

$$\text{res } \bar{L}^k - k \bar{D} D \mathcal{H}_k = \text{res } L^k + k D \bar{D} \mathcal{H}_k. \quad (2.29)$$

In this last equation, the left-hand side is an antichiral superfield, whereas the right-hand side is a chiral superfield. Then both sides must be equal to a constant, and we get

$$H_k = \int d^3 \underline{x} \mathcal{H}_k \quad (2.30)$$

This last expression of the conserved charges involves an integration on the whole $N = 2$ superspace, which ensures that they are invariant under supersymmetry.

Hamiltonian structure. We turn now to the problem of constructing a Poisson bracket for the flows (2.14). From the analysis of the third series of $N = 2$ KdV hierarchies [2, 3], we expect the existence of a single quadratic hamiltonian structure corresponding to the $N = 2$ \mathcal{W}_n algebra. Since the flows (2.14) do not have the Lax form, we do not expect this quadratic bracket to be of the Adler-Gelfand-Dickey or the *abcd* type as is the case for the two first series of $N = 2$ KdV hierarchies. Nevertheless, it is possible to use the quadratic hamiltonian structures found in [4] in order to get one for the hierarchy we are dealing with. Let us give a brief account of some of the results in [4].

We consider the associative algebra $\check{\mathcal{C}}$ of pseudo-differential operators \check{L} preserving chirality of the form

$$\check{L} = D \mathcal{L} \bar{D}, \quad \mathcal{L} = \sum_{i < M} u_i \partial^i \in \mathcal{D}. \quad (2.31)$$

The coefficient functions u_i again are commuting $N = 2$ superfields²⁾. We define the residue of the pseudo-differential operator \check{L} by $\text{Res } \check{L} = \text{res } \mathcal{L}$. The residue of a commutator is a total derivative in $N = 2$ superspace, $\text{Res } [\check{L}, \check{L}'] = D\bar{\omega} + \bar{D}\omega$. The trace of \check{L} is the integral of the residue

$$\text{Tr } \check{L} = \int d^3 \underline{x} \text{Res } \check{L}, \quad \text{Tr } [\check{L}, \check{L}'] = 0. \quad (2.32)$$

We define a classical r -matrix in $\check{\mathcal{C}}$ by $\check{R}(\check{L}) = DR(\mathcal{L})\bar{D}^3$. It is not skew symmetric, but satisfies the equation

$$\text{Tr } (\check{R}(\check{L})\check{L}' + \check{L}\check{R}(\check{L}')) = - \int d^3 \underline{x} \text{Res } \check{L} \text{Res } \check{L}'. \quad (2.33)$$

To this r -matrix correspond two a priori different quadratic Poisson brackets, which however are related by a Poisson map. In this article we shall use the first of these brackets. Let \check{X} be some Ψ DO in $\check{\mathcal{C}}$ with coefficients independent of the phase space fields $\{u_i\}$, then define the linear functional $l_{\check{X}}(\check{L}) = \text{Tr } (\check{L}\check{X})$. We also define projections Φ and $\bar{\Phi}$ on the chiral and antichiral parts of a general $N = 2$ superfield H by⁴⁾

$$H = \Phi(H) + \bar{\Phi}(H), \quad D\Phi(H) = 0, \quad \bar{D}\bar{\Phi}(H) = 0. \quad (2.34)$$

The first quadratic bracket in [4] then reads

$$\begin{aligned} \{l_{\check{X}}, l_{\check{Y}}\}_{(2)}^a(\check{L}) &= \text{Tr } (\check{L}\check{X}\check{R}(\check{L}\check{Y}) - \check{X}\check{L}\check{R}(\check{Y}\check{L})) \\ &\quad + \Phi(\text{Res } [\check{L}, \check{Y}])\check{L}\check{X} + \check{X}\check{L}\bar{\Phi}(\text{Res } [\check{L}, \check{Y}]), \end{aligned} \quad (2.35)$$

We now wish to rewrite this hamiltonian structure directly on \mathcal{L} . We consider linear functionals on the phase space $l_X[\mathcal{L}] = \text{tr } (\mathcal{L}X)$ where X is a pseudo-differential operator in \mathcal{D} independent of the phase space fields u_k . We shall use the relation between linear functionals of \mathcal{L} and \check{L}

$$\check{L} = D\mathcal{L}\bar{D}, \quad \check{X} = D\bar{D}\partial^{-1}X\partial^{-1}D\bar{D}, \quad (2.36)$$

$$l_X[\mathcal{L}] = \text{tr } (\mathcal{L}X) = \text{Tr } (\check{L}\check{X}) = l_{\check{X}}[\check{L}]. \quad (2.37)$$

One then obtains the hamiltonian structure

$$\begin{aligned} \{l_X, l_Y\}(\mathcal{L}) &= \text{tr } (\mathcal{L}X\partial(\mathcal{L}Y)_+ - \mathcal{L}X([D[\bar{D}\mathcal{L}]Y])_+ + [D\mathcal{L}[\bar{D}X]](\mathcal{L}Y)_+ \\ &\quad - X\mathcal{L}\partial(Y\mathcal{L})_+ + X\mathcal{L}([\bar{D}[DY]\mathcal{L}])_+ - [\bar{D}X[D\mathcal{L}]](Y\mathcal{L})_+ \\ &\quad + \mathcal{L}X\Phi(\text{res } [\mathcal{L}, Y]) + X\mathcal{L}\bar{\Phi}(\text{res } [\mathcal{L}, Y])). \end{aligned} \quad (2.38)$$

Let us insist here that the Poisson bracket (2.38) is in fact identical to (2.35), which properties, and in particular the Jacobi identities, have been studied in [4].

²⁾Although there is an obvious bijection between operators in \mathcal{D} and operators in $\check{\mathcal{C}}$, the products in the two algebras differ.

³⁾The classical Yang-Baxter equation satisfied by \check{R} does not follow from this same equation for R . It requires an independent proof.

⁴⁾An explicit expression of the map Φ may be found in [4].

We shall now show that this hamiltonian structure, together with the hamiltonian H_p , generates the flows (2.14). For any functional $F[\mathcal{L}]$, we define the functional derivative $\frac{\delta F}{\delta \mathcal{L}} \in \mathcal{D}$ as follows. Under a small variation $\delta \mathcal{L}$ of \mathcal{L} , the variation of the functional is

$$\delta F[\mathcal{L}] = \int d^3 \underline{x} \operatorname{res} \left(\frac{\delta F}{\delta \mathcal{L}} \delta \mathcal{L} \right) = \operatorname{tr} \left(\frac{\delta F}{\delta \mathcal{L}} \delta \mathcal{L} \right). \quad (2.39)$$

Then we consider the hamiltonian H_p defined in equation (2.26). Using the fact that L is chiral and \bar{L} is anti-chiral, we obtain the functional derivative

$$\frac{\delta H_p}{\delta \mathcal{L}} = \bar{L}^{p-1} \mathcal{L}^{-1} = \mathcal{L}^{-1} L^{p-1}. \quad (2.40)$$

In order to compute the hamiltonian vector field associated with the functional H_p , we use the following identities

$$\begin{aligned} \left[\mathcal{L}, \frac{\delta H_p}{\delta \mathcal{L}} \right] &= L^{p-1} - \bar{L}^{p-1} \\ \Rightarrow \quad \Phi \left(\operatorname{res} \left[\mathcal{L}, \frac{\delta H_p}{\delta \mathcal{L}} \right] \right) &= \operatorname{res} L^{p-1}, \quad \bar{\Phi} \left(\operatorname{res} \left[\mathcal{L}, \frac{\delta H_p}{\delta \mathcal{L}} \right] \right) = -\operatorname{res} \bar{L}^{p-1}, \\ (\partial L^{p-1})_+ &= \partial(L^{p-1})_+ + \operatorname{res} L^{p-1} \end{aligned} \quad (2.41)$$

Using the expressions (2.11), a fairly easy computation then leads to

$$\partial_p l_X[\mathcal{L}] = \{l_X, H_p\}(\mathcal{L}) = \operatorname{tr} (X(R(L^p)\mathcal{L} - \mathcal{L}R(\bar{L}^p))) \quad (2.42)$$

which is the desired equation of motion. The Poisson bracket of two hamiltonians reads

$$\{H_p, H_q\} = \operatorname{tr} (L^{p-1} R(L^q) - \bar{L}^{p-1} R(\bar{L}^q)) = 0. \quad (2.43)$$

The last equality follows from the fact that the superspace integral of a chiral or antichiral superfield vanishes. We thus proved the integrability of this hierarchy.

Reductions. In order to find reductions of the KP type hierarchy, we need to find Poisson submanifolds of the KP type phase space. The Poisson submanifolds of the quadratic bracket (2.38) correspond to those of the quadratic brackets $\{, \}_{(2)}^a$ which were given in [4]. In particular the constraint

$$\mathcal{L} = \mathcal{L}_+ \quad (2.44)$$

defines a first type of Poisson submanifold. For an operator \mathcal{L} of order $n-1$, the corresponding Poisson algebra is the $N=2$ \mathcal{W}_n algebra and the hierarchy thus obtained is the third $N=2$ n -KdV hierarchy. The simplest example of such a hierarchy⁵⁾ is provided by the choice

$$\mathcal{L} = \partial + J. \quad (2.45)$$

The first non trivial flow is

$$\partial_2 J = ([D, \bar{D}]J - J^2)_x, \quad (2.46)$$

⁵⁾Many examples of the first flows of these hierarchies can be found in [11].

where one recognizes the second flow of the so-called $N = 2$ $a = 4$ KdV hierarchy. Two alternative Lax operator for this hierarchy are already known [1, 13, 4], but the reason why these three operators give the same conserved quantities is not clear to us. Notice that the transformation T (2.24) is given in this example by

$$J \rightarrow -J, \quad D \leftrightarrow \bar{D}, \quad t_2 \rightarrow -t_2 \quad (2.47)$$

and is a symmetry of the flow (2.46). The Poisson algebra for this hierarchy can be directly worked out from the general hamiltonian structure (2.38). With the choice $X = f\partial^{-1}$, $Y = g\partial^{-1}$ where f and g are superfields independent on J , one obtains the following Poisson bracket between the two linear functionals $l_X(\mathcal{L}) = \int d^3\mathbf{x} Jf$, $l_Y(\mathcal{L}) = \int d^3\mathbf{x} Jg$

$$\left\{ \int Jf, \int Jg \right\} = \int d^3\mathbf{x} f \left((Jg)_x - DJ\bar{D}g - \bar{D}JDg - [D, \bar{D}]g_x \right). \quad (2.48)$$

One recognizes in this expression the classical $N = 2$ superconformal algebra.

Another possible reduction is to take \mathcal{L} of the form

$$\mathcal{L} = \mathcal{L}_+ + \phi\partial^{-1}\bar{\phi}, \quad D\phi = \bar{D}\bar{\phi} = 0, \quad (2.49)$$

where ϕ and $\bar{\phi}$ are Grassmann even or odd superfields. In this case, the Poisson algebra is an extension of the $N = 2$ \mathcal{W}_n algebra for \mathcal{L} of order $n - 1$. This extension is non local if ϕ and $\bar{\phi}$ are Grassmann odd [4]. The simplest example of such a hierarchy is provided by the choice

$$\mathcal{L} = 1 + \phi\partial^{-1}\bar{\phi} \quad (2.50)$$

where ϕ and $\bar{\phi}$ are Grassmann odd. Krivonos and Sorin [11] noticed that the first non trivial flow belongs to the $N = 2$ NLS hierarchy for which an alternative Lax operator exists [14, 4]. The relation between both operators is unclear. An other simple example is provided by the choice

$$\mathcal{L} = \partial + J + \phi\partial^{-1}\bar{\phi}, \quad (2.51)$$

where ϕ and $\bar{\phi}$ are Grassmann even. The Poisson algebra for this hierarchy is the “small” $N = 4$ superconformal algebra (SCA). The computation of the first flows for this operator [11] shows that this hierarchy is neither the “small” $N = 4$ KdV hierarchy, nor the “quasi” $N = 4$ KdV [4, 15]. Hence, this is an other example of integrable hierarchy having the $N = 4$ SCA as a hamiltonian structure but which respects only $N = 2$ supersymmetry.

Bosonic limit. For completeness, we study the bosonic limit of the $N = 2$ n -KdV hierarchy described before, which was already considered in [11]. One finds, as it has been conjectured in [7], that this limit is actually the $(1, n)$ KdV hierarchy [11, 8, 9, 10].

From now on, we restrict to operators \mathcal{K} in \mathcal{D} satisfying the conditions $D\mathcal{K}|_0 = \bar{D}\mathcal{K}|_0 = 0$, where the limit $|_0$ means that θ and $\bar{\theta}$ are set to zero. This defines a subspace \mathcal{D}_B of \mathcal{D} which is closed under the usual product. To an operator \mathcal{K} in \mathcal{D}_B we can associate two ordinary bosonic operators in \mathcal{D}_0 by

$$\pi_1(\mathcal{K}) = \mathcal{K}_1 = \mathcal{K}|_0, \quad (2.52)$$

$$\pi_2(\mathcal{K}) = \mathcal{K}_2 = \mathcal{K}|_0\partial + [\bar{D}D\mathcal{K}]|_0. \quad (2.53)$$

Remark that, if k is the order of \mathcal{K} , then the respective orders of \mathcal{K}_1 and \mathcal{K}_2 are k and $k+1$. It is easily checked that π_1 is a morphism from \mathcal{D}_B to \mathcal{D}_0 , that is to say $(\mathcal{K}\mathcal{K}')_1 = \mathcal{K}_1\mathcal{K}'_1$, and that the following property holds

$$(\mathcal{K}_+)_1 = (\mathcal{K}_1)_+, \quad (\mathcal{K}_-)_1 = (\mathcal{K}_1)_-. \quad (2.54)$$

It may be checked, from their definition (2.10), that the limit of the supersymmetric operators L and \bar{L} is given by

$$L_1 = \mathcal{L}_2 \mathcal{L}_1^{-1}, \quad \bar{L}_1 = \mathcal{L}_1^{-1} \mathcal{L}_2. \quad (2.55)$$

Using these properties, it may be shown that the flows (2.14) have the following limit

$$\partial_p \mathcal{L}_1 = R(L_1^p) \mathcal{L}_1 - \mathcal{L}_1 R(\bar{L}_1^p), \quad (2.56)$$

$$\partial_p \mathcal{L}_2 = R(L_1^p) \mathcal{L}_2 - \mathcal{L}_2 R(\bar{L}_1^p) \quad (2.57)$$

which are precisely the defining flows of the $(1, n)$ hierarchy, once one has restricted \mathcal{L} to be a differential operator of order $n-1$. The hamiltonian structure for this system can be recovered from the supersymmetric Poisson bracket (2.38) following the lines of [4].

Wronskian solutions. Our first goal in this paragraph will be to construct solutions of the nonlinear equations (2.15) in terms of a set of functions satisfying linear equations. First remark that the flow equations (2.13) and (2.14) for L and \bar{L} are just the standard KP flows. Simply, the coefficient functions in L and \bar{L} are not ordinary functions, but rather constrained superfields. Then, it is reasonable to introduce a set of P chiral superfields Y_i and a set of Q antichiral superfields \bar{Y}_i satisfying

$$\begin{aligned} DY_i &= 0, & \partial_k Y_i &= \partial^k Y_i, & i &= 1, \dots, P, \\ \bar{D}\bar{Y}_i &= 0, & \partial_k \bar{Y}_i &= \partial^k \bar{Y}_i, & i &= 1, \dots, Q. \end{aligned} \quad (2.58)$$

We require the functions Y_i (respectively the functions \bar{Y}_i) to be independent, that is to say that the Wronskians $\mathcal{W}(Y_1, \dots, Y_P)$ and $\mathcal{W}(\bar{Y}_1, \dots, \bar{Y}_Q)$ do not vanish. Next, we introduce the differential operators

$$\begin{aligned} \Phi &= \frac{1}{\mathcal{W}(Y_1, \dots, Y_P)} \begin{vmatrix} Y_1 & \cdots & Y_P & 1 \\ Y_1^{(1)} & \cdots & Y_P^{(1)} & \partial \\ \vdots & \ddots & \vdots & \vdots \\ Y_1^{(P)} & \cdots & Y_P^{(P)} & \partial^P \end{vmatrix}, & [D, \Phi] &= 0, \\ \bar{\Phi} &= \frac{1}{\mathcal{W}(\bar{Y}_1, \dots, \bar{Y}_Q)} \begin{vmatrix} \bar{Y}_1 & \cdots & \bar{Y}_Q & 1 \\ \bar{Y}_1^{(1)} & \cdots & \bar{Y}_Q^{(1)} & \partial \\ \vdots & \ddots & \vdots & \vdots \\ \bar{Y}_1^{(Q)} & \cdots & \bar{Y}_Q^{(Q)} & \partial^Q \end{vmatrix}, & [\bar{D}, \bar{\Phi}] &= 0, \end{aligned} \quad (2.59)$$

for which we derive in the usual way [12] the flow equations

$$\partial_k \Phi = (\Phi \partial^k \Phi^{-1})_- \Phi, \quad \partial_k \bar{\Phi} = (\bar{\Phi} \partial^k \bar{\Phi}^{-1})_- \bar{\Phi}. \quad (2.60)$$

Then we consider the dressed operator

$$\mathcal{L} = \Phi \partial^p \bar{\Phi}^{-1}, \quad (2.61)$$

where the order of \mathcal{L} is n and $p = n + Q - P$ is a positive integer, and derive the expressions

$$L = \{D, \mathcal{L} \bar{D} \mathcal{L}^{-1}\} = \Phi \partial \Phi^{-1}, \quad \bar{L} = \{\bar{D}, \mathcal{L}^{-1} D \mathcal{L}\} = \bar{\Phi} \partial \bar{\Phi}^{-1}. \quad (2.62)$$

We conclude, using (2.60), that the operator \mathcal{L} defined in equation (2.61), satisfy the KP type flow equations (2.15).

We now wish to restrict to KdV type flows. In other words we wish to find conditions which ensure that \mathcal{L} is a differential operator. In order to do this, we borrow from [16] the formula

$$\bar{\Phi}^{-1} = \sum_{i=1}^Q \bar{Y}_i \partial^{-1} \bar{Z}_i, \quad \bar{Z}_i = (-)^{Q-i} \frac{\mathcal{W}(\bar{Y}_1, \dots, \bar{Y}_{i-1}, \bar{Y}_{i+1}, \dots, \bar{Y}_Q)}{\mathcal{W}(\bar{Y}_1, \dots, \bar{Y}_Q)} \quad (2.63)$$

from which we deduce that the pseudo-differential part of \mathcal{L} in (2.61) reads

$$\mathcal{L}_- = \sum_{i=1}^Q [\Phi \bar{Y}_i^{(p)}] \partial^{-1} \bar{Z}_i. \quad (2.64)$$

It is then clear that sufficient conditions for \mathcal{L} to be a differential operator are

$$[\Phi \bar{Y}_i^{(p)}] = \mathcal{W}(Y_1, \dots, Y_P, \bar{Y}_i^{(p)}) = 0, \quad i = 1, \dots, Q. \quad (2.65)$$

It is to be noted that these equations mix chiral (Y_i) and antichiral (\bar{Y}_i) superfields.

We shall now find a τ -function for the $N = 2$ KP type hierarchy, that is to say a generating function for the conserved quantities [12]. The starting point is the relation (2.29) between the hamiltonian density \mathcal{H}_k and the residues of powers of L and \bar{L} which can be written as

$$k \partial \mathcal{H}_k = \text{res } \bar{L}^k - \text{res } L^k. \quad (2.66)$$

Our goal is then to obtain expressions for $\text{res } L^k$ and $\text{res } \bar{L}^k$ from the dressing relation

$$L^k = \Phi \partial^k \Phi^{-1}, \quad \bar{L}^k = \bar{\Phi} \partial^k \bar{\Phi}^{-1}. \quad (2.67)$$

For this we apply the formula (2.63) to Φ and $\bar{\Phi}$ and deduce

$$\text{res } L^k = \sum_{i=1}^P (-)^{P-i} [\Phi Y_i^{(k)}] \frac{\mathcal{W}[Y_1, \dots, Y_{i-1}, Y_{i+1}, \dots, Y_P]}{\mathcal{W}[Y_1, \dots, Y_P]} \quad (2.68)$$

together with a similar result for $\text{res } \bar{L}^k$. We then make use of the Wronskian identity

$$\frac{\mathcal{W}[f_1, \dots, f_{P+1}] \mathcal{W}[f_1, \dots, f_{P-1}]}{\mathcal{W}[f_1, \dots, f_P]^2} = \left(\frac{\mathcal{W}[f_1, \dots, f_{P-1}, f_{P+1}]}{\mathcal{W}[f_1, \dots, f_{P-1}, f_P]} \right)_x \quad (2.69)$$

and obtain

$$\text{res } L^k = \sum_{i=1}^P \left(\frac{\mathcal{W}[Y_1, \dots, Y_{i-1}, Y_i^{(k)}, Y_{i+1}, \dots, Y_P]}{\mathcal{W}[Y_1, \dots, Y_P]} \right)_x. \quad (2.70)$$

The result is then obtained by using the condition $\partial_k Y_i = \partial^k Y_i$ and reads

$$\text{res} L^k = \partial \partial_k \ln \mathcal{W}[Y_1, \dots, Y_P], \quad \text{res} \bar{L}^k = \partial \partial_k \ln \mathcal{W}[\bar{Y}_1, \dots, \bar{Y}_Q]. \quad (2.71)$$

Finally, we get the following expression for the τ -function

$$\tau(t, \theta, \bar{\theta}) = \frac{\mathcal{W}[\bar{Y}_1, \dots, \bar{Y}_Q]}{\mathcal{W}[Y_1, \dots, Y_P]}, \quad (2.72)$$

which generates the hamiltonian densities according to

$$\mathcal{H}_k = \frac{1}{k} \partial_k \ln \tau. \quad (2.73)$$

In the following, we shall give an example of a soliton solution constructed as described before for the $N = 2$ $a = 4$ KdV hierarchy. This case, which has been presented before, corresponds to the choice of the differential operator $\mathcal{L} = \partial + J$. Here and further we use the notations

$$\zeta(t, \theta, \bar{\theta}; z, \mu) = z \left(t_1 - \frac{1}{2} \theta \bar{\theta} \right) + \sum_{k=2}^{\infty} z^k t_k + \mu \bar{\theta}, \quad D\zeta = 0, \quad (2.74)$$

$$\bar{\zeta}(t, \theta, \bar{\theta}; z, \mu) = z \left(t_1 + \frac{1}{2} \theta \bar{\theta} \right) + \sum_{k=2}^{\infty} z^k t_k + \mu \theta, \quad \bar{D}\bar{\zeta} = 0, \quad (2.75)$$

where z is a real number, μ an odd Grassmann variable and t_k are the times of the KdV hierarchy with the space variable x identified with t_1 .

We shall choose the number of chiral functions Y to be $N = 2$ and the number of antichiral functions \bar{Y} to be $Q = 1$ so that the integer $p = n + Q - P$ has value zero. We then define the two chiral functions to be

$$Y_1 = e^{\zeta(t, \theta, \bar{\theta}; z_1, \mu_1)} + e^{\zeta(t, \theta, \bar{\theta}; z_2, \mu_2)}, \quad Y_2 = (Y_1)_x \quad (2.76)$$

and the antichiral one to be

$$\bar{Y}_1 = e^{\bar{\zeta}(t, \theta, \bar{\theta}; z_1, \bar{\mu}_1)} + e^{\bar{\zeta}(t, \theta, \bar{\theta}; z_2, \bar{\mu}_2)}. \quad (2.77)$$

A computation shows that this set of functions satisfies the conditions (2.58) and (2.65). The components of the KdV superfield

$$J = J_0 + \theta J_\theta + \bar{\theta} J_{\bar{\theta}} + \theta \bar{\theta} J_{\theta \bar{\theta}} \quad (2.78)$$

then read

$$J_0 = \sqrt{z_1 z_2} \frac{\text{ch}(\eta_1 + \ln z_1 - \eta_2 - \ln z_2)/2}{\text{ch}(\eta_1 - \eta_2)/2}, \quad J_{\theta \bar{\theta}} = \frac{(z_1 - z_2)^2}{8 \text{ch}^2(\eta_1 - \eta_2)/2}, \quad (2.79)$$

$$J_\theta = \frac{(z_2 - z_1)(\bar{\mu}_1 - \bar{\mu}_2)}{4 \text{ch}^2(\eta_1 - \eta_2)/2}, \quad J_{\bar{\theta}} = 0.$$

In this expressions, we used the short-hand notation

$$\eta_k = \zeta(t, 0, 0; z_k, 0) = \bar{\zeta}(t, 0, 0; z_k, 0) \quad (2.80)$$

for $k = 1, 2$. It is to be noted that, since only the difference $\eta_1 - \eta_2$ appears in the above equation, this solution represents a one-soliton for the $N = 2$ $a = 4$ KdV hierarchy. It may also be noted that if one sets $\bar{\mu}_1$ and $\bar{\mu}_2$ to zero in equation (2.79), one obtains a soliton solution of the $(1, 2)$ KdV hierarchy. Finally, the τ -function associated with this solution (2.72) reads

$$\tau(t, \theta, \bar{\theta}) = \frac{e^{\bar{\zeta}(t, \theta, \bar{\theta}; z_1, \bar{\mu}_1)} + e^{\bar{\zeta}(t, \theta, \bar{\theta}; z_2, \bar{\mu}_2)}}{(z_1 - z_2)^2} e^{-\zeta(t, \theta, \bar{\theta}; z_1, \mu_1) - \zeta(t, \theta, \bar{\theta}; z_2, \mu_2)} \quad (2.81)$$

and inspection of equation (2.73) shows that the hamiltonians are

$$H_k = \frac{|z_1^k - z_2^k|}{k}. \quad (2.82)$$

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