Journal of Nonlinear Mathematical Physics 1999, V.6, N 4, 411–447. Article

# $r$-Matrices for Relativistic Deformations of Integrable Systems

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*Received June 06, 1999; Revised July 05, 1999; Accepted August 05, 1999*

**Abstract**

We include the relativistic lattice KP hierarchy, introduced by Gibbons and Kupershmidt, into the $r$-matrix framework. An $r$-matrix account of the nonrelativistic lattice KP hierarchy is also provided for the reader's convenience. All relativistic constructions are regular one-parameter perturbations of the nonrelativistic ones. We derive in a simple way the linear Hamiltonian structure of the relativistic lattice KP, and find for the first time its quadratic Hamiltonian structure. Amasingly, the latter turns out to coincide with its nonrelativistic counterpart (a phenomenon, known previously only for the simplest case of the relativistic Toda lattice).

## 1 Introduction

One of the basic objects in the theory of integrable lattices (or differential-difference systems) is the so called *lattice KP hierarchy* [4, 6]. This is a system of an infinite number of commuting flows on the phase space consisting of an infinite number of fields

$$b = (b_k)_{k\in\mathbb{Z}}, \qquad a^{(j)} = \left(a_k^{(j)}\right)_{k\in\mathbb{Z}} \quad (j \ge 1). \tag{1.1}$$

The simplest ("first") flow of this hierarchy is governed by the following equations of motion:

$$\left\{\begin{array}{l} \dot b_k = a_k^{(1)} - a_{k-1}^{(1)}, \\ \dot a_k^{(j)} = a_k^{(j)}(b_{k+j} - b_k) + \left(a_k^{(j+1)} - a_{k-1}^{(j+1)}\right). \end{array}\right. \tag{1.2}$$

A proper language for such infinite-dimensional systems is the differential-difference calculus developed in [4, 6]. However, all essential properties of this hierarchy hold also for its finite-dimensional reductions described as follows.

First of all, we shall consider here only systems with a finite number of fields. We obtain an $m$-field system, if we set $a_k^{(j)} = 0$ for all $j \ge m$. We shall use the notation $\mathrm{TL}_m$ for this reduced system, because it is a direct generalization of the *Toda lattice*, which appears in this framework as the first nontrivial case $\mathrm{TL}=\mathrm{TL}_2$ with two fields.

Further, we restrict ourselves here to the case of periodic or open-end boundary conditions, when all fields contain only a finite number $N$ of variables, so that in (1.1) one should replace $k\in\mathbb{Z}$ through $1\le k\le N$. More precisely, in case of periodic boundary conditions all subscripts in (1.2) are supposed to belong to $\mathbb{Z}/N\mathbb{Z}$, and in case of open-end boundary conditions we enforce

$$b_k=0 \quad \text{for} \quad k<1 \quad \text{and} \quad \text{for} \quad k>N,$$
$$a_k^{(j)}=0 \quad \text{for} \quad k<1 \quad \text{and} \quad \text{for} \quad k>N-j.$$

One of the outstanding features of this hierarchy is integrability; another one is its bi-Hamiltonian structure. What we want to address in the present paper, is an $r$-matrix theory as the (probably most direct) route towards understanding these both properties.

Actually, the $r$-matrix interpretation of the lattice KP hierarchy is a more or less established thing nowadays [1, 11, 15, 9]. What is really new in this paper, is a similar interpretation of the *relativistic lattice KP hierarchy*. This appeared in [10, 7] as a natural generalization of the relativistic Toda lattice, invented by Ruijsenaars [12]. The relativistic ansatz of [7] (referred to later on as "the first construction") leads to a one-parameter perturbation of the lattice KP hierarchy. An interesting "splitting" phenomenon takes place: to each (polynomial) flow of the lattice KP hierarchy there correspond two flows of its relativistic conterpart, one of them being still polynomial, and another one rational in coordinates. It turns out that the rational relativistic perturbation of the "first" flow (1.2) remains nice and elegant:

$$\left\{\begin{array}{l} \dot b_k=\dfrac{a_k^{(1)}}{1+\alpha b_{k+1}}-\dfrac{a_{k-1}^{(1)}}{1+\alpha b_{k-1}}, \\[3mm] \dot a_k^{(j)}=a_k^{(j)}\left(\dfrac{b_{k+j}}{1+\alpha b_{k+j}}-\dfrac{b_k}{1+\alpha b_k}\right)+\left(\dfrac{a_k^{(j+1)}}{1+\alpha b_{k+j+1}}-\dfrac{a_{k-1}^{(j+1)}}{1+\alpha b_{k-1}}\right), \end{array}\right. \tag{1.3}$$

while the polynomial relativistic perturbation is described by a system of messy equations, right-hand side of each one depending on all fields. In the $m$-field situation we shall denote the flow (1.3) and the whole hierarchy attached to it by $\mathrm{RTL}_m^{(-)}(\alpha)$. It is a direct generalization of the "minus first" flow of the relativistic Toda hierarchy.

It is known that this relativistic lattice KP hierarchy consists of commuting flows, and also one Hamiltonian structure was known for it previously [7, 5, 6]. In the present work we find also the second compatible Hamiltonian structure, and, moreover, give an $r$-matrix account for both. Needless to say, that this automatically implies also the integrability of this hierarchy. Amasingly, while the first Hamiltonian structure of the relativistic hierarchy is a perturbation of its nonrelativistic counterpart, the second Hamiltonian structures of the both hierarchies literally coincide!

The relativistic ansatz of [7] leads to Lax equations of a somewhat non-standard type. We introduce a certain (invertible) gauge transformation, such that the transformed hierarchy possesses the standard Lax representations. In other words, it may be considered as included into the lattice KP hierarchy. It is a highly non-trivial situation, if one takes into account that the lattice KP may be seen also as a limiting case of the relativistic lattice KP! This gauge transformed relativistic hierarchy (referred to as "the second construction") also contains two perturbations, a polynomial one and a rational one, for each nonrelativistic flow, but this time the polynomial flows are simpler. For example, the polynomial "first" flow in the second relativistic construction reads:

$$\begin{cases} \dot{b}_k = (1+\alpha b_k)\left(a_k^{(1)} - a_{k-1}^{(1)}\right), \\ \dot{a}_k^{(j)} = a_k^{(j)}\left(b_{k+j} - b_k + \alpha a_{k+j}^{(1)} - \alpha a_{k-1}^{(1)}\right) + \left(a_k^{(j+1)} - a_{k-1}^{(j+1)}\right). \end{cases} \tag{1.4}$$

We denote this flow, as well as the whole corresponding hierarchy, in the $m$-field situation as $\mathrm{RTL}_m^{(+)}(\alpha)$; it serves as a direct generalization of the "first" flow of the relativistic Toda hierarchy.

Remarkably, in the case $m = 2$, i.e. of the relativistic Toda, both constructions lead to one and the same hierarchy, but this is no more the case for $m > 2$!

Also the second relativistic lattice KP turns out to be bi-Hamiltonian and integrable; we give in this paper an $r$-matrix interpretation of one of its Hamiltonian structures.

The paper is built as follows. In Sect. 2–5 we recall several notions and results from the $r$-matrix theory (those of Sect. 3 being to a certain extent new). Sect. 6 is devoted to fixing the basic notations. The lattice KP hierarchy and its linear and quadratic invariant Poisson structures are dealt with in Sect. 7–9. The similar discussion of the first relativistic construction is contained in Sect. 11–13 (so that the presentation of really new results starts in Sect. 12). Further, we introduce the second relativistic construction in Sect. 14, 15, and discuss its Hamiltonian properties in Sect. 16. As illustrations, we provide concrete results for the three-field systems in Sect. 10, 17. A final discussion takes place in Sect. 18.

## 2 Linear $r$-matrix Poisson structure

We first recall the (by now classical) construction of the linear $r$-matrix structure on the dual to a Lie algebra. Let $\boldsymbol{g}$ be a Lie algebra, carrying a nondegenerate scalar product $\langle\cdot,\cdot\rangle$, allowing to identify the dual space $\boldsymbol{g}^*$ with $\boldsymbol{g}$. In what follows we shall not distinguish between $\boldsymbol{g}$ and $\boldsymbol{g}^*$, but try to use the letters $X$, $Y$, $Z$ consequently for the elements of $\boldsymbol{g}$, and the letters $L$, $M$ for the elements of $\boldsymbol{g}^*$. Suppose that the scalar product is invariant with respect to the Lie bracket in $\boldsymbol{g}$, i.e.

$$\langle X, [Y,Z]\rangle = \langle [X,Y], Z\rangle \qquad \forall\ X,Y,Z \in \boldsymbol{g}.$$

Recall that the gradient $\nabla\varphi : \boldsymbol{g} \mapsto \boldsymbol{g}$ of a smooth function $\varphi$ on $\boldsymbol{g}$ is defined via

$$\langle \nabla\varphi(L), M\rangle = \frac{d}{d\varepsilon}\varphi(L+\varepsilon M)\Big|_{\varepsilon=0} \qquad \forall\ M \in \boldsymbol{g}.$$

Recall also that Ad-invariant functions $\varphi$ on $\boldsymbol{g}$ are characterized by the property

$$[\nabla\varphi(L), L] = 0 \qquad \forall\ L \in \boldsymbol{g}. \tag{2.1}$$

(Of course, we mean here actually $\mathrm{Ad}^*$-invariant functions on $\boldsymbol{g}^*$, characterized by

$$\mathrm{ad}^*\nabla\varphi(L)\cdot L = 0;$$

we shall consequently avoid such remarks from now on).

**Definition 2.1** [13] *Let* R *be a linear operator on* $\boldsymbol{g}$. *A* ***linear r-matrix bracket*** *on* $\boldsymbol{g}$ *corresponding to the operator* R *is defined by:*

$$\{\varphi,\psi\}_1(L)=\frac{1}{2}\langle[\mathrm{R}(\nabla\varphi(L)),\nabla\psi(L)]+[\nabla\varphi(L),\mathrm{R}(\nabla\psi(L))],L\rangle. \tag{2.2}$$

*If this is indeed a Poisson bracket, it will be denoted by* $\mathrm{PB}_1(\mathrm{R})$.

**Theorem 2.2** [13] *A sufficient condition for (2.2) to define a Poisson bracket is given by the* ***modified Yang–Baxter equation*** *for the operator* R:

$$[\mathrm{R}(X),\mathrm{R}(Y)]-\mathrm{R}\left([\mathrm{R}(X),Y]+[X,\mathrm{R}(Y)]\right)=-\alpha[X,Y]\qquad\forall X,Y\in\boldsymbol{g}, \tag{2.3}$$

*where* $\alpha$ *is an arbitrary constant. This equation is denoted* $\mathrm{mYB}(\mathrm{R};\alpha)$.

Actually, (2.3) is a sufficient condition for

$$[X,Y]_{\mathrm{R}}=\frac{1}{2}\left([\mathrm{R}(X),Y]+[X,\mathrm{R}(Y)]\right) \tag{2.4}$$

to define a new Lie bracket on $\boldsymbol{g}$, and then (2.2) is nothing but a Lie–Poisson bracket on $\boldsymbol{g}^*$ corresponding to this new Lie algebra structure on $\boldsymbol{g}$.

One of the most important properties of the linear $r$-matrix bracket is the following one.

**Theorem 2.3** [13] *a) Hamiltonian equations of motion on* $\boldsymbol{g}$ *corresponding to an* Ad-*invariant Hamilton function* $\varphi$, *have the Lax form*

$$\dot{L}=[L,C],\qquad C=\frac{1}{2}\mathrm{R}(\nabla\varphi(L)). \tag{2.5}$$

*b)* Ad-*invariant functions on* $\boldsymbol{g}$ *are in involution with respect to the bracket* $\mathrm{PB}_1(\mathrm{R})$.

# 3 Generalized linear $r$-matrix Poisson structure

Let now $\boldsymbol{g}$ have an additional structure of an associative algebra, with the multiplication $(X,Y)\mapsto X\cdot Y$. The standard definition of the Lie bracket on $\boldsymbol{g}$ is then $[X,Y]=X\cdot Y-Y\cdot X$. Let $\boldsymbol{g}$ carry a *bi-invariant* scalar product, the property expressed by the equalities

$$\langle X,Y\cdot Z\rangle=\langle X\cdot Y,Z\rangle=\langle Z\cdot X,Y\rangle\qquad\forall\ X,Y,Z\in\boldsymbol{g}$$

(this assures the invariance of the scalar product in the previous sense relevant to the Lie algebra structure). We again identify $\boldsymbol{g}^*$ with $\boldsymbol{g}$ with the help of this scalar product.

Actually, there exist infinitely many ways to turn $\boldsymbol{g}$ into a Lie algebra, not only the standard one mentioned above. Indeed, fix an arbitrary element $\mathcal{F}\in\boldsymbol{g}$, and define the corresponding Lie bracket as

$$[X,Y]^{(\mathcal{F})}=X\cdot\mathcal{F}\cdot Y-Y\cdot\mathcal{F}\cdot X. \tag{3.1}$$

We can replace the Lie bracket $[\cdot,\cdot]$ in all constructions of the previous Section by the Lie bracket $[\cdot,\cdot]^{(\mathcal{F})}$, and come in this way to the following definition.

**Definition 3.1** *Let* R *be a linear operator on* $\boldsymbol{g}$*. A* ***generalized linear r-matrix bracket*** *on* $\boldsymbol{g}$ *corresponding to the operator* R *and the element* $\mathcal{F}$ *is defined by:*

$$\{\varphi,\psi\}_1(L)=\frac{1}{2}\langle[\mathrm{R}(\nabla\varphi(L)),\nabla\psi(L)]^{(\mathcal{F})}+[\nabla\varphi(L),\mathrm{R}(\nabla\psi(L))]^{(\mathcal{F})},L\rangle. \tag{3.2}$$

*If this is indeed a Poisson bracket, it will be denoted by* $\mathrm{PB}_1(\mathrm{R};\mathcal{F})$*.*

Of course, a sufficient condition for (3.2) to be a Poisson bracket is that the following expression defines a new structure of a Lie algebra on $\boldsymbol{g}$:

$$[X,Y]_{\mathrm{R}}^{(\mathcal{F})}=\frac{1}{2}\left([\mathrm{R}(X),Y]^{(\mathcal{F})}+[X,\mathrm{R}(Y)]^{(\mathcal{F})}\right). \tag{3.3}$$

A sufficient condition for this is given, clearly, by the $\mathrm{mYB}(\mathrm{R};\alpha)$, where the Lie bracket $[\cdot,\cdot]$ is replaced through $[\cdot,\cdot]^{(\mathcal{F})}$, i.e.

$$[\mathrm{R}(X),\mathrm{R}(Y)]^{(\mathcal{F})}-\mathrm{R}\left([\mathrm{R}(X),Y]^{(\mathcal{F})}+[X,\mathrm{R}(Y)]^{(\mathcal{F})}\right)=-\alpha\,[X,Y]^{(\mathcal{F})}\quad\forall\, X,Y\in\boldsymbol{g}. \tag{3.4}$$

Notice that Ad-invariant functions $\psi$ of the Lie algebra $\left(\boldsymbol{g},[\cdot,\cdot]^{(\mathcal{F})}\right)$ are characterized by the equation

$$L\cdot\nabla\psi(L)\cdot\mathcal{F}-\mathcal{F}\cdot\nabla\psi(L)\cdot L=0\qquad\forall\, L\in\boldsymbol{g}. \tag{3.5}$$

The analog of Theorem 2.3 can now be formulated.

**Theorem 3.2** *a) Let* $\psi$ *be an* Ad*-invariant function of the Lie algebra* $\left(\boldsymbol{g},[\cdot,\cdot]^{(\mathcal{F})}\right)$*. Then the Hamiltonian equation on* $\boldsymbol{g}$ *generated by the Hamilton function* $\psi$ *in the Poisson bracket* $\mathrm{PB}_1(\mathrm{R};\mathcal{F})$ *reads:*

$$\dot{L}=L\cdot C_2-C_1\cdot L, \tag{3.6}$$

*where*

$$C_1=\frac{1}{2}\mathcal{F}\cdot\mathrm{R}(\nabla\psi(L)),\qquad C_2=\frac{1}{2}\mathrm{R}(\nabla\psi(L))\cdot\mathcal{F}. \tag{3.7}$$

*b)* Ad*-invariant functions of the algebra* $\left(\boldsymbol{g},[\cdot,\cdot]^{(\mathcal{F})}\right)$ *are in involution with respect to the bracket* $\mathrm{PB}_1(\mathrm{R};\mathcal{F})$*.*

The above notions simplify under some additional structural assumptions. Let $\boldsymbol{g}$ be an associative algebra with unit, and let $\mathcal{F}\in\boldsymbol{g}$ be an *invertible element*. Then:

**Proposition 3.3** *a) The function* $\psi$ *is an* Ad*-invariant of the algebra* $\left(\boldsymbol{g},[\cdot,\cdot]^{(\mathcal{F})}\right)$*, i.e.* (3.5) *is fulfilled, if and only if*

$$\psi(L)=\varphi\left(L\cdot\mathcal{F}^{-1}\right)=\varphi\left(\mathcal{F}^{-1}\cdot L\right),$$

*where* $\varphi$ *is an* Ad*-invariant of the algebra* $\boldsymbol{g}$ *with the standard Lie bracket. In this case*

$$\nabla\psi(L)=\mathcal{F}^{-1}\cdot\nabla\varphi\left(L\cdot\mathcal{F}^{-1}\right)=\nabla\varphi\left(\mathcal{F}^{-1}\cdot L\right)\cdot\mathcal{F}^{-1}.$$

*b) The modified Yang-Baxter equation (3.4) in the Lie algebra* $\left(\boldsymbol{g},[\cdot,\cdot]^{(\mathcal{F})}\right)$ *for the operator* R *is equivalent to the modified Yang–Baxter equation (2.3) in the algebra* $\boldsymbol{g}$ *with the standard Lie bracket for either of the two linear operators on* $\boldsymbol{g}$*,*

$$\mathrm{R}_1(X)=\mathcal{F}\cdot\mathrm{R}\left(\mathcal{F}^{-1}\cdot X\right)\quad\textit{and}\quad\mathrm{R}_2(X)=\mathrm{R}\left(X\cdot\mathcal{F}^{-1}\right)\cdot\mathcal{F}. \tag{3.8}$$

*c) Under the map* $L\mapsto L\cdot\mathcal{F}^{-1}$ *the bracket* $\mathrm{PB}_1(\mathrm{R};\mathcal{F})$ *is pushed to* $\mathrm{PB}_1(\mathrm{R}_1)$*, and under the map* $L\mapsto\mathcal{F}^{-1}\cdot L$ *the bracket* $\mathrm{PB}_1(\mathrm{R};\mathcal{F})$ *is pushed to* $\mathrm{PB}_1(\mathrm{R}_2)$*.*

# 4 Quadratic $r$-matrix Poisson structure

Let again $\boldsymbol{g}$ be an associative algebra with a nondegenerate invariant scalar product. Let $\mathrm{A}_1$, $\mathrm{A}_2$, $\mathrm{S}$ be three linear operators on $\boldsymbol{g}$, $\mathrm{A}_1$ and $\mathrm{A}_2$ being skew-symmetric:

$$\mathrm{A}_1^* = -\mathrm{A}_1, \qquad \mathrm{A}_2^* = -\mathrm{A}_2. \tag{4.1}$$

**Definition 4.1** **[15]** *A* ***quadratic $r$-matrix bracket*** *on $\boldsymbol{g}$ corresponding to the triple $\mathrm{A}_1$, $\mathrm{A}_2$, $\mathrm{S}$ is defined by:*

$$\begin{aligned}\{\varphi,\psi\}_2(L) = \frac{1}{2}\langle \mathrm{A}_1(d'\varphi(L)), d'\psi(L)\rangle - \frac{1}{2}\langle \mathrm{A}_2(d\varphi(L)), d\psi(L)\rangle \\ +\frac{1}{2}\langle \mathrm{S}(d\varphi(L)), d'\psi(L)\rangle - \frac{1}{2}\langle \mathrm{S}^*(d'\varphi(L)), d\psi(L)\rangle,\end{aligned} \tag{4.2}$$

*where we denote for brevity*

$$d\varphi(L) = L\cdot\nabla\varphi(L), \qquad d'\varphi(L) = \nabla\varphi(L)\cdot L. \tag{4.3}$$

*If this expression indeed defines a Poisson bracket, we shall denote it by* $\mathrm{PB}_2(\mathrm{A}_1,\mathrm{A}_2,\mathrm{S})$.

In what follows we usually suppose the following condition to be satisfied:

$$\mathrm{A}_1 + \mathrm{S} = \mathrm{A}_2 + \mathrm{S}^* = \mathrm{R}. \tag{4.4}$$

Then a linearization of $\mathrm{PB}_2(\mathrm{A}_1,\mathrm{A}_2,\mathrm{S})$ in the unit element of $\boldsymbol{g}$ coincides with $\mathrm{PB}_1(\mathrm{R})$, and we call the former a *quadratization* of the latter.

**Theorem 4.2** **[15]** *A sufficient condition for (4.2) to be a Poisson bracket is given by the equations (4.4) and*

$$\mathrm{mYB}(\mathrm{R};\alpha), \quad \mathrm{mYB}(\mathrm{A}_1;\alpha), \quad \mathrm{mYB}(\mathrm{A}_2;\alpha). \tag{4.5}$$

*Under these conditions the bracket* $\mathrm{PB}_2(\mathrm{A}_1,\mathrm{A}_2,\mathrm{S})$ *is compatible with* $\mathrm{PB}_1(\mathrm{R})$.

If the operator R is skew-symmetric and satisfies $\mathrm{mYB}(\mathrm{R};\alpha)$, then the Poisson bracket $\mathrm{PB}_2(\mathrm{R},\mathrm{R},0)$ is called *Sklyanin bracket* [13]. The brackets $\mathrm{PB}_2(\mathrm{A},\mathrm{A},\mathrm{S})$ with a skew-symmetric operator A and a symmetric operator S were introduced in [8, 10].

One of the most important properties of the $r$-matrix brackets is the following one.

**Theorem 4.3** **[15]** *Let the condition (4.4) be satisfied. Then:*

*a) Hamiltonian equations of motion on $\boldsymbol{g}$ corresponding to an* Ad*-invariant Hamilton function $\varphi$, have the Lax form*

$$\dot{L} = [L, C], \qquad C = \frac{1}{2}\mathrm{R}(d\varphi(L)). \tag{4.6}$$

*b)* Ad*-invariant functions on $\boldsymbol{g}$ are in involution with respect to* $\mathrm{PB}_2(\mathrm{A}_1,\mathrm{A}_2,\mathrm{S})$.

## 5 Quadratic $r$-matrix structures on direct products

Quadratic $r$-matrix brackets have interesting and important features when considered on a "big" algebra $\mathbf{g} = \bigotimes_{j=1}^{n} \boldsymbol{g}$. This algebra carries a (nondegenerate, bi-invariant) scalar product

$$\langle\langle \boldsymbol{L}, \boldsymbol{M} \rangle\rangle = \sum_{k=1}^{n} \langle L_k, M_k \rangle.$$

Working with linear operators on $\mathbf{g}$, we use the following natural notations. Let $\mathbf{A} : \mathbf{g} \mapsto \mathbf{g}$ be a linear operator, let $(\mathbf{A}(\boldsymbol{L}))_i$ be the $i$th component of $\mathbf{A}(\boldsymbol{L})$; then we set

$$(\mathbf{A}(\boldsymbol{L}))_i = \sum_{j=1}^{n} (\mathbf{A})_{ij}(L_j). \tag{5.1}$$

For a smooth function $\Phi(\boldsymbol{L})$ on $\mathbf{g}$ we also denote by $\nabla_j \Phi$, $d_j \Phi$, $d'_j \Phi$ the $j$th components of the corresponding objects. We define also the *monodromy maps* $T_j : \mathbf{g} \mapsto \boldsymbol{g}$, $1 \le j \le n$, by the formula

$$T_j(\boldsymbol{L}) = L_j \cdot \ldots \cdot L_1 \cdot L_n \cdot \ldots \cdot L_{j+1}. \tag{5.2}$$

Now let $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{S}$ be linear operators on $\mathbf{g}$ satisfying conditions analogous to (4.1) and to (4.5). One has, obviously:

$$((\mathbf{A}_1)_{ij})^* = -(\mathbf{A}_1)_{ji}, \qquad ((\mathbf{A}_2)_{ij})^* = -(\mathbf{A}_2)_{ji}, \qquad ((\mathbf{S})_{ij})^* = (\mathbf{S}^*)_{ji}.$$

Then one can define the bracket $\mathrm{PB}_2(\mathbf{A}_1, \mathbf{A}_2, \mathbf{S})$ on $\mathbf{g}$. In components it reads:

$$\begin{aligned} \{\Phi, \Psi\}_2(\boldsymbol{L}) = {} & \frac{1}{2} \sum_{i,j=1}^{n} \langle (\mathbf{A}_1)_{ij}(d'_j \Phi), d'_i \Psi \rangle - \frac{1}{2} \sum_{i,j=1}^{n} \langle (\mathbf{A}_2)_{ij}(d_j \Phi), d_i \Psi \rangle \\ & + \frac{1}{2} \sum_{i,j=1}^{n} \langle (\mathbf{S})_{ij}(d_j \Phi), d'_i \Psi \rangle - \frac{1}{2} \sum_{i,j=1}^{n} \langle (\mathbf{S}^*)_{ij}(d'_j \Phi), d_i \Psi \rangle. \end{aligned} \tag{5.3}$$

**Theorem 5.1** **[16]** *Let* $\mathbf{g}$ *be equipped with the Poisson bracket* $\mathrm{PB}_2(\mathbf{A}_1, \mathbf{A}_2, \mathbf{S})$. *Suppose that the following relations hold:*

$$(\mathbf{A}_1)_{j+1,j+1} + (\mathbf{S})_{j+1,j} = (\mathbf{A}_2)_{j,j} + (\mathbf{S}^*)_{j,j+1} = \mathrm{R} \quad \text{for all} \quad 1 \le j \le n; \tag{5.4}$$

$$(\mathbf{A}_1)_{i+1,j+1} = -(\mathbf{S})_{i+1,j} = (\mathbf{S}^*)_{i,j+1} = -(\mathbf{A}_2)_{i,j} \quad \text{for} \quad i \ne j. \tag{5.5}$$

*Then each map* $T_j : \mathbf{g} \mapsto \boldsymbol{g}$ *(5.2) is Poisson, if the target space* $\boldsymbol{g}$ *is equipped with the Poisson bracket*

$$\mathrm{PB}_2\left((\mathbf{A}_1)_{j+1,j+1}, (\mathbf{A}_2)_{j,j}, (\mathbf{S})_{j+1,j}\right).$$

*Hamilton function of the form* $\Phi(\boldsymbol{L}) = \varphi(L_n \cdot \ldots \cdot L_1)$, *where* $\varphi$ *is an* $\mathrm{Ad}$*-invariant function on* $\boldsymbol{g}$, *generates Hamiltonian equations of motion on* $\mathbf{g}$ *having the form of Lax triads:*

$$\dot{L}_j = L_j C_{j-1} - C_j L_j, \qquad C_j = \frac{1}{2} \mathrm{R}(d\varphi(T_j)). \tag{5.6}$$

*(In all formulas the subscripts should be taken* $(\mathrm{mod}\ n)$*).*

This theorem is a far-reaching generalization of the corresponding result for the Sklyanin bracket $\mathrm{PB}_2(\mathbf{R},\mathbf{R},0)$ on $\mathbf{g}$, which arises when $\mathbf{S}=0$, $\mathbf{A}_1=\mathbf{A}_2=\mathbf{R}$, and $(\mathbf{R})_{ij}=\mathrm{R}\delta_{ij}$. In this case each map $T_j:\mathbf{g}\mapsto\boldsymbol{g}$ (5.2) is Poisson, if the target space $\boldsymbol{g}$ is equipped with the Sklyanin bracket $\mathrm{PB}_2(\mathrm{R},\mathrm{R},0)$ [13]. Certain generalizations of the latter result appeared also, e.g., in [13, 8, 14], but in all previously known formulations only few nonvanishing "operator entries" for the operators $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{S}$ were allowed, namely "diagonal" ones for $\mathbf{A}_1$, $\mathbf{A}_2$, and "subdiagonal" ones for $\mathbf{S}$. In other words, all operators in (5.5) had to vanish. In the present paper, however, we shall need this Theorem in its full generality (albeit only for $n=2$). For the first time this general bracket was applied in [15].

We have discussed above the $r$-matrix origin of Lax equations, such as (2.5), (4.6), or (5.6). If one is concerned with a Lax representation of a certain Hamiltonian flow

$$\dot{x}=\{\mathrm{H},x\} \tag{5.7}$$

on a Poisson manifold $(\mathcal{X},\{\cdot,\cdot\})$, then finding an $r$-matrix interpretation for it consists of finding an $r$-matrix bracket on $\boldsymbol{g}$ (or on $\mathbf{g}$) such that the Lax matrix map $L:\mathcal{X}\mapsto\boldsymbol{g}$ (resp. $\boldsymbol{L}:\mathcal{X}\mapsto\mathbf{g}$) is a Poisson map. Then the manifold consisting of the Lax matrices is a Poisson submanifold.

# 6 Notations

## 6.1 Algebras

Two concrete algebras play the basic role in our presentation. They are well suited to describe various lattice systems with the so called open-end and periodic boundary conditions, respectively. Here are the relevant definitions.

For the *open-end case* we always set $\boldsymbol{g}=gl(N)$, the algebra of $N\times N$ matrices with the usual matrix product, the Lie bracket $[X,Y]=XY-YX$, and the nondegenerate bi-invariant scalar product $\langle X,Y\rangle=\mathrm{tr}(XY)$. As a linear space, $\boldsymbol{g}$ may be represented as a direct sum

$$\boldsymbol{g}=\bigoplus_{p=-N+1}^{N-1}\boldsymbol{g}_p,$$

where the subspace $\boldsymbol{g}_p$ consists of matrices

$$X=\sum_{j-k=p}x_{jk}E_{jk}.$$

(Here and below $E_{jk}$ stands for the matrix whose only nonzero entry is on the intersection of the $j$th row and the $k$th column and is equal to 1). The following sets are *subalgebras* of $\boldsymbol{g}$:

$$\boldsymbol{g}_{>0}=\bigoplus_{p=1}^{N-1}\boldsymbol{g}_p,\qquad \boldsymbol{g}_{\geq 0}=\bigoplus_{p=0}^{N-1}\boldsymbol{g}_p,\qquad \boldsymbol{g}_{<0}=\bigoplus_{p=-N+1}^{-1}\boldsymbol{g}_p,\qquad \boldsymbol{g}_{\leq 0}=\bigoplus_{p=-N+1}^{0}\boldsymbol{g}_p,$$

so that, for instance, $\boldsymbol{g}_{\geq 0}$ consists of lower triangular matrices, $\boldsymbol{g}_{<0}$ consists of strictly upper triangular matrices, and $\boldsymbol{g}_0$ consists of diagonal matrices. We shall always set

$$\boldsymbol{g}_+=\boldsymbol{g}_{\geq 0},\qquad \boldsymbol{g}_-=\boldsymbol{g}_{<0}.$$

Notice that, as a linear space,

$$\boldsymbol{g} = \boldsymbol{g}_+ \oplus \boldsymbol{g}_-,$$

In the *periodic case* we always choose $\boldsymbol{g}$ as a certain *twisted loop algebra* over $gl(N)$. A loop algebra over $gl(N)$ is an algebra of Laurent polynomials with coefficients from $gl(N)$ and a natural commutator $\left[X\lambda^j, Y\lambda^k\right] = [X,Y]\lambda^{j+k}$. Our twisted algebra $\boldsymbol{g}$ is a subalgebra singled out by the additional condition

$$\boldsymbol{g} = \left\{X(\lambda) \in gl(N)\left[\lambda, \lambda^{-1}\right] : \Omega X(\lambda)\Omega^{-1} = X(\omega\lambda)\right\},$$

where $\Omega = \operatorname{diag}\left(1, \omega, \ldots, \omega^{N-1}\right)$, $\omega = \exp(2\pi i/N)$. In other words, elements of $\boldsymbol{g}$ satisfy

$$X(\lambda) = \sum_p \lambda^p \sum_{\substack{j-k\equiv p \\ (\mathrm{mod}\, N)}} x_{jk}^{(p)} E_{jk}. \tag{6.1}$$

The nondegenerate bi-invariant scalar product is chosen as

$$\langle X(\lambda), Y(\lambda)\rangle = \operatorname{tr}(X(\lambda)Y(\lambda))_0, \tag{6.2}$$

the subscript 0 denoting the free term of the formal Laurent series.

As a linear space, $\boldsymbol{g}$ is again a direct sum

$$\boldsymbol{g} = \bigoplus_{p=-\infty}^{\infty} \boldsymbol{g}_p,$$

where $\boldsymbol{g}_p$ consists of matrices

$$X = \lambda^p \sum_{\substack{j-k\equiv p \\ (\mathrm{mod}\, N)}} x_{jk} E_{jk}.$$

We have the subalgebras

$$\boldsymbol{g}_{>0} = \bigoplus_{p>0} \boldsymbol{g}_p, \qquad \boldsymbol{g}_{\geq 0} = \bigoplus_{p\geq 0} \boldsymbol{g}_p, \qquad \boldsymbol{g}_{<0} = \bigoplus_{p<0} \boldsymbol{g}_p, \qquad \boldsymbol{g}_{\leq 0} = \bigoplus_{p\leq 0} \boldsymbol{g}_p.$$

Again, we set

$$\boldsymbol{g}_+ = \boldsymbol{g}_{\geq 0}, \qquad \boldsymbol{g}_- = \boldsymbol{g}_{<0},$$

so that, as a linear space,

$$\boldsymbol{g} = \boldsymbol{g}_+ \oplus \boldsymbol{g}_-.$$

## 6.2 Operators

In both the open-end and the periodic cases we shall denote by $\mathrm{P}_{>0}$, $\mathrm{P}_{\geq 0}$, etc. the projections from $\boldsymbol{g}$ onto the corresponding subspace $\boldsymbol{g}_{>0}$, $\boldsymbol{g}_{\geq 0}$, etc., along the complementing subspace. For two such projections we use special notations:

$$\pi_+ = \mathrm{P}_{\geq 0}, \qquad \pi_- = \mathrm{P}_{<0}.$$

The basic operator governing the hierarchies of Lax equations, is

$$\mathrm{R} = \pi_+ - \pi_-. \tag{6.3}$$

An important property of this operator is given by the following general statement:

**Theorem 6.1 [1, 13]** *Let $\boldsymbol{g}$ as a linear space be a direct sum of its two subalgebras:*

$$\boldsymbol{g} = \boldsymbol{g}_+ \oplus \boldsymbol{g}_-,$$

*and let $\pi_\pm$ denote projections from $\boldsymbol{g}$ onto $\boldsymbol{g}_\pm$ along the complementary subspace. Then the operator*

$$\mathrm{R} = \pi_+ - \pi_-$$

*satisfies the modified Yang–Baxter equation* $\mathrm{mYB}(\mathrm{R}, \alpha)$ *with* $\alpha = 1$.

Let us remark that for the operators (6.3) the Lax equations (2.5), (4.6) are equivalent to

$$\dot{L} = [L, B] = [A, L], \quad \text{where} \quad B = \pi_+(f(L)), \quad A = \pi_-(f(L)) \tag{6.4}$$

with $f(L) = \nabla\varphi(L)$ and $f(L) = d\varphi(L)$, respectively. Similarly, the Lax equations on the "big" algebra $\mathbf{g}$ (5.6) for the operator (6.3) are equivalent to:

$$\dot{L}_j = L_j B_{j-1} - B_j L_j = A_j L_j - L_j A_{j-1}, \tag{6.5}$$

where

$$B_j = \pi_+(f(T_j)), \qquad A_j = \pi_-(f(T_j)).$$

We shall call such Lax equations *standard* ones.

Denote by $\mathrm{R}_0$, $\mathrm{P}_0$ the skew-symmetric and the symmetric parts of the operator (6.3), respectively:

$$\mathrm{R}_0 = (\mathrm{R} - \mathrm{R}^*)/2, \qquad \mathrm{P}_0 = (\mathrm{R} + \mathrm{R}^*)/2.$$

It is easy to see that

$$\mathrm{R}_0 = \mathrm{P}_{>0} - \mathrm{P}_{<0},$$

and the notation $\mathrm{P}_0$ agrees with the previously defined projection to $\boldsymbol{g}_0$: in the open-end case $\mathrm{P}_0$ assigns to each matrix $X$ its diagonal part, and in the periodic case $\mathrm{P}_0$ assigns to each Laurent series $X(\lambda)$ its free term.

Let the skew-symmetric operator W act on $\boldsymbol{g}_0$ according to

$$\mathrm{W}(E_{kk}) = \sum_{j<k} E_{jj} - \sum_{j>k} E_{jj} = \sum_{j=1}^{N} w_{kj} E_{jj}, \tag{6.6}$$

where

$$w_{kj} = \operatorname{sgn}(k-j) = \begin{cases} 1, & k > j \\ 0, & k = j \\ -1, & k < j \end{cases} \tag{6.7}$$

and extend W on the rest of $\boldsymbol{g}$ according to $\mathrm{W} = \mathrm{W} \circ \mathrm{P}_0$. Finally, define:

$$\mathrm{A}_1 = \mathrm{R}_0 + \mathrm{W}, \quad \mathrm{A}_2 = \mathrm{R}_0 - \mathrm{W}, \quad \mathrm{S} = \mathrm{P}_0 - \mathrm{W}, \quad \mathrm{S}^* = \mathrm{P}_0 + \mathrm{W}. \tag{6.8}$$

These operators will be basic building blocks in all quadratic $r$-matrix brackets appearing in this paper.

# 7 Multi-field analog of the Toda lattice

The $(m+1)$-field phase space is, in the periodic case,

$$\mathcal{T}_{m+1} = \mathbb{R}^{(m+1)N}\left(b, a^{(1)}, \ldots, a^{(m)}\right), \tag{7.1}$$

where the vectors

$$b = (b_1, \ldots, b_N) \quad \text{and} \quad a^{(j)} = \left(a_1^{(j)}, \ldots, a_N^{(j)}\right) \quad (j = 1, \ldots, m)$$

represent the $m+1$ fields. In the open-end case the fields $a^{(j)}$ consist of $N-j$ variables $\left(a_1^{(j)}, \ldots, a_{N-j}^{(j)}\right)$ only.

Consider the Lax matrix

$$T\left(b, a^{(1)}, \ldots, a^{(m)}, \lambda\right) = \lambda \sum_{k=1}^{N} E_{k+1,k} + \sum_{k=1}^{N} b_k E_{kk} + \sum_{j=1}^{m} \lambda^{-j} \sum_{k=1}^{N} a_k^{(j)} E_{k,k+j}. \tag{7.2}$$

The subspace of $\boldsymbol{g}$ consisting of all such matrices will be denoted

$$\mathbf{T}_{m+1} = \mathcal{E} \oplus \bigoplus_{j=0}^{m} \boldsymbol{g}_{-j}. \tag{7.3}$$

(We introduced here the notation $\mathcal{E} = \lambda \sum\limits_{k=1}^{N} E_{k+1,k}$; recall that in the periodic case all subscripts are considered (mod $N$), so that $E_{N+1,N} = E_{1,N}$).

In the open-end case the Lax matrix, by convention, is given by

$$T\left(b, a^{(1)}, \ldots, a^{(m)}\right) = \sum_{k=1}^{N-1} E_{k+1,k} + \sum_{k=1}^{N} b_k E_{kk} + \sum_{j=1}^{m} \sum_{k=1}^{N-j} a_k^{(j)} E_{k,k+j},$$

and the space of all such matrices is denoted still by (7.3). (So, in the open-end case $\mathcal{E} = \sum\limits_{k=1}^{N-1} E_{k+1,k}$). In what follows we shall formulate the results and give the proofs mainly for the periodic case, since for the open-end case they appear to be parallel and only more simple.

**Proposition 7.1** [4] *The Lax equation*

$$\dot{T} = [T, B], \tag{7.4}$$

*with the Lax matrix (7.2) and the auxiliary matrix*

$$B\left(b, a^{(1)}, \ldots, a^{(m)}, \lambda\right) = \sum_{k=1}^{N} b_k E_{kk} + \lambda \sum_{k=1}^{N} E_{k+1,k}, \tag{7.5}$$

*is equivalent to the following system of differential equations:*

$$\begin{cases} \dot{b}_k = a_k^{(1)} - a_{k-1}^{(1)}, \\ \dot{a}_k^{(j)} = a_k^{(j)} \left(b_{k+j} - b_k\right) + \left(a_k^{(j+1)} - a_{k-1}^{(j+1)}\right), \quad 1 \le j \le m-1, \\ \dot{a}_k^{(m)} = a_k^{(m)} \left(b_{k+m} - b_k\right). \end{cases} \tag{7.6}$$

**Proof** – an elementary matrix calculation. ■

The system (7.6) is what will be called $\mathrm{TL}_{m+1}$ – the generalized $(m+1)$-field Toda lattice. It serves as a direct generalization of the usual Toda lattice TL. A consistent notation for TL would be $\mathrm{TL}_2$, i.e. it corresponds to the value $m = 1$.

## 8 Linear $r$-matrix structure for $\mathrm{TL}_{m+1}$

Obviously, the matrix $B$ from (7.5) is nothing but the projection

$$B = \pi_+(T) \tag{8.1}$$

in the standard decomposition of the algebra $\boldsymbol{g}$. So, the Lax representation of the system $\mathrm{TL}_{m+1}$ is of the standard form (6.4) with $f(T) = T$. We now give an $r$-matrix interpretation to this Lax representation, defining simultaneosly the whole hierarchy $\mathrm{TL}_{m+1}$. We start with the linear $r$-matrix Poisson structure on $\boldsymbol{g}$.

**Theorem 8.1** [13, 11] *The set $\mathbf{T}_{m+1}$ is a Poisson submanifold in the algebra $\boldsymbol{g}$ equipped with the linear bracket* $\mathrm{PB}_1(\mathrm{R})$.

**Proof.** We have to demonstrate that at each point $T \in \mathbf{T}_{m+1}$ every Hamiltonian vector field $\{\varphi, \cdot\}_1$ is tangent to $\mathbf{T}_{m+1}$. To do this, notice that the value of the general Hamiltonian vector field in the point $T$ is given by:

$$2\{\varphi, T\}_1 = [T, \mathrm{R}(\nabla\varphi)] + \mathrm{R}^*([T, \nabla\varphi]).$$

Obviously, this can be represented in two equivalent forms:

$$\{\varphi, T\}_1 = [T, \mathrm{P}_{\ge 0}(\nabla\varphi)] - \mathrm{P}_{>0}([T, \nabla\varphi]), \tag{8.2}$$

and

$$\{\varphi, T\}_1 = -[T, \mathrm{P}_{<0}(\nabla\varphi)] + \mathrm{P}_{\le 0}([T, \nabla\varphi]). \tag{8.3}$$

Now take into account that $T \in \bigoplus_{j=-m}^{1} \boldsymbol{g}_j$. Then, according to (8.2), the value of $\{\varphi, T\}_1$ belongs to $\bigoplus_{j\geq -m} \boldsymbol{g}_j$, and, according to (8.3), this value belongs also to $\bigoplus_{j\leq 0} \boldsymbol{g}_j$. Hence it belongs to $\bigoplus_{j=-m}^{0} \boldsymbol{g}_j$, which is the tangent space to the manifold $\mathbf{T}_{m+1}$ in $\boldsymbol{g}$. ■

It remains to calculate the induced bracket on $\mathcal{T}_{m+1}$. In what follows we always define Poisson brackets by writing down all nonvanishing brackets among the coordinate functions.

**Theorem 8.2** *The coordinate representation of the restriction of* $\mathrm{PB}_1(\mathrm{R})$ *to the submanifold* $\mathbf{T}_{m+1}$ *is given by the formulas:*

$$\left\{b_k, a_k^{(j)}\right\}_1 = -a_k^{(j)}, \qquad \left\{a_k^{(j)}, b_{k+j}\right\}_1 = -a_k^{(j)}, \tag{8.4}$$

$$\left\{a_k^{(i)}, a_{k+i}^{(j)}\right\}_1 = -a_k^{(i+j)} \quad (i+j \leq m). \tag{8.5}$$

**Proof.** We have, obviously:

$$\nabla b_k = \mathrm{R}(\nabla b_k) = E_{kk}, \qquad \nabla a_k^{(i)} = \mathrm{R}\left(\nabla a_k^{(i)}\right) = \lambda^i E_{k+i,k}. \tag{8.6}$$

Hence

$$\left\{b_k, a_\ell^{(j)}\right\}_1 = \langle T, \lambda^j [E_{kk}, E_{\ell+j,\ell}]\rangle = \langle T, \lambda^j E_{\ell+j,\ell}(\delta_{k,\ell+j} - \delta_{k,\ell})\rangle = a_\ell^{(j)}(\delta_{k,\ell+j} - \delta_{k,\ell}),$$

which coincides with (8.4). Analogously,

$$\begin{aligned}\left\{a_k^{(i)}, a_\ell^{(j)}\right\}_1 &= \langle T, \lambda^{i+j}[E_{k+i,k}, E_{\ell+j,\ell}]\rangle = \langle T, \lambda^{i+j} E_{k+i,\ell}\delta_{k,\ell+j} - E_{\ell+j,k}\delta_{\ell,k+i}\rangle \\ &= a_\ell^{(i+j)}\delta_{k,\ell+j} - a_k^{(i+j)}\delta_{\ell,k+i},\end{aligned}$$

which coincides with (8.5). One shows similarly that the brackets $\{b_k, b_j\}_1$ vanish. This finishes the proof. ■

**Corollary.** [4] *The system* $\mathrm{TL}_{m+1}$ *(7.6) is Hamiltonian with respect to the bracket (8.4), (8.5), with the Hamilton function*

$$\mathrm{H}_2 = \frac{1}{2}\sum_{k=1}^{N} b_k^2 + \sum_{k=1}^{N} a_k^{(1)} = \frac{1}{2}\left(\mathrm{tr}\left(T^2\right)\right)_0. \tag{8.7}$$

Theorem 8.1 not only provides us with the Hamiltonian structure for the system $\mathrm{TL}_{m+1}$, but gives also an $r$-matrix explanation to the Lax representation from Theorem 7.1, and allows to define the $\mathrm{TL}_{m+1}$ hierarchy as a set of flows

$$\dot{T} = [T, \pi_+(\nabla\varphi(T))]$$

with Ad-invariant functions $\varphi$.

Also the complete integrability of $\mathrm{TL}_{m+1}$ with respect to the linear Poisson bracket can be studied on the base of Theorem 8.1, because it guarantees involutivity of all spectral invariants of the matrix $T\left(b, a^{(1)}, \ldots, a^{(m)}, \lambda\right)$. Therefore, the complete integrability will follow as soon as it is proved that the number of independent spectral invariants is large enough. This is almost obvious in the case $m=1$, i.e. for the usual TL, but for $m>1$ it is no longer evident, and, moreover, is no longer true in general. However, it may be shown that in the periodic case one gets the sufficient number of independent spectral invariants. In the open-end case the situation is somewhat more delicate. One has to introduce integrals which are not spectral invariants [2, 3].

## 9 Quadratic $r$-matrix structure for $\mathbf{TL}_{m+1}$

**Theorem 9.1 [15, 9]** *The set $\mathbf{T}_{m+1}$ is a Poisson submanifold in the algebra $\boldsymbol{g}$ equipped with the quadratic bracket* $\mathrm{PB}_2(\mathrm{A}_1, \mathrm{A}_2, \mathrm{S})$.

**Proof.** The Hamiltonian vector field on $\boldsymbol{g}$ with an arbitrary Hamilton function $\varphi$ with respect to the bracket $\mathrm{PB}_2(\mathrm{A}_1, \mathrm{A}_2, \mathrm{S})$ is given by:

$$2\{\varphi, T\}_2 = T \cdot \mathrm{A}_1(\nabla\varphi T) - \mathrm{A}_2(T\nabla\varphi) \cdot T + T \cdot \mathrm{S}(T\nabla\varphi) - \mathrm{S}^*(\nabla\varphi T) \cdot T. \tag{9.1}$$

For the operators (6.8) this can be represented in two equivalent forms:

$$\begin{aligned} 2\{\varphi, T\}_2 = {} & 2T\mathrm{P}_{>0}(\nabla\varphi T) - 2\mathrm{P}_{>0}(T\nabla\varphi)T \\ & + T\mathrm{P}_0(\nabla\varphi T) - \mathrm{P}_0(T\nabla\varphi)T + T\mathrm{P}_0(T\nabla\varphi) - \mathrm{P}_0(\nabla\varphi T)T \\ & + T\mathrm{W}(\nabla\varphi T) + \mathrm{W}(T\nabla\varphi)T - T\mathrm{W}(T\nabla\varphi) - \mathrm{W}(\nabla\varphi T)T, \end{aligned} \tag{9.2}$$

and

$$\begin{aligned} 2\{\varphi, T\}_2 = {} & -2T\mathrm{P}_{<0}(\nabla\varphi T) + 2\mathrm{P}_{<0}(T\nabla\varphi)T \\ & - T\mathrm{P}_0(\nabla\varphi T) + \mathrm{P}_0(T\nabla\varphi)T + T\mathrm{P}_0(T\nabla\varphi) - \mathrm{P}_0(\nabla\varphi T)T \\ & + T\mathrm{W}(\nabla\varphi T) + \mathrm{W}(T\nabla\varphi)T - T\mathrm{W}(T\nabla\varphi) - \mathrm{W}(\nabla\varphi T)T. \end{aligned} \tag{9.3}$$

The first expression assures that for $T \in \mathbf{T}_{m+1}$ this vector field takes the value in $\bigoplus_{j \ge -m} \boldsymbol{g}_j$, and the second expression yields that that this vector field belongs also to $\bigoplus_{j \le 1} \boldsymbol{g}_j$. Hence it belongs to $\overset{1}{\underset{j=-m}{\oplus}} \boldsymbol{g}_j$. It remains to prove that the $\boldsymbol{g}_1$ component of this vector field vanishes. Up to now everything held true for an arbitrary operator W with the values in $\boldsymbol{g}_0$. Now the concrete expression (6.6) for W becomes crucial. From (7.2), (9.3) we have the following expression for the component in question:

$$\begin{aligned} 2\mathrm{P}_1\left(\{\varphi, T\}_2\right) = {} & -\mathcal{E}\mathrm{P}_0(\nabla\varphi T) + \mathrm{P}_0(T\nabla\varphi)\mathcal{E} + \mathcal{E}\mathrm{P}_0(T\nabla\varphi) - \mathrm{P}_0(\nabla\varphi T)\mathcal{E} \\ & + \mathcal{E}\mathrm{W}(\nabla\varphi T) + \mathrm{W}(T\nabla\varphi)\mathcal{E} - \mathcal{E}\mathrm{W}(T\nabla\varphi) - \mathrm{W}(\nabla\varphi T)\mathcal{E}. \end{aligned} \tag{9.4}$$

Due to $\mathrm{W} = \mathrm{W} \circ \mathrm{P}_0$ the condition for (9.4) to vanish may be presented as

$$\mathrm{W}(D)\mathcal{E} - \mathcal{E}\mathrm{W}(D) + D\mathcal{E} + \mathcal{E}D = \mathrm{W}(D')\mathcal{E} - \mathcal{E}\mathrm{W}(D') + D'\mathcal{E} + \mathcal{E}D', \tag{9.5}$$

where

$$D = \mathrm{P}_0(T\nabla\varphi), \qquad D' = \mathrm{P}_0(\nabla\varphi T). \tag{9.6}$$

Now a direct calculation shows that for the operator W in (6.6) and for an arbitrary diagonal matrix $D$ one has:

$$\mathrm{W}(D)\mathcal{E} - \mathcal{E}\mathrm{W}(D) + D\mathcal{E} + \mathcal{E}D = \begin{cases} 2\lambda\,\mathrm{tr}(D)\,E_{1,N}, & \text{periodic case} \\ 0, & \text{open} - \text{end case} \end{cases} \tag{9.7}$$

This proves (9.5) for the matrices (9.6), since $\mathrm{tr}(D) = \mathrm{tr}(T\nabla\varphi) = \mathrm{tr}(\nabla\varphi T) = \mathrm{tr}(D')$ (note that in the open-end case the last argumentation is superfluos). ■

Theorem 9.1 gives the second Hamiltonian interpretation of the system $\mathrm{TL}_{m+1}$, and yields simultaneously the complete integrability of $\mathrm{TL}_{m+1}$ in this Hamiltonian formulation (at least in the periodic case). Moreover, it delivers an alternative $r$-matrix explaination of the Lax representation from Theorem 7.1. It remains to calculate the coordinate representation of the restriction of $\mathrm{PB}_2(\mathrm{A}_1, \mathrm{A}_2, \mathrm{S})$ to $\mathbf{T}_{m+1}$. In the sequel we often use in our formulas the following notational convention:

$$a_k^{(0)} = b_k; \qquad a_k^{(i)} = 0 \quad \text{for} \quad i < 0 \quad \text{or} \quad i > m, \tag{9.8}$$

whenever applicable.

**Theorem 9.2** *The bracket induced on* $\mathbf{T}_{m+1}$ *by* $\mathrm{PB}_2(\mathrm{A}_1, \mathrm{A}_2, \mathrm{S})$ *is given by the following formulas (in (9.13) we suppose that* $i < j$*, and in (9.14), (9.15) we suppose that* $i \le j$*):*

$$\{b_k, b_{k+1}\}_2 = -a_k^{(1)}, \tag{9.9}$$

$$\left\{b_k, a_{k+1}^{(j)}\right\}_2 = -a_k^{(j+1)}, \qquad \left\{a_k^{(j)}, b_{k+j+1}\right\}_2 = -a_k^{(j+1)}, \tag{9.10}$$

$$\left\{b_k, a_k^{(j)}\right\}_2 = -b_k a_k^{(j)}, \qquad \left\{a_k^{(j)}, b_{k+j}\right\}_2 = -a_k^{(j)} b_{k+j}, \tag{9.11}$$

$$\left\{a_k^{(i)}, a_{k+i+1}^{(j)}\right\}_2 = -a_k^{(i+j+1)}, \tag{9.12}$$

$$\left\{a_k^{(i)}, a_k^{(j)}\right\}_2 = -a_k^{(i)} a_k^{(j)}, \qquad \left\{a_k^{(j)}, a_{k+j-i}^{(i)}\right\}_2 = -a_k^{(j)} a_{k+j-i}^{(i)}, \tag{9.13}$$

$$\left\{a_k^{(i)}, a_{k+\ell}^{(j)}\right\}_2 = -a_k^{(i)} a_{k+\ell}^{(j)} - a_k^{(j+\ell)} a_{k+\ell}^{(i-\ell)} \qquad (1 \le \ell \le i), \tag{9.14}$$

$$\left\{a_k^{(j)}, a_{k+\ell}^{(i)}\right\}_2 = -a_k^{(j)} a_{k+\ell}^{(i)} - a_k^{(i+\ell)} a_{k+\ell}^{(j-\ell)} \qquad (j - i + 1 \le \ell \le j). \tag{9.15}$$

**Proof.** Notice that without the convention (9.8) the second term on the right-hand side of (9.14) would have to be multiplied by $\chi_\ell\,(1, \min(i, m-j))$, and, likewise, the second term on the right-hand side of (9.15) would have to be multiplied by $\chi_\ell\,(j-i+1, \min(j, m-i))$, where

$$\chi_\ell(\alpha, \beta) = \begin{cases} 1, & \alpha \le \ell \le \beta \\ 0, & \text{else} \end{cases}$$

is the characteristic function of the interval $[\alpha, \beta]$. Notice also that under the convention (9.8) the formulas (9.9), (9.10) might be seen as particular cases of (9.12). Similarly, (9.11) might be seen as a particular case of (9.13).

Toward the calculation of the induced bracket, we use the defining formula

$$\begin{aligned}2\{\varphi,\psi\}_2 &= 2\langle \mathrm{P}_{>0}(\nabla\varphi T),\nabla\psi T\rangle - 2\langle \mathrm{P}_{>0}(T\nabla\varphi),T\nabla\psi\rangle\\ &\quad+\langle \mathrm{P}_0(\nabla\varphi T),\nabla\psi T\rangle - \langle \mathrm{P}_0(T\nabla\varphi),T\nabla\psi\rangle + \langle \mathrm{P}_0(T\nabla\varphi),\nabla\psi T\rangle - \langle \mathrm{P}_0(\nabla\varphi T),T\nabla\psi\rangle\\ &\quad+\langle \mathrm{W}(\nabla\varphi T),\nabla\psi T\rangle + \langle \mathrm{W}(T\nabla\varphi),T\nabla\psi\rangle - \langle \mathrm{W}(T\nabla\varphi),\nabla\psi T\rangle - \langle \mathrm{W}(\nabla\varphi T),T\nabla\psi\rangle.\end{aligned} \tag{9.16}$$

Take here

$$\varphi(T)=a_k^{(i)},\qquad \psi(T)=a_{k+\ell}^{(j)},$$

so that

$$\nabla\varphi=\lambda^i E_{k+i,i},\qquad \nabla\psi=\lambda^j E_{k+\ell+j,k+\ell}.$$

Further, we find:

$$\nabla\varphi T=\lambda^{i+1}E_{k+i,k-1}+\sum_{\beta=0}^{m}\lambda^{i-\beta}a_k^{(\beta)}E_{k+i,k+\beta}, \tag{9.17}$$

$$T\nabla\varphi=\lambda^{i+1}E_{k+i+1,k}+\sum_{\beta=0}^{m}\lambda^{i-\beta}a_{k+i-\beta}^{(\beta)}E_{k+i-\beta,k}. \tag{9.18}$$

We consider first the contribution to the Poisson bracket $\{\varphi,\psi\}_2=\left\{a_k^{(i)},a_{k+\ell}^{(j)}\right\}_2$ from the first line in (9.16):

$$\left\langle \lambda^{i+1}E_{k+i,k-1}+\sum_{\beta=0}^{i-1}\lambda^{i-\beta}a_k^{(\beta)}E_{k+i,k+\beta},\ \sum_{\gamma=j+1}^{m}\lambda^{j-\gamma}a_{k+\ell}^{(\gamma)}E_{k+\ell+j,k+\ell+\gamma}\right\rangle$$
$$-\left\langle \lambda^{i+1}E_{k+i+1,k}+\sum_{\beta=0}^{i-1}\lambda^{i-\beta}a_{k+i-\beta}^{(\beta)}E_{k+i-\beta,k},\ \sum_{\gamma=j+1}^{m}\lambda^{j-\gamma}a_{k+\ell+j-\gamma}^{(\gamma)}E_{k+\ell+j-\gamma,k+\ell}\right\rangle.$$

Calculating these scalar products, we find:

$$\begin{aligned}&=\sum_{\gamma=j+1}^{m}\left(a_{k+\ell}^{(\gamma)}\delta_{-1,\ell+j}\delta_{i,\ell+\gamma}-a_{k+\ell+j-\gamma}^{(\gamma)}\delta_{0,\ell+j-\gamma}\delta_{i+1,\ell}\right)\\ &\quad+\sum_{\beta=0}^{i-1}\sum_{\gamma=j+1}^{m}\left(a_k^{(\beta)}a_{k+\ell}^{(\gamma)}\delta_{\beta,\ell+j}\delta_{i,\ell+\gamma}-a_{k+i-\beta}^{(\beta)}a_{k+\ell+j-\gamma}^{(\gamma)}\delta_{0,\ell+j-\gamma}\delta_{i-\beta,\ell}\right)\\ &\quad=a_{k+\ell}^{(i+j+1)}\,\delta_{\ell,-j-1}-a_k^{(i+j+1)}\delta_{\ell,i+1}+\varkappa_{k\ell}^{(ij)}a_k^{(\ell+j)}a_{k+\ell}^{(i-\ell)},\end{aligned}$$

where

$$\begin{aligned}\varkappa_{k\ell}^{(ij)}&=\chi_\ell(-j,i-j-1)\chi_\ell(i-m,i-j-1)-\chi_\ell(1,m-j)\chi_\ell(1,i)\\ &=\chi_\ell\left(-\min(j,m-i),i-j-1\right)-\chi_\ell\left(1,\min(i,m-j)\right).\end{aligned} \tag{9.19}$$

Assuming, for the sake of definiteness, that $i \le j$, we see that the intervals of the two characteristic functions in the last line do not intersect. So, we found the contributions to the Poisson bracket $\left\{a_k^{(i)}, a_{k+\ell}^{(j)}\right\}_2$ described by the formulas (9.9), (9.10), (9.12), and the second terms on the right-hand sides of (9.14), (9.15). Now we turn to the contribution of the remaining part of (9.16). To this end, we notice that

$$\mathrm{P}_0(\nabla\varphi T) = a_k^{(i)} E_{k+i,k+i}, \qquad \mathrm{P}_0(T\nabla\varphi) = a_k^{(i)} E_{kk}.$$

Using also (6.6), we find that the contribution under consideration is equal to

$$\varepsilon_{k\ell}^{(ij)} a_k^{(i)} a_{k+\ell}^{(j)}, \tag{9.20}$$

where

$$\begin{aligned}\varepsilon_{k\ell}^{(ij)} = \frac{1}{2}(&\delta_{i,\ell+j} - \delta_{0,\ell} + \delta_{0,\ell+j} - \delta_{i,\ell} \\ &+ w_{k+i,k+\ell+j} + w_{k,k+\ell} - w_{k,k+\ell+j} - w_{k+i,k+\ell}).\end{aligned} \tag{9.21}$$

A direct analysis allows to find a compact expression for this coefficient. In the case $i \le j$ we have:

$$\varepsilon_{k\ell}^{(ij)} = \chi_\ell(-j, i-j) - \chi_\ell(0, i). \tag{9.22}$$

Again, by $i < j$ the intervals of these two characteristic functions do not intersect, and by $i = j$ their only common point is $\ell = 0$, so that the last formula exactly describes the contribution into the Poisson bracket made by (9.11), (9.13), and the first terms on the right-hand sides of (9.14), (9.15). This finishes the proof. ∎

**Corollary.** [4] *The system* $\mathrm{TL}_{m+1}$ *(7.6) is Hamiltonian with respect to the bracket given in Theorem* 9.2*, with the Hamilton function*

$$\mathrm{H}_1 = \sum_{k=1}^{N} b_k = (\mathrm{tr}(T))_0\,. \tag{9.23}$$

## 10 Example: $\mathrm{TL}_3$, the three-field analog of the Toda lattice

In order to illustrate the above results, we give their specialization for the case next in complexity after the usual Toda lattice TL, i.e. for $m = 2$. The equations of motion of the system $\mathrm{TL}_3$ read:

$$\begin{cases} \dot b_k = a_k^{(1)} - a_{k-1}^{(1)}, \\ \dot a_k^{(1)} = a_k^{(1)}\,(b_{k+1} - b_k) + \left(a_k^{(2)} - a_{k-1}^{(2)}\right), \\ \dot a_k^{(2)} = a_k^{(2)}\,(b_{k+2} - b_k)\,. \end{cases} \tag{10.1}$$

The linear invariant Poisson bracket of this system is given by:

$$\begin{aligned} &\left\{b_k, a_k^{(1)}\right\}_1 = -a_k^{(1)}, \qquad \left\{a_k^{(1)}, b_{k+1}\right\}_1 = -a_k^{(1)}, \\ &\left\{b_k, a_k^{(2)}\right\}_1 = -a_k^{(2)}, \qquad \left\{a_k^{(2)}, b_{k+2}\right\}_1 = -a_k^{(2)}, \qquad \left\{a_k^{(1)}, a_{k+1}^{(1)}\right\}_1 = -a_k^{(2)}. \end{aligned} \tag{10.2}$$

The quadratic invariant Poisson bracket, compatible with the previous one, is given by:

$$\begin{gathered}
\{b_k, b_{k+1}\}_2 = -a_k^{(1)}, \qquad \left\{b_k, a_k^{(1)}\right\}_2 = -b_k a_k^{(1)}, \qquad \left\{a_k^{(1)}, b_{k+1}\right\}_2 = -a_k^{(1)} b_{k+1}, \\
\left\{b_k, a_{k+1}^{(1)}\right\}_2 = -a_k^{(2)}, \qquad \left\{a_k^{(1)}, b_{k+2}\right\}_2 = -a_k^{(2)}, \qquad \left\{b_k, a_k^{(2)}\right\}_2 = -b_k a_k^{(2)}, \\
\left\{a_k^{(2)}, b_{k+2}\right\}_2 = -a_k^{(2)} b_{k+2}, \qquad \left\{a_k^{(1)}, a_{k+1}^{(1)}\right\}_2 = -a_k^{(1)} a_{k+1}^{(1)} - a_k^{(2)} b_{k+2}, \\
\left\{a_k^{(1)}, a_k^{(2)}\right\}_2 = -a_k^{(1)} a_k^{(2)}, \qquad \left\{a_k^{(2)}, a_{k+1}^{(1)}\right\}_2 = -a_k^{(2)} a_{k+1}^{(1)}, \\
\left\{a_k^{(1)}, a_{k+1}^{(2)}\right\}_2 = -a_k^{(1)} a_{k+1}^{(2)}, \qquad \left\{a_k^{(2)}, a_{k+2}^{(1)}\right\}_2 = -a_k^{(2)} a_{k+2}^{(1)}, \\
\left\{a_k^{(2)}, a_{k+1}^{(2)}\right\}_2 = -a_k^{(2)} a_{k+1}^{(2)}, \qquad \left\{a_k^{(2)}, a_{k+2}^{(2)}\right\}_2 = -a_k^{(2)} a_{k+2}^{(2)}.
\end{gathered} \tag{10.3}$$

# 11 Multi-field analog of the relativistic Toda lattice: the first construction

The essence of the relativistic ansatz of [7] consists in multiplying the nonrelativistic Lax matrix by $\mathcal{F}^{-1}(\lambda)$, where

$$\mathcal{F}(\lambda) = I - \alpha\lambda \sum_{k=1}^{N} E_{k+1,k}\,, \tag{11.1}$$

and the (small) parameter $\alpha$ is the relativistic parameter (the inverse speed of light). So the Lax matrix of the relativistic analog of $\mathrm{TL}_{m+1}$ is

$$T\mathcal{F}^{-1} \quad \text{or} \quad \mathcal{F}^{-1}T,$$

where $T$ is the matrix from (7.2). It will be convenint to consider also the matrices

$$\begin{aligned}
\mathcal{T}_1\left(b, a^{(1)}, \dots, a^{(m)}, \lambda\right) &= I + \alpha T\left(b, a^{(1)}, \dots, a^{(m)}, \lambda\right) \cdot \mathcal{F}^{-1}(\lambda) \\
&= \mathcal{L}\left(b, a^{(1)}, \dots, a^{(m)}, \lambda\right) \cdot \mathcal{F}^{-1}(\lambda)\,,
\end{aligned} \tag{11.2}$$

$$\begin{aligned}
\mathcal{T}_2\left(b, a^{(1)}, \dots, a^{(m)}, \lambda\right) &= I + \alpha \mathcal{F}^{-1}(\lambda) \cdot T\left(b, a^{(1)}, \dots, a^{(m)}, \lambda\right) \\
&= \mathcal{F}^{-1}(\lambda) \cdot \mathcal{L}\left(b, a^{(1)}, \dots, a^{(m)}, \lambda\right)\,,
\end{aligned} \tag{11.3}$$

where

$$\begin{aligned}
\mathcal{L}\left(b, a^{(1)}, \dots, a^{(m)}, \lambda\right) &= \mathcal{F}(\lambda) + \alpha T\left(b, a^{(1)}, \dots, a^{(m)}, \lambda\right) \\
&= \sum_{k=1}^{N} (1 + \alpha b_k) E_{kk} + \alpha \sum_{j=1}^{m} \lambda^{-j} \sum_{k=1}^{N} a_k^{(j)} E_{k,k+j}.
\end{aligned} \tag{11.4}$$

The arising hierarchy, as well as its simplest (“minus first”) flow will be denoted $\mathrm{RTL}_{m+1}^{(-)}(\alpha)$. We shall use the notation

$$\mathcal{RT}_{m+1}^{(-)} = \mathbb{R}^{(m+1)N}\left(b, a^{(1)}, \dots, a^{(m)}\right) \tag{11.5}$$

for the phase variables of this hierarchy.

**Proposition 11.1** [6] *Consider the matrix differential equation*

$$\dot{T} = T\mathcal{B}_2 - \mathcal{B}_1 T, \tag{11.6}$$

*with the auxiliary matrices*

$$\mathcal{B}_1 = \sum_{k=1}^{N} \frac{b_k}{1+\alpha b_k} E_{kk} + \lambda \sum_{k=1}^{N} \frac{1}{1+\alpha b_k} E_{k+1,k}, \tag{11.7}$$

$$\mathcal{B}_2 = \sum_{k=1}^{N} \frac{b_k}{1+\alpha b_k} E_{kk} + \lambda \sum_{k=1}^{N} \frac{1}{1+\alpha b_{k+1}} E_{k+1,k}. \tag{11.8}$$

*The matrices $\mathcal{B}_i$ satisfy the relation*

$$\mathcal{F}\mathcal{B}_2 - \mathcal{B}_1\mathcal{F} = 0, \tag{11.9}$$

*which assures that the equation* (11.6) *is equivalent to either of the following two Lax equations:*

$$\dot{\mathcal{T}}_i = [\mathcal{T}_i, \mathcal{B}_i], \qquad i = 1, 2. \tag{11.10}$$

*These matrix differential equations are equivalent to the following system:*

$$\begin{cases} \dot{b}_k = \dfrac{a_k^{(1)}}{1+\alpha b_{k+1}} - \dfrac{a_{k-1}^{(1)}}{1+\alpha b_{k-1}}, \\ \dot{a}_k^{(j)} = a_k^{(j)} \left( \dfrac{b_{k+j}}{1+\alpha b_{k+j}} - \dfrac{b_k}{1+\alpha b_k} \right) + \left( \dfrac{a_k^{(j+1)}}{1+\alpha b_{k+j+1}} - \dfrac{a_{k-1}^{(j+1)}}{1+\alpha b_{k-1}} \right), \\ \dot{a}_k^{(m)} = a_k^{(m)} \left( \dfrac{b_{k+m}}{1+\alpha b_{k+m}} - \dfrac{b_k}{1+\alpha b_k} \right), \qquad 1 \le j \le m-1. \end{cases} \tag{11.11}$$

In [6] this system was called "the shadow relativistic lattice KP". Obviously, the system (11.11) is a one-parameter perturbation of (7.6). In order to define the hierarchy attached to the flow $\mathrm{RTL}_{m+1}^{(-)}(\alpha)$ and to find its Hamiltonian structure, we shall use the $r$-matrix theory.

## 12 Linear $r$-matrix structure for $\mathrm{RTL}_{m+1}^{(-)}(\alpha)$

Let us consider the generalized $r$-matrix bracket $\mathrm{PB}_1(\mathrm{R};\mathcal{F})$ introduced in Sect. 3, with the standard operator

$$\mathrm{R} = \pi_+ - \pi_-. \tag{12.1}$$

First of all, we have to be sure that this is indeed a Poisson bracket. Notice that the matrix $\mathcal{F}$ is invertible. Therefore the constructions of Proposition 3.3 are applicable.

**Lemma 12.1** *Let the linear operators $\mathrm{R}_{1,2}$ on $\boldsymbol{g}$ be defined by the formulas*

$$\mathrm{R}_1(X) = \mathcal{F} \cdot \mathrm{R}\left(\mathcal{F}^{-1}X\right), \qquad \mathrm{R}_2(X) = \mathrm{R}\left(X\mathcal{F}^{-1}\right) \cdot \mathcal{F}. \tag{12.2}$$

*Then these operators satisfy the modified Yang–Baxter equation.*

**Proof.** The proofs for $\mathrm{R}_1$ and $\mathrm{R}_2$ are similar, therefore we restrict ourselves to the $\mathrm{R}_1$ case. We prove that this operator can be represented as

$$\mathrm{R}_1 = \mathrm{P}_{(+)} - \mathrm{P}_{(-)}, \tag{12.3}$$

where $\mathrm{P}_{(\pm)}$ denote the projections from $\boldsymbol{g}$ to the corresponding subspaces $\boldsymbol{g}_{(\pm)}$ in a certain decomposition of $\boldsymbol{g}$ (as a linear space) into a direct sum

$$\boldsymbol{g} = \boldsymbol{g}_{(+)} \oplus \boldsymbol{g}_{(-)}, \tag{12.4}$$

each one of the subspaces $\boldsymbol{g}_{(\pm)}$ being also a subalgebra and a Lie subalgebra of $\boldsymbol{g}$. Then the statement of the Lemma is assured by the general construction of Theorem 6.1. To prove the above claim, set

$$\boldsymbol{g}_{(+)} = \mathcal{F}\boldsymbol{g}_+, \qquad \boldsymbol{g}_{(-)} = \mathcal{F}\boldsymbol{g}_-,$$

and

$$\mathrm{P}_{(+)}(X) = \mathcal{F}\pi_+\left(\mathcal{F}^{-1}X\right), \qquad \mathrm{P}_{(-)}(X) = \mathcal{F}\pi_-\left(\mathcal{F}^{-1}X\right).$$

Obviously, from these definitions there follows the direct decomposition (12.4), the representation (12.3), as well as the following statements:

$$\mathrm{P}_{(+)}(X) + \mathrm{P}_{(-)}(X) = X, \qquad \mathrm{P}_{(+)}(X) \in \boldsymbol{g}_{(+)}, \qquad \mathrm{P}_{(-)}(X) \in \boldsymbol{g}_{(-)}.$$

It remains to show that $\boldsymbol{g}_{(\pm)}$ are subalgebras. But this follows from the obvious relations

$$X\mathcal{F}Y \in \boldsymbol{g}_+ \quad \forall\, X, Y \in \boldsymbol{g}_+ \quad \Rightarrow \quad \mathcal{F}X \cdot \mathcal{F}Y \in \boldsymbol{g}_{(+)} \quad \forall\, \mathcal{F}X, \mathcal{F}Y \in \boldsymbol{g}_{(+)},$$

and

$$X\mathcal{F}Y \in \boldsymbol{g}_- \quad \forall\, X, Y \in \boldsymbol{g}_- \quad \Rightarrow \quad \mathcal{F}X \cdot \mathcal{F}Y \in \boldsymbol{g}_{(-)} \quad \forall\, \mathcal{F}X, \mathcal{F}Y \in \boldsymbol{g}_{(-)}.$$

The Lemma is proved. ∎

**Remark.** It is important to notice that the operators $\mathrm{R}_1$ and $\mathrm{R}_2$ correspond to *different* splittings (12.4). In both cases one has $\boldsymbol{g}_{(+)} = \boldsymbol{g}_+ = \boldsymbol{g}_{\geq 0}$, but $\boldsymbol{g}_{(-)}$ are different subalgebras of $\boldsymbol{g}_{\leq 0}$ for the two cases. Let us give also somewhat more explicit formulas for the operators $\mathrm{R}_{1,2}$. We use the following notation for the negative part of $X$:

$$\pi_-(X) = \sum_{\ell>0} \lambda^{-\ell} \sum_{k=1}^{N} x_k^{(-\ell)} E_{k,k+\ell}. \tag{12.5}$$

Then for the operators $\mathrm{R}_{1,2} = \mathrm{P}_{(+)} - \mathrm{P}_{(-)}$ we have:

$$\mathrm{P}_{(+)}(X) = \pi_+(X) + \sigma_{1,2}(X), \qquad \mathrm{P}_{(-)}(X) = \pi_-(X) - \sigma_{1,2}(X), \tag{12.6}$$

respectively, where the operators $\sigma_{1,2} : \boldsymbol{g} \mapsto \boldsymbol{g}_0$ act according to the formula

$$\sigma_1(X) = \sum_{\ell>0} \alpha^{\ell} \sum_{k=1}^{N} x_k^{(-\ell)} E_{k+\ell,k+\ell}, \qquad \sigma_2(X) = \sum_{\ell>0} \alpha^{\ell} \sum_{k=1}^{N} x_k^{(-\ell)} E_{kk}. \tag{12.7}$$

In words: the operators $\sigma_{1,2}$ assign the weights $\alpha^{\ell}$ to the elements of $\boldsymbol{g}_{-\ell}$, and shift all entries of $\pi_-(X)$ to the diagonal positions: $\sigma_1$ along the rows where they stand, and $\sigma_2$ along the columns where they stand. In this form the operators $\sigma_{1,2}$ appeared in [10, 7] (in the latter reference – under the name "relativistic symbol").

**Theorem 12.2** *a) The set* $\mathbf{T}_{m+1}$ *is a Poisson submanifold in* $\boldsymbol{g}$ *equipped with the linear* $r$*-matrix bracket* $\mathrm{PB}_1(\mathrm{R};\mathcal{F})$*.*

*b) Let* $\varphi$ *be an* Ad*-invariant function on* $\boldsymbol{g}$*, and let* $\psi(T) = \varphi\left(T\mathcal{F}^{-1}\right) = \varphi\left(\mathcal{F}^{-1}T\right)$*. Then the Hamiltonian equation on* $\boldsymbol{g}$ *with respect to* $\mathrm{PB}_1(\mathrm{R},\mathcal{F})$ *with the Hamilton function* $\psi(T)$ *reads:*

$$\dot{T} = T\mathcal{C}_2 - \mathcal{C}_1 T, \tag{12.8}$$

*where*

$$\mathcal{C}_1 = \frac{1}{2}\mathrm{R}_1\left(\nabla\varphi\left(T\mathcal{F}^{-1}\right)\right) = \frac{1}{2}\mathcal{F}\cdot\mathrm{R}\left(\mathcal{F}^{-1}\nabla\varphi\left(T\mathcal{F}^{-1}\right)\right), \tag{12.9}$$

$$\mathcal{C}_2 = \frac{1}{2}\mathrm{R}_2\left(\nabla\varphi\left(\mathcal{F}^{-1}T\right)\right) = \frac{1}{2}\mathrm{R}\left(\nabla\varphi\left(\mathcal{F}^{-1}T\right)\mathcal{F}^{-1}\right)\cdot\mathcal{F}; \tag{12.10}$$

*this equation may be properly restricted to* $\mathbf{T}_{m+1}$*. The evolution of the matrices* $\mathcal{T}_1 = I + \alpha T\mathcal{F}^{-1}$ *and* $\mathcal{T}_2 = I + \alpha\mathcal{F}^{-1}T$ *is described by the usual Lax equations*

$$\dot{\mathcal{T}}_1 = [\mathcal{T}_1, \mathcal{C}_1], \qquad \dot{\mathcal{T}}_2 = [\mathcal{T}_2, \mathcal{C}_2]. \tag{12.11}$$

*c) For two* Ad*-invariant functions* $\varphi_1$*,* $\varphi_2$ *on* $\boldsymbol{g}$ *the functions* $\psi_1(T) = \varphi_1\left(T\mathcal{F}^{-1}\right) = \varphi_1\left(\mathcal{F}^{-1}T\right)$ *and* $\psi_2(T) = \varphi_2\left(T\mathcal{F}^{-1}\right) = \varphi_2\left(\mathcal{F}^{-1}T\right)$ *are in involution with respect to* $\mathrm{PB}_1(\mathrm{R},\mathcal{F})$*.*

**Proof** follows closely the arguments used in Theorem 8.1. We denote by $\{\cdot,\cdot\}_1$ the Poisson bracket $\mathrm{PB}_1(\mathrm{R};\mathcal{F})$ on $\boldsymbol{g}$. For an arbitrary Hamilton function $\psi$ on $\boldsymbol{g}$, the value of the Hamiltonian vector field $\{\psi,\cdot\}_1$ in the point $T\in\boldsymbol{g}$ is given by:

$$2\{\psi,T\}_1 = T\mathrm{R}(\nabla\psi)\mathcal{F} - \mathcal{F}\mathrm{R}(\nabla\psi)T + \mathrm{R}^*\left(T\nabla\psi\mathcal{F} - \mathcal{F}\nabla\psi T\right).$$

This may be represented in the following two alternative forms:

$$\{\psi,T\}_1 = T\mathrm{P}_{\geq 0}(\nabla\psi)\mathcal{F} - \mathcal{F}\mathrm{P}_{\geq 0}(\nabla\psi)T - \mathrm{P}_{>0}\left(T\nabla\psi\mathcal{F} - \mathcal{F}\nabla\psi T\right), \tag{12.12}$$

and

$$\{\psi,T\}_1 = -T\mathrm{P}_{<0}(\nabla\psi)\mathcal{F} + \mathcal{F}\mathrm{P}_{<0}(\nabla\psi)T + \mathrm{P}_{\leq 0}\left(T\nabla\psi\mathcal{F} - \mathcal{F}\nabla\psi T\right). \tag{12.13}$$

Now take into account that $T\in\bigoplus_{j=-m}^{1}\boldsymbol{g}_j$ and $\mathcal{F}\in\boldsymbol{g}_0\oplus\boldsymbol{g}_1$. Then, according to (12.12), the value of $\{\psi,T\}_1$ belongs to $\sum_{j\geq -m}\boldsymbol{g}_j$. From the first glance, from (12.13) one can derive only that $\{\psi,T\}_1$ belongs to $\sum_{j\leq 1}\boldsymbol{g}_j$, but, rewriting (12.13) as

$$\alpha\{\psi,T\}_1 = -\mathcal{L}\mathrm{P}_{<0}(\nabla\psi)\mathcal{F} + \mathcal{F}\mathrm{P}_{<0}(\nabla\psi)\mathcal{L} + \mathrm{P}_{\leq 0}\left(\mathcal{L}\nabla\psi\mathcal{F} - \mathcal{F}\nabla\psi\mathcal{L}\right),$$

we see that actually the stronger inclusion $\{\psi,T\}_1\in\sum_{j\leq 0}\boldsymbol{g}_j$ holds. Hence, the value of the vector field under consideration belongs to $\bigoplus_{j=-m}^{0}\boldsymbol{g}_j$, which means that $\mathbf{T}_{m+1}$ is a Poisson submanifold in $\boldsymbol{g}$. Other statements of the Theorem follow from the general constructions of Sect. 3. ■

In the formulation and in the proof of the next Theorem we use the conventions (9.8).

**Theorem 12.3** *The bracket induced on* $\mathbf{T}_{m+1}$ *by* $\mathrm{PB}_1(\mathrm{R},\mathcal{F})$ *has the following coordinate representation:*

$$\{b_k, b_{k+1}\}_1 = \alpha a_k^{(1)}, \tag{12.14}$$

$$\left\{b_k, a_k^{(j)}\right\}_1 = -a_k^{(j)}, \qquad \left\{a_k^{(j)}, b_{k+j}\right\}_1 = -a_k^{(j)}, \tag{12.15}$$

$$\left\{b_k, a_{k+1}^{(j)}\right\}_1 = \alpha a_k^{(j+1)}, \qquad \left\{a_k^{(j)}, b_{k+j+1}\right\}_1 = \alpha a_k^{(j+1)}, \tag{12.16}$$

$$\left\{a_k^{(i)}, a_{k+i}^{(j)}\right\}_1 = -a_k^{(i+j)}, \tag{12.17}$$

$$\left\{a_k^{(i)}, a_{k+i+1}^{(j)}\right\}_1 = \alpha a_k^{(i+j+1)}. \tag{12.18}$$

**Proof.** To calculate the induced bracket, we denote

$$\varphi(T) = a_k^{(i)}, \qquad \psi(T) = a_{k+\ell}^{(j)}, \tag{12.19}$$

so that

$$\nabla\varphi = \mathrm{R}(\nabla\varphi) = \lambda^i E_{k+i,k}, \qquad \nabla\psi = \mathrm{R}(\nabla\psi) = \lambda^j E_{k+\ell+j,k+\ell}. \tag{12.20}$$

We find the following value for the Poisson bracket $\{\varphi,\psi\}_1 = \left\{a_k^{(i)}, a_{k+\ell}^{(j)}\right\}_1$:

$$\begin{aligned}
&\Big\langle T, \nabla\varphi\cdot\mathcal{F}\cdot\nabla\psi - \nabla\psi\cdot\mathcal{F}\cdot\nabla\varphi\Big\rangle\\
&\quad = \Big\langle T, \lambda^{i+j}E_{k+i,k}\cdot\Big(I - \alpha\lambda\sum_{n=1}^{N}E_{n+1,n}\Big)\cdot E_{k+\ell+j,k+\ell}\Big\rangle\\
&\quad -\Big\langle T, \lambda^{i+j}E_{k+\ell+j,k+\ell}\cdot\Big(I - \alpha\lambda\sum_{n=1}^{N}E_{n+1,n}\Big)\cdot E_{k+i,k}\Big\rangle\\
&\quad = \Big\langle T, \lambda^{i+j}(\delta_{\ell,-j}E_{k+i,k+\ell} - \delta_{\ell,i}E_{k+\ell+j,k})\\
&\quad -\alpha\lambda^{i+j+1}(\delta_{\ell,-j-1}E_{k+i,k+\ell} - \delta_{\ell,i+1}E_{k+\ell+j,k})\Big\rangle\\
&\quad = a_{k+\ell}^{(i+j)}\delta_{\ell,-j} - a_k^{(i+j)}\delta_{\ell,i} - \alpha a_{k+\ell}^{(i+j+1)}\delta_{\ell,-j-1} + \alpha a_k^{(i+j+1)}\delta_{\ell,i+1}.
\end{aligned}$$

This proves the Theorem. ∎

The Theorems above deliver a Hamiltonian interpretation of the flow (11.11), as well as an explanation of the Lax representations of this flow given in Proposition 11.1. Indeed, as a Corollary be find the following statement.

**Proposition 12.4** [5, 6] *The flow (11.11) on $\mathcal{RT}_{m+1}^{(-)}$ is Hamiltonian with respect to the linear bracket (12.14)–(12.18), with the Hamilton function*

$$\mathrm{H}_0 = -\alpha^{-2} \sum_{k=1}^{N} \log(1+\alpha b_k). \tag{12.21}$$

Of course, this function is singular in $\alpha$, but this can be repaired by adding the following Casimir function:

$$\alpha^{-1}\mathrm{H}_1 = \alpha^{-1} \sum_{k=1}^{N} b_k + \sum_{k=1}^{N} a_k^{(1)}; \tag{12.22}$$

obviously, the resulting Hamilton function is regular in $\alpha$, and, moreover, has an asymtotics

$$\mathrm{H}_0 + \alpha^{-1}\mathrm{H}_1 = \frac{1}{2} \sum_{k=1}^{N} b_k^2 + \sum_{k=1}^{N} a_k^{(1)} + O(\alpha).$$

The integrals of motion (12.21), (12.22) may be represented as the following functions of $T$:

$$\mathrm{H}_0 = \varphi\left(\mathcal{F}^{-1}T\right) = \varphi\left(T\mathcal{F}^{-1}\right) = -\alpha^{-2}\left(\log\det(\mathcal{T}_{1,2})\right)_0, \qquad \mathrm{H}_1 = \left(\mathrm{tr}(\mathcal{T}_{1,2})\right)_0. \tag{12.23}$$

For the Hamilton function $\mathrm{H}_0$ in (12.23) we find:

$$\mathcal{F}^{-1}\nabla\varphi\left(T\mathcal{F}^{-1}\right) = \nabla\varphi\left(\mathcal{F}^{-1}T\right)\mathcal{F}^{-1} = -\alpha^{-1}(\mathcal{F}+\alpha T)^{-1} = -\alpha^{-1}\mathcal{L}^{-1}.$$

This explains the expressions (11.7), (11.8) for the auxiliary matrices $\mathcal{B}_{1,2}$: indeed, up to adding a matrix proportional to $I$,

$$\mathcal{B}_1 = -\alpha^{-1}\mathcal{F}\cdot\pi_+\left(\mathcal{L}^{-1}\right), \qquad \mathcal{B}_2 = -\alpha^{-1}\pi_+\left(\mathcal{L}^{-1}\right)\cdot\mathcal{F},$$

in agreement with (12.9), (12.10). The formulas (12.8), (12.9), (12.10) define various members of the hierarchy $\mathrm{RTL}_{m+1}^{(-)}(\alpha)$. The equations of motion of all other members, except (11.11), are much more long and complicated. The reader should verify that this is the case already for the Hamilton function $\psi(T) = \varphi\left(\mathcal{F}^{-1}T\right)$ such that $\nabla\varphi(T) = T$, which corresponds to the “first” flow of the hierarchy, as opposed to the “minus first” flow $\mathrm{RTL}_{m+1}^{(-)}(\alpha)$. As an example, we give here the formulas of this “first” flow for $m = 2$:

$$\begin{aligned}
\dot{b}_k &= (1+\alpha b_k)\left(a_k^{(1)} - a_{k-1}^{(1)} + \alpha a_k^{(2)} - \alpha a_{k-2}^{(2)}\right), \\
\dot{a}_k^{(1)} &= a_k^{(1)}\left(b_{k+1} - b_k + \alpha a_{k+1}^{(1)} - \alpha a_{k-1}^{(1)} + \alpha^2 a_{k+1}^{(2)} + \alpha^2 a_k^{(2)} - \alpha^2 a_{k-1}^{(2)} - \alpha^2 a_{k-2}^{(2)}\right) \\
&\quad + a_k^{(2)}\left(1+\alpha b_{k+1}\right) - a_{k-1}^{(2)}\left(1+\alpha b_k\right), \\
\dot{a}_k^{(2)} &= a_k^{(2)}\left(b_{k+2} - b_k + \alpha a_{k+2}^{(1)} - \alpha a_{k-1}^{(1)} + \alpha^2 a_{k+2}^{(2)} + \alpha^2 a_{k+1}^{(2)} - \alpha^2 a_{k-1}^{(2)} - \alpha^2 a_{k-2}^{(2)}\right).
\end{aligned}$$

# 13 Quadratic $r$-matrix structure for $\mathrm{RTL}_{m+1}^{(-)}(\alpha)$

Now we prove a very remarkable Theorem. According to it, the quadratic invariant Poisson bracket of the usual Toda lattice is at the same time invariant with respect to the flows of the relativistic Toda hierarchy.

**Theorem 13.1** *Supply $\boldsymbol{g}$ with the quadratic $r$-matrix bracket* $\mathrm{PB}_2(\mathrm{A}_1,\mathrm{A}_2,\mathrm{S})$ *defined by the operators (6.8). Let $\varphi$ be an* Ad*-invariant function on $\boldsymbol{g}$, and consider the Hamiltonian flow on $\boldsymbol{g}$ with the Hamilton function*

$$\psi(T)=\varphi\left(T\mathcal{F}^{-1}\right)=\varphi\left(\mathcal{F}^{-1}T\right). \tag{13.1}$$

*The equations of motion of this flow read:*

$$\dot{T}=T\mathcal{C}_2-\mathcal{C}_1T, \tag{13.2}$$

*where*

$$\mathcal{C}_1=\frac{1}{2}\mathrm{R}_1\left(d\varphi\left(T\mathcal{F}^{-1}\right)\right)=\frac{1}{2}\mathcal{F}\cdot\mathrm{R}\left(\mathcal{F}^{-1}d\varphi\left(T\mathcal{F}^{-1}\right)\right), \tag{13.3}$$

$$\mathcal{C}_2=\frac{1}{2}\mathrm{R}_2\left(d\varphi\left(\mathcal{F}^{-1}T\right)\right)=\frac{1}{2}\mathrm{R}\left(d\varphi\left(\mathcal{F}^{-1}T\right)\mathcal{F}^{-1}\right)\cdot\mathcal{F}. \tag{13.4}$$

**Proof.** For an arbitrary Hamilton function $\psi(T)$, the Hamiltonian equations of motion have the form (13.2) with the matrices

$$2\mathcal{C}_1=\mathrm{A}_2\left(T\nabla\psi(T)\right)+\mathrm{S}^*\left(\nabla\psi(T)T\right), \tag{13.5}$$

$$2\mathcal{C}_2=\mathrm{A}_1\left(\nabla\psi(T)T\right)+\mathrm{S}\left(T\nabla\psi(T)\right). \tag{13.6}$$

For the functions $\psi(T)=\varphi\left(T\mathcal{F}^{-1}\right)$ with Ad-invariant $\varphi$ we have, obviously:

$$\nabla\psi(T)=\mathcal{F}^{-1}\cdot\nabla\varphi\left(T\mathcal{F}^{-1}\right)=\nabla\varphi\left(\mathcal{F}^{-1}T\right)\cdot\mathcal{F}^{-1},$$

so that

$$T\nabla\psi(T)=d\varphi\left(T\mathcal{F}^{-1}\right),\qquad \nabla\psi(T)T=d\varphi\left(\mathcal{F}^{-1}T\right).$$

Hence, denoting

$$X=\mathcal{F}^{-1}\cdot d\varphi\left(T\mathcal{F}^{-1}\right)=d\varphi\left(\mathcal{F}^{-1}T\right)\cdot\mathcal{F}^{-1},$$

we can rewrite (13.5), (13.6) as

$$2\mathcal{C}_1=\mathrm{A}_2(\mathcal{F}X)+\mathrm{S}^*(X\mathcal{F}), \tag{13.7}$$

$$2\mathcal{C}_2=\mathrm{A}_1(X\mathcal{F})+\mathrm{S}(\mathcal{F}X). \tag{13.8}$$

We now prove the following two identities. For an arbitrary matrix $X\in\boldsymbol{g}$ let

$$\mathrm{P}_{-1}(X)=X_{-1}=\lambda^{-1}\sum_{k=1}^{N}x_k^{(-1)}E_{k,k+1}. \tag{13.9}$$

Then

$$\mathrm{A}_2(\mathcal{F}X)+\mathrm{S}^*(X\mathcal{F})=\mathcal{F}\mathrm{R}(X)-2\alpha x_N^{(-1)}I, \tag{13.10}$$

$$\mathrm{A}_1(X\mathcal{F})+\mathrm{S}(\mathcal{F}X)=\mathrm{R}(X)\mathcal{F}-2\alpha x_N^{(-1)}I. \tag{13.11}$$

This will, obviously, imply the statement of the Theorem. Since (13.10), (13.11) are proved analogously, we do this only for the first one of them. According to (6.8), and $\mathcal{F}=I-\alpha\mathcal{E}$, we have:

$$\begin{aligned}\mathrm{A}_2(\mathcal{F}X)+\mathrm{S}^*(X\mathcal{F})&=\mathrm{R}_0(\mathcal{F}X)+\mathrm{P}_0(X\mathcal{F})-\mathrm{W}(\mathcal{F}X)+\mathrm{W}(X\mathcal{F})\\ &=\mathrm{R}(X)-\alpha\left(\mathrm{R}_0(\mathcal{E}X)+\mathrm{P}_0(X\mathcal{E})-\mathrm{W}(\mathcal{E}X)+\mathrm{W}(X\mathcal{E})\right).\end{aligned} \tag{13.12}$$

The term proportional to $\alpha$ is equal to:

$$\begin{aligned}&\mathrm{P}_{>0}(\mathcal{E}X)-\mathrm{P}_{<0}(\mathcal{E}X)+\mathrm{P}_0(X\mathcal{E})+\mathrm{W}\left([X_{-1},\mathcal{E}]\right)\\ &\qquad=\mathcal{E}\mathrm{P}_{\geq 0}(X)-\mathcal{E}\mathrm{P}_{<-1}(X)+X_{-1}\mathcal{E}+\mathrm{W}\left([X_{-1},\mathcal{E}]\right)\\ &\qquad=\mathcal{E}\mathrm{R}(X)+\mathcal{E}X_{-1}+X_{-1}\mathcal{E}+\mathrm{W}\left([X_{-1},\mathcal{E}]\right).\end{aligned}$$

Now a direct calculation shows that for an arbitrary matrix $X_{-1}\in\boldsymbol{g}_{-1}$ there holds:

$$\mathcal{E}X_{-1}+X_{-1}\mathcal{E}+\mathrm{W}\left([X_{-1},\mathcal{E}]\right)=2x_N^{(-1)}I,$$

and the proof is finished. (Notice that in the open-end case the left-hand side of the last equation simply vanishes). ■

The Theorem just proved yields that the matrix differential equations (13.2) can be properly restricted to an arbitrary Poisson submanifold of the quadratic bracket $\mathrm{PB}_2(\mathrm{A}_1,\mathrm{A}_2,\mathrm{S})$, and the resulting equations are Hamiltonian with respect to the restriction of this bracket. In particular, this holds for the Poisson submanifold $\mathbf{T}_{m+1}$, and we find the following statement.

**Theorem 13.2** *The hierarchy* $\mathrm{RTL}_{m+1}^{(-)}(\alpha)$ *is Hamiltonian with respect to the quadratic invariant Poisson bracket (9.9)–(9.15) of the non-relativistic hierarchy* $\mathrm{TL}_{m+1}$*. This bracket is compatible with the linear bracket of Theorem 12.3.*

In particular:

**Proposition 13.3** *The system (11.11) is Hamiltonian with respect to the bracket* $\{\cdot,\cdot\}_2$ *on* $\mathcal{RT}_{m+1}^{(-)}$ *given by the formulas (9.9)–(9.15). The corresponding Hamilton function is equal to*

$$-\alpha\mathrm{H}_0=\alpha^{-1}\sum_{k=1}^{N}\log(1+\alpha b_k).$$

# 14 Introducing the gauge transformed hierarchy

The Lax equations of $\mathrm{RTL}_{m+1}^{(-)}(\alpha)$ are of a somewhat nonstandard type. We now find a gauge transformation bringing the Lax equations into the standard form. To this end, we start with the matrices $\mathcal{F}(\lambda)$, $\mathcal{L}(a,b,\lambda)$ for the $\mathrm{RTL}_{m+1}^{(-)}(-\alpha)$ hierarchy, i.e. we change $\alpha$ to $-\alpha$ in the formulas (11.1), (11.4). As it follows from Theorems 12.2, 13.1, we have the following Lax representations for the flows of the $\mathrm{RTL}_{m+1}^{(-)}(-\alpha)$ hierarchy:

$$\dot{\mathcal{T}}_i = [\mathcal{T}_i, \mathcal{B}_i] = [\mathcal{A}_i, \mathcal{T}_i], \qquad i = 1, 2, \tag{14.1}$$

where

$$\mathcal{B}_1 = \mathcal{F}\cdot\pi_+\left(\mathcal{F}^{-1}f\left(\mathcal{L}\mathcal{F}^{-1}\right)\right), \qquad \mathcal{A}_1 = \mathcal{F}\cdot\pi_-\left(\mathcal{F}^{-1}f\left(\mathcal{L}\mathcal{F}^{-1}\right)\right), \tag{14.2}$$

$$\mathcal{B}_2 = \pi_+\left(f\left(\mathcal{F}^{-1}\mathcal{L}\right)\mathcal{F}^{-1}\right)\cdot\mathcal{F}, \qquad \mathcal{A}_2 = \pi_-\left(f\left(\mathcal{F}^{-1}\mathcal{L}\right)\mathcal{F}^{-1}\right)\cdot\mathcal{F}, \tag{14.3}$$

with an Ad-covariant function $f : \boldsymbol{g} \mapsto \boldsymbol{g}$ (we have $f(\mathcal{T}) = \nabla\varphi((\mathcal{T}-I)/\alpha)$ and $f(\mathcal{T}) = d\varphi((\mathcal{T}-I)/\alpha)$ for the Hamiltonian flows with the Hamilton function $\varphi\big(T\mathcal{F}^{-1}\big) = \varphi\big(\mathcal{F}^{-1}T\big)$ in the linear and the quadratic bracket, respectively).

Now introduce two arbitrary diagonal matrices

$$\Omega_1 = \mathrm{diag}(\omega_1, \omega_2, \ldots, \omega_N), \qquad \Omega_2 = \mathrm{diag}(\omega_1', \omega_2', \ldots, \omega_N'). \tag{14.4}$$

It will be convenient for us to denote

$$\Omega_i\cdot\mathcal{T}_i\cdot\Omega_i^{-1} = T_i^{-1}, \qquad i = 1, 2. \tag{14.5}$$

The gauge transformed Lax equations for the matrices $T_i$ are easily derived from (14.1) and (14.5):

$$\dot{T}_i = [T_i, B_i] = [A_i, T_i], \qquad i = 1, 2, \tag{14.6}$$

where

$$B_i = \Omega_i\mathcal{B}_i\Omega_i^{-1} - \dot{\Omega}_i\Omega_i^{-1}, \qquad A_i = \Omega_i\mathcal{A}_i\Omega_i^{-1} + \dot{\Omega}_i\Omega_i^{-1}. \tag{14.7}$$

¿From the expressions (14.2), (14.3) one sees immediately that

$$\mathcal{B}_i \in \boldsymbol{g}_{\geq 0}, \qquad \mathcal{A}_i \in \boldsymbol{g}_{\leq 0},$$

and the same holds for the matrices $B_i$, $A_i$:

$$B_i \in \boldsymbol{g}_{\geq 0}, \qquad A_i \in \boldsymbol{g}_{\leq 0}. \tag{14.8}$$

Our aim is to assure that

$$B_i \in \boldsymbol{g}_{\geq 0} = \boldsymbol{g}_+, \qquad A_i \in \boldsymbol{g}_{<0} = \boldsymbol{g}_-. \tag{14.9}$$

To this end it is necessary only to kill the diagonal entries of $A_i$, i.e. to define the entries of the matrices $\Omega_i$ in such a way that

$$\dot{\Omega}_i\Omega_i^{-1} = -\mathrm{P}_0(\mathcal{A}_i). \tag{14.10}$$

So, all we need to do, is to determine the diagonal parts of the matrices $\mathcal{A}_i$.

**Lemma 14.1** *If*

$$\mathrm{P}_0(\mathcal{A}_1) = \mathrm{diag}(\mathbf{a}_1, \ldots, \mathbf{a}_N), \qquad \mathrm{P}_0(\mathcal{A}_2) = \mathrm{diag}(\mathbf{a}'_1, \ldots, \mathbf{a}'_N),$$

*then*

$$\mathbf{a}_k = C + \sum_{j=1}^{k-1} \frac{\alpha \dot{b}_j}{1 - \alpha b_j}, \qquad \mathbf{a}'_k = C + \sum_{j=1}^{k} \frac{\alpha \dot{b}_j}{1 - \alpha b_j} \ ,$$

*where $C$ is some constant (independent on $k$).*

**Proof.** Consider the matrix equations

$$\mathcal{A}_1 \mathcal{F} - \mathcal{F} \mathcal{A}_2 = 0, \qquad \dot{\mathcal{L}} = \mathcal{A}_1 \mathcal{L} - \mathcal{L} \mathcal{A}_2. \tag{14.11}$$

Due to $\mathcal{A}_i \in \boldsymbol{g}_{\leq 0}$ and the definition of $\mathcal{F}$, we find from the first equation in (14.11):

$$\mathbf{a}'_k = \mathbf{a}_{k+1}. \tag{14.12}$$

On the other hand, since $\mathcal{L} \in \boldsymbol{g}_{\leq 0}$, we find, considering the $\boldsymbol{g}_0$ part of the second equation in (14.11):

$$-\alpha \dot{b}_k = (1 - \alpha b_k)(\mathbf{a}_k - \mathbf{a}'_k) \quad \Rightarrow \quad \mathbf{a}_{k+1} - \mathbf{a}_k = \frac{\alpha \dot{b}_k}{1 - \alpha b_k}.$$

This implies the statement of the Lemma. ∎

So, let us define the entries of the diagonal matrices $\Omega_i$, $i = 1, 2$, by the following formulas:

$$\omega_k = \delta^{k-1} \prod_{j=1}^{k-1} (1 - \alpha b_j), \qquad \omega'_k = \delta^{k-1} \prod_{j=1}^{k} (1 - \alpha b_j), \tag{14.13}$$

where we set in the periodic case

$$\delta^N = \prod_{j=1}^{N} (1 - \alpha b_j)^{-1},$$

and in the open-end case simply $\delta = 1$. Of course, in the periodic case $\delta$ is a spectral invariant of the Lax matrix, and therefore an integral of motion of an arbitrary flow of the hierarchy.

**Theorem 14.2** *Fix an* Ad*-covariant function $f : \boldsymbol{g} \mapsto \boldsymbol{g}$ and consider the corresponding flow of the hierarchy* $\mathrm{RTL}_{m+1}^{(-)}(-\alpha)$*. Then the evolution of the matrices $T_i$ is described by the standard Lax equations*

$$\dot{T}_i = \left[T_i, \pi_+ \left(f\left(T_i^{-1}\right)\right)\right] = \left[\pi_- \left(f\left(T_i^{-1}\right)\right), T_i\right], \qquad i = 1, 2.$$

**Proof.** From Lemma 14.1 and the definitions (14.13) we find:

$$\mathbf{a}_k = C - \frac{\dot\omega_k}{\omega_k}, \qquad \mathbf{a}'_k = C - \frac{\dot\omega'_k}{\omega'_k},$$

which means that

$$\mathrm{P}_0(\mathcal{A}_i) + \dot\Omega_i\Omega_i^{-1} = C\cdot I. \tag{14.14}$$

Therefore, renaming $A_i - C\cdot I$ by $A_i$, and $B_i + C\cdot I$ by $B_i$ (which does not affect the Lax equations (14.6)), we come to the desired property (14.9). Further, from (14.2), (14.3) we derive:

$$\mathcal{B}_i + \mathcal{A}_i = f(\mathcal{T}_i)$$

and it follows that

$$B_i + A_i = f\left(T_i^{-1}\right). \tag{14.15}$$

Therefore $B_i$ and $A_i$ are with necessity the $\pi_+$ and $\pi_-$ projections of $f(T_i^{-1})$, respectively. ∎

In particular, the flow $\mathrm{RTL}_{m+1}^{(-)}(-\alpha)$ is characterized, as we have seen, by $f(\mathcal{T}) = \alpha^{-1}\mathcal{T}^{-1}$. Therefore its gauge transformed Lax representation reads:

$$\dot T_i = \alpha^{-1}[T_i, \pi_+(T_i)].$$

In other words, the image of the flow $\mathrm{RTL}_{m+1}^{(-)}(-\alpha)$ in the gauge transformed hierarchy becomes the "first" flow.

# 15 Multi-field analog of the relativistic Toda lattice: the second construction

An easy calculation shows that (we still consider $\mathcal{L}$, $\mathcal{F}$ attached to the $\mathrm{RTL}_{m+1}^{(-)}(-\alpha)$ hierarchy):

$$\Omega_1\cdot\mathcal{F}(\lambda)\cdot\Omega_2^{-1} = L\left(d, c^{(1)},\ldots,c^{(m)},\delta\lambda\right),$$

$$\Omega_1\cdot\mathcal{L}\left(b, a^{(1)},\ldots,a^{(m)},\lambda\right)\cdot\Omega_2^{-1} = U\left(d, c^{(1)},\ldots,c^{(m)},\delta\lambda\right),$$

where

$$L\left(d, c^{(1)},\ldots,c^{(m)},\lambda\right) = \sum_{k=1}^{N}(1+\alpha d_k)E_{kk} + \alpha\lambda\sum_{k=1}^{N}E_{k+1,k}, \tag{15.1}$$

$$U\left(d, c^{(1)},\ldots,c^{(m)},\lambda\right) = I - \alpha\sum_{j=1}^{m}\lambda^{-j}\sum_{k=1}^{N}c_k^{(j)}E_{k,k+j}, \tag{15.2}$$

and the variables $d$, $c^{(j)}$ are defined by the formulas

$$\mathbf{B}(\alpha): \quad d_k = \frac{b_k}{1-\alpha b_k}, \qquad c_k^{(j)} = \frac{a_k^{(j)}}{\prod\limits_{i=0}^{j}(1-\alpha b_{k+i})}. \tag{15.3}$$

We want now to study the gauge transformed hierarchy in its own rights. The phase space of this second version of the generalized relativistic Toda lattice with $m+1$ fields, abbreviated $\mathrm{RTL}_{m+1}^{(+)}(\alpha)$, is, in the periodic case, the space

$$\mathcal{RT}_{m+1}^{(+)} = \mathbb{R}^{(m+1)N}\left(d, c^{(1)}, \ldots, c^{(m)}\right). \tag{15.4}$$

Here

$$d = (d_1, \ldots, d_N) \quad \text{and} \quad c^{(j)} = \left(c_1^{(j)}, \ldots, c_N^{(j)}\right) \quad (j = 1, \ldots, m)$$

are the $m+1$ fields. The Lax matrix map $(L, U^{-1}) : \mathcal{RT}_{m+1}^{(+)} \mapsto \boldsymbol{g} \otimes \boldsymbol{g}$ is defined by the formulas (15.1), (15.2). We set also

$$T_1 = LU^{-1}, \qquad T_2 = U^{-1}L. \tag{15.5}$$

**Proposition 15.1** *The Lax triads*

$$\dot{L} = LB_2 - B_1L, \qquad \dot{U} = UB_2 - B_1U \tag{15.6}$$

*with the Lax matrices (15.1), (15.2) and the auxiliary matrices*

$$B_1\left(d, c^{(1)}, \ldots, c^{(m)}, \lambda\right) = \sum_{k=1}^{N}\left(d_k + \alpha c_{k-1}^{(1)}\right)E_{kk} + \lambda\sum_{k=1}^{N}E_{k+1,k}, \tag{15.7}$$

$$B_2\left(d, c^{(1)}, \ldots, c^{(m)}, \lambda\right) = \sum_{k=1}^{N}\left(d_k + \alpha c_{k}^{(1)}\right)E_{kk} + \lambda\sum_{k=1}^{N}E_{k+1,k}, \tag{15.8}$$

*are equivalent to the following system of differential equations:*

$$\begin{cases} \dot{d}_k = (1+\alpha d_k)\left(c_k^{(1)} - c_{k-1}^{(1)}\right), \\ \dot{c}_k^{(j)} = c_k^{(j)}\left(d_{k+j} - d_k + \alpha c_{k+j}^{(1)} - \alpha c_{k-1}^{(1)}\right) + \left(c_k^{(j+1)} - c_{k-1}^{(j+1)}\right), \\ \dot{c}_k^{(m)} = c_k^{(m)}\left(d_{k+m} - d_k + \alpha c_{k+m}^{(1)} - \alpha c_{k-1}^{(1)}\right), \quad 1 \le j \le m-1. \end{cases} \tag{15.9}$$

**Proof** – an elementary matrix calculation. ■

Notice that

$$B_1 = \pi_+\left(\left(LU^{-1} - I\right)/\alpha\right), \qquad B_2 = \pi_+\left(\left(U^{-1}L - I\right)/\alpha\right). \tag{15.10}$$

So, the above Lax representation is indeed of a standard type.

One should compare the equations of motion (15.9) of $\mathrm{RTL}_{m+1}^{(+)}(\alpha)$ with the "first" flow of the $\mathrm{RTL}_{m+1}^{(-)}(\alpha)$ hierarchy (given for $m = 2$ at the end of Sect. 12).

Returning to the relation between the hierarchies $\mathrm{RTL}_{m+1}^{(+)}(\alpha)$ and $\mathrm{RTL}_{m+1}^{(-)}(-\alpha)$, we see that it is established via the change of variables

$$\mathbf{B}(\alpha): \quad \mathcal{RT}_{m+1}^{(-)}\left(b, a^{(1)}, \ldots, a^{(m)}\right) \mapsto \mathcal{RT}_{m+1}^{(+)}\left(d, c^{(1)}, \ldots, c^{(m)}\right)$$

given by the formulas (15.3). Clearly, this transformation is invertible, and its inverse is given by

$$\mathbf{B}^{-1}(\alpha) = \mathbf{B}(-\alpha): \quad b_k = \frac{d_k}{1+\alpha d_k}, \qquad a_k^{(j)} = \frac{c_k^{(j)}}{\prod_{i=0}^{j}(1+\alpha d_{k+i})}. \tag{15.11}$$

This transformation relates, as it follows from Theorem 14.2, the whole hierarchies. In particular, this holds for the simplest flows: the change of variables given by the formulas (15.3) brings the system $\mathrm{RTL}_{m+1}^{(-)}(-\alpha)$ (11.11) into the system $\mathrm{RTL}_{m+1}^{(+)}(\alpha)$ (15.9), and vice versa.

## 16 Quadratic $r$-matrix structure for $\mathrm{RTL}_{m+1}^{(+)}(\alpha)$

Consider the subset

$$\mathbf{RT}_{m+1} = (\mathcal{E} \oplus \boldsymbol{g}_0) \times \left(I \oplus \bigoplus_{j=1}^{m} \boldsymbol{g}_{-j}\right)^{-1}$$

of $\boldsymbol{g} \otimes \boldsymbol{g}$, consisting of pairs $(L, U^{-1})$ of matrices (15.1), (15.2). We show now that this subset is a Poisson submanifold, if $\boldsymbol{g} \otimes \boldsymbol{g}$ is equipped with a certain quadratic $r$-matrix bracket. Actually, this bracket was used already in [15, 16] to give an $r$-matrix interpretation of the usual (two-field) relativistic Toda lattice and of the Volterra lattice.

**Theorem 16.1** *a) Supply the algebra $\boldsymbol{g} \otimes \boldsymbol{g}$ with the bracket $\alpha\mathrm{PB}_2(\mathbf{A}_1, \mathbf{A}_2, \mathbf{S})$ defined by the operators*

$$\mathbf{A}_1 = \begin{pmatrix} \mathrm{A}_1 & -\mathrm{S} \\ \mathrm{S}^* & \mathrm{A}_1 \end{pmatrix}, \qquad \mathbf{A}_2 = \begin{pmatrix} \mathrm{A}_2 & -\mathrm{S}^* \\ \mathrm{S} & \mathrm{A}_2 \end{pmatrix}, \tag{16.1}$$

$$\mathbf{S} = \begin{pmatrix} \mathrm{S} & \mathrm{S} \\ \mathrm{S} & -\mathrm{S}^* \end{pmatrix}, \qquad \mathbf{S}^* = \begin{pmatrix} \mathrm{S}^* & \mathrm{S}^* \\ \mathrm{S}^* & -\mathrm{S} \end{pmatrix}, \tag{16.2}$$

*where the operators $\mathrm{A}_1$, $\mathrm{A}_2$, $\mathrm{S}$, $\mathrm{S}^*$ are as in (6.8). Then the set $\mathbf{RT}_{m+1}$ is a Poisson submanifold in $\boldsymbol{g} \otimes \boldsymbol{g}$.*

*b) Consider the monodromy maps $\mathbf{M}_{1,2} : \boldsymbol{g} \otimes \boldsymbol{g} \mapsto \boldsymbol{g}$*

$$\mathbf{M}_1 : (L, U^{-1}) \mapsto T_1 = LU^{-1}, \qquad \mathbf{M}_2 : (L, U^{-1}) \mapsto T_2 = U^{-1}L. \tag{16.3}$$

*Both maps $\mathbf{M}_{1,2}$ are Poisson, if the target space $\boldsymbol{g}$ is equipped with the Poisson bracket $\alpha\mathrm{PB}_2(\mathrm{A}_1, \mathrm{A}_2, \mathrm{S})$.*

*c) Let $\varphi$ be an* Ad*-invariant function on $\boldsymbol{g}$. Then the Hamiltonian equations of motion on $\boldsymbol{g}\otimes\boldsymbol{g}$ with the Hamilton function $\varphi\circ\mathbf{M}_{1,2}$ may be presented in the form of the "Lax triads"*

$$\dot{L}=LC_2-C_1L,\qquad \dot{U}=UC_2-C_1U, \tag{16.4}$$

*where*

$$C_1=\frac{1}{2}\mathrm{R}\left(d\varphi\left(LU^{-1}\right)\right),\qquad C_2=\frac{1}{2}\mathrm{R}\left(d\varphi\left(U^{-1}L\right)\right). \tag{16.5}$$

**Proof.** As usual, we have to prove that an arbitrary Hamiltonian vector field on

$$(\boldsymbol{g}\otimes\boldsymbol{g},\mathrm{PB}_2(\mathbf{A}_1,\mathbf{A}_2,\mathbf{S}))$$

is tangent to $\mathbf{RT}_{m+1}$. It is enough to consider separately vector fields of Hamilton functions $\phi(L)$ and $\varphi(U)$, depending only on the first, resp. second, factor in the tensor product, because an arbitrary Hamiltonian vector field is a linear combination of such ones. Further, the calculations are simplified, if we regard the values of these vector fields, parametrizing the second factors via $U$ instead of $U^{-1}$. With these conventions, the value of the vector field of the Hamilton function $\phi(L)$ in the point of $\boldsymbol{g}\otimes\boldsymbol{g}$ parametrized as $(L,U)$, is given by

$$2\alpha^{-1}\{\phi(L),L\}_{2\alpha}=L\cdot\mathrm{A}_1(\nabla\phi L)-\mathrm{A}_2(L\nabla\phi)\cdot L+L\cdot\mathrm{S}(L\nabla\phi)-\mathrm{S}^*(\nabla\phi L)\cdot L, \tag{16.6}$$

$$2\alpha^{-1}\{\phi(L),U\}_{2\alpha}=U\cdot\mathrm{S}^*(\nabla\phi L)-\mathrm{S}(L\nabla\phi)\cdot U+U\cdot\mathrm{S}(L\nabla\phi)-\mathrm{S}^*(\nabla\phi L)\cdot U. \tag{16.7}$$

Similarly, for the vector field of the Hamilton function $\varphi(U)$ in the point of $\boldsymbol{g}\otimes\boldsymbol{g}$ parametrized as $(L,U)$, is given by

$$2\alpha^{-1}\{\varphi(U),L\}_{2\alpha}=-L\cdot\mathrm{S}(\nabla\varphi U)+\mathrm{S}^*(U\nabla\varphi)\cdot L+L\cdot\mathrm{S}(U\nabla\varphi)-\mathrm{S}^*(\nabla\varphi U)\cdot L, \tag{16.8}$$

$$2\alpha^{-1}\{\varphi(U),U\}_{2\alpha}=-U\cdot\mathrm{A}_2(\nabla\varphi U)+\mathrm{A}_1(U\nabla\varphi)\cdot U+U\cdot\mathrm{S}(U\nabla\varphi)-\mathrm{S}^*(\nabla\varphi U)\cdot U. \tag{16.9}$$

Consider the first vector field $\{\phi(L),\cdot\}_{2\alpha}$. Its $L$-component (16.6) lies in $\boldsymbol{g}_0$: this is a particular case of Theorem 9.1. The value of its $U$-component (16.7) lies, obviously, in $\sum\limits_{j=0}^{m}\boldsymbol{g}_{-j}$, because the range of the operators S, $\mathrm{S}^*$ is $\boldsymbol{g}_0$. Moreover, it is easy to see that the $\boldsymbol{g}_0$-component of (16.7) vanishes. Henceforth, the vector fields $\{\phi(L),\cdot\}_{2\alpha}$ are tangent to $\mathbf{RT}_{m+1}$.

Let us turn now to the second vector field $\{\varphi(U),\cdot\}_{2\alpha}$. Its $L$-component, (16.8), may be rewritten as

$$\begin{aligned}2\alpha^{-1}\{\varphi(U),L\}_{2\alpha}&=LD+DL+\mathrm{W}(D)L-L\mathrm{W}(D)\\&\quad-LD'-D'L-\mathrm{W}(D')L+L\mathrm{W}(D'),\end{aligned} \tag{16.10}$$

where

$$D=\mathrm{P}_0(U\nabla\varphi),\qquad D'=\mathrm{P}_0(\nabla\varphi U).$$

The vanishing of the $\boldsymbol{g}_1$-component of this expression follows from (9.7). Finally, for the $U$-component (16.9) of the vector field under consideration, we have the following two equivalent expressions:

$$\begin{aligned}2\alpha^{-1}\{\varphi, U\}_{2\alpha} &= -2U\mathrm{P}_{>0}(\nabla\varphi U) + 2\mathrm{P}_{>0}(U\nabla\varphi)U \\ &\quad -U\mathrm{P}_0(\nabla\varphi U) + \mathrm{P}_0(U\nabla\varphi)U + U\mathrm{P}_0(U\nabla\varphi) - \mathrm{P}_0(\nabla\varphi U)U \\ &\quad +U\mathrm{W}(\nabla\varphi U) + \mathrm{W}(U\nabla\varphi)U - U\mathrm{W}(U\nabla\varphi) - \mathrm{W}(\nabla\varphi U)U,\end{aligned} \tag{16.11}$$

and

$$\begin{aligned}2\alpha^{-1}\{\varphi, U\}_{2\alpha} &= 2U\mathrm{P}_{<0}(\nabla\varphi U) - 2\mathrm{P}_{<0}(U\nabla\varphi)U \\ &\quad +U\mathrm{P}_0(\nabla\varphi U) - \mathrm{P}_0(U\nabla\varphi)U + U\mathrm{P}_0(U\nabla\varphi) - \mathrm{P}_0(\nabla\varphi U)U \\ &\quad +U\mathrm{W}(\nabla\varphi U) + \mathrm{W}(U\nabla\varphi)U - U\mathrm{W}(U\nabla\varphi) - \mathrm{W}(\nabla\varphi U)U.\end{aligned} \tag{16.12}$$

(as compared with the formulas (9.2) and (9.3), respectively, in our present formulas the overall signs in the first lines and in the first two terms on the second lines are opposite). The first expression above assures that for $U \in I \oplus \bigoplus_{j=-m}^{-1} \boldsymbol{g}_j$ this vector belongs to $\bigoplus_{j\ge -m} \boldsymbol{g}_j$. The second expression yields that this vector belongs also to $\bigoplus_{j\le 0} \boldsymbol{g}_j$; moreover, since the $\boldsymbol{g}_0$ component of $U$ is equal to $I$, it is easy to see that the $\boldsymbol{g}_0$ component of (16.12) vanishes, so that actually it belongs to $\bigoplus_{j\le -1} \boldsymbol{g}_j$. Hence, it belongs to $\bigoplus_{j=-m}^{-1} \boldsymbol{g}_j$. This finishes the proof of the first statement of the Theorem. The parts b) and c) follow from Theorem 5.1 with $n = 2$. ■

**Theorem 16.2** *The coordinate representation of the bracket induced by* $\alpha\mathrm{PB}_2(\mathbf{A}_1, \mathbf{A}_2, \mathbf{S})$ *on the submanifold* $\mathbf{RT}_{m+1}$*, is given by the following formulas: (in (16.15), (16.16) below we assume that* $i \le j$*):*

$$\left\{d_k, c_k^{(j)}\right\}_{2\alpha} = -c_k^{(j)}(1+\alpha d_k), \qquad \left\{c_k^{(j)}, d_{k+j}\right\}_{2\alpha} = -c_k^{(j)}(1+\alpha d_{k+j}), \tag{16.13}$$

$$\left\{c_k^{(i)}, c_{k+i}^{(j)}\right\}_{2\alpha} = -c_k^{(i+j)} - \alpha c_k^{(i)} c_{k+i}^{(j)}, \tag{16.14}$$

$$\left\{c_k^{(i)}, c_{k+\ell}^{(j)}\right\}_{2\alpha} = -\alpha c_k^{(i)} c_{k+\ell}^{(j)} + \alpha c_k^{(j+\ell)} c_{k+\ell}^{(i-\ell)} \qquad (1 \le \ell \le i-1), \tag{16.15}$$

$$\left\{c_k^{(j)}, c_{k+\ell}^{(i)}\right\}_{2\alpha} = -\alpha c_k^{(j)} c_{k+\ell}^{(i)} + \alpha c_k^{(i+\ell)} c_{k+\ell}^{(j-\ell)} \qquad (j-i+1 \le \ell \le j-1). \tag{16.16}$$

**Proof.** The vanishing of all Poisson brackets $\{d_k, d_\ell\}_{2\alpha}$ is a light consequence of (16.6). Similarly, (16.13) follows simply by considering the $\boldsymbol{g}_0$ component of (16.10). The longest is the calculation of Poisson brackets among the $c_k^{(j)}$ variables. To this end we use the defining formula, following from (16.11):

$$\begin{aligned}2\alpha^{-1}\{\varphi(U), \psi(U)\}_{2\alpha} &= -2\langle \mathrm{P}_{>0}(\nabla\varphi U), \nabla\psi U\rangle + 2\langle \mathrm{P}_{>0}(U\nabla\varphi), U\nabla\psi\rangle \\ &-\langle \mathrm{P}_0(\nabla\varphi U), \nabla\psi U\rangle + \langle \mathrm{P}_0(U\nabla\varphi), U\nabla\psi\rangle + \langle \mathrm{P}_0(U\nabla\varphi), \nabla\psi U\rangle - \langle \mathrm{P}_0(\nabla\varphi U), U\nabla\psi\rangle \\ &+\langle \mathrm{W}(\nabla\varphi U), \nabla\psi U\rangle + \langle \mathrm{W}(U\nabla\varphi), U\nabla\psi\rangle - \langle \mathrm{W}(U\nabla\varphi), \nabla\psi U\rangle - \langle \mathrm{W}(\nabla\varphi U), U\nabla\psi\rangle.\end{aligned} \tag{16.17}$$

Set in this formula

$$\varphi(U) = -\alpha c_k^{(i)}, \qquad \psi(U) = -\alpha c_{k+\ell}^{(j)},$$

so that

$$\nabla\varphi = \lambda^i E_{k+i,i}, \qquad \nabla\psi = \lambda^j E_{k+\ell+j,k+\ell}.$$

We find:

$$\nabla\varphi U = \lambda^i E_{k+i,k} - \alpha \sum_{\beta=1}^{m} \lambda^{i-\beta} c_k^{(\beta)} E_{k+i,k+\beta}, \tag{16.18}$$

$$U\nabla\varphi = \lambda^i E_{k+i,k} - \alpha \sum_{\beta=1}^{m} \lambda^{i-\beta} c_{k+i-\beta}^{(\beta)} E_{k+i-\beta,k}. \tag{16.19}$$

We consider first the contribution to the Poisson bracket $\alpha^{-1}\{\varphi,\psi\}_{2\alpha} = \alpha \left\{c_k^{(i)}, c_{k+\ell}^{(j)}\right\}_{2\alpha}$ from the first line in (16.17):

$$\left\langle \lambda^i E_{k+i,k} - \alpha \sum_{\beta=1}^{i-1} \lambda^{i-\beta} c_k^{(\beta)} E_{k+i,k+\beta}, \alpha \sum_{\gamma=j+1}^{m} \lambda^{j-\gamma} c_{k+\ell}^{(\gamma)} E_{k+\ell+j,k+\ell+\gamma} \right\rangle$$
$$- \left\langle \lambda^i E_{k+i,k} - \alpha \sum_{\beta=1}^{i-1} \lambda^{i-\beta} c_{k+i-\beta}^{(\beta)} E_{k+i-\beta,k}, \alpha \sum_{\gamma=j+1}^{m} \lambda^{j-\gamma} c_{k+\ell+j-\gamma}^{(\gamma)} E_{k+\ell+j-\gamma,k+\ell} \right\rangle.$$

Calculating these scalar products, we find:

$$\begin{aligned} &= \alpha \sum_{\gamma=j+1}^{m} \left( c_{k+\ell}^{(\gamma)} \delta_{0,\ell+j} \delta_{i,\ell+\gamma} - c_{k+\ell+j-\gamma}^{(\gamma)} \delta_{0,\ell+j-\gamma} \delta_{i,\ell} \right) \\ &\quad -\alpha^2 \sum_{\beta=1}^{i-1} \sum_{\gamma=j+1}^{m} \left( c_k^{(\beta)} c_{k+\ell}^{(\gamma)} \delta_{\beta,\ell+j} \delta_{i,\ell+\gamma} - c_{k+i-\beta}^{(\beta)} c_{k+\ell+j-\gamma}^{(\gamma)} \delta_{0,\ell+j-\gamma} \delta_{i-\beta,\ell} \right) \\ &\quad = \alpha c_{k+\ell}^{(i+j)} \delta_{\ell,-j} - \alpha c_k^{(i+j)} \delta_{\ell,i} - \alpha^2 \bar{\varkappa}_{k\ell}^{(ij)} c_k^{(\ell+j)} c_{k+\ell}^{(i-\ell)}, \end{aligned}$$

where

$$\begin{aligned} \bar{\varkappa}_{k\ell}^{(ij)} &= \chi_\ell(-j+1, i-j-1)\chi_\ell(i-m, i-j-1) - \chi_\ell(1, m-j)\chi_\ell(1, i-1) \\ &= \chi_\ell\left(-\min(j-1, m-i), i-j-1\right) - \chi_\ell\left(1, \min(i-1, m-j)\right). \end{aligned} \tag{16.20}$$

Assuming, for the sake of definiteness, that $i \le j$, we see that the intervals of the two characteristic functions in the last line do not intersect. So, we found the contributions to the Poisson bracket $\alpha \left\{c_k^{(i)}, c_{k+\ell}^{(j)}\right\}_{2\alpha}$ described by the first term on the right-hand side of the formula (16.14), and the second terms on the right-hand sides of (16.15), (16.16).

Finally, calculating the contribution of the remaining part of (16.17), we get:

$$\alpha^2 \bar{\varepsilon}_{k\ell}^{(ij)} c_k^{(i)} c_{k+\ell}^{(j)}, \tag{16.21}$$

where

$$\bar{\varepsilon}_{k\ell}^{(ij)} = \tfrac{1}{2}(-\delta_{i,\ell+j} + \delta_{0,\ell} + \delta_{0,\ell+j} - \delta_{i,\ell} \\ +w_{k+i,k+\ell+j} + w_{k,k+\ell} - w_{k,k+\ell+j} - w_{k+i,k+\ell}). \tag{16.22}$$

This coefficient differs from (9.21) by $-\delta_{i,\ell+j} + \delta_{0,\ell}$, so that the result (for $i \le j$) may be written down immediately:

$$\bar{\varepsilon}_{k\ell}^{(ij)} = \chi_\ell(-j, -j+i-1) - \chi_\ell(1, i). \tag{16.23}$$

This describes the first terms on the right-hand sides of (16.15), (16.16), which finishes the proof of the part a) of Theorem 16.1 and the proof of Theorem 16.2. The parts b) and c) of Theorem 16.1 follow from Theorem 5.1 with $m = 2$. ■

**Corollary.** *The system* $\mathrm{RTL}_{m+1}^{(+)}(\alpha)$ *(15.9) is Hamiltonian with respect to the bracket (16.13)–(16.16), with the Hamilton function*

$$\alpha^{-1}\mathrm{H}_1 = \alpha^{-1}\sum_{k=1}^{N} d_k + \sum_{k=1}^{N} c_k^{(1)} = \alpha^{-1}\left(\mathrm{tr}(LU^{-1} - I)\right)_0. \tag{16.24}$$

Unfortunately, the $\mathrm{RTL}_{m+1}^{(+)}(\alpha)$ hierarchy seems not to have an invariant linear Poisson bracket. Nevertheless, it is bi-Hamiltonian, and the most direct way to find its second Hamiltonian formulation is to push forward under $\mathbf{B}(\alpha)$ the known invariant Poisson brackets of the $\mathrm{RTL}_{m+1}^{(-)}(-\alpha)$ hierarchy. It can be verified that the bracket $\{\cdot,\cdot\}_{2\alpha}$ found in Theorem 16.2 is recovered by applying this construction to the bracket $\{\cdot,\cdot\}_1 - \alpha\{\cdot,\cdot\}_2$ of the hierarchy $\mathrm{RTL}_{m+1}^{(-)}(-\alpha)$. A compatible bracket appears, if we start just with $\{\cdot,\cdot\}_1$. We formulate this as a separate statement.

**Theorem 16.3** *The hierarchy* $\mathrm{RTL}_{m+1}^{(+)}(\alpha)$ *has an invariant cubic Poisson bracket* $\{\cdot,\cdot\}_{3\alpha}$*, which can be defined as the push-forward under the change of variables* $\mathbf{B}(\alpha)$ *(15.3) of the invariant bracket* $\{\cdot,\cdot\}_1$ *of the* $\mathrm{RTL}_{m+1}^{(-)}(-\alpha)$ *hierarchy. The brackets* $\{\cdot,\cdot\}_{2\alpha}$ *and* $\{\cdot,\cdot\}_{3\alpha}$ *are compatible.*

We shall not try to write down the general formulas for this pushed bracket, as they turn out to be complicated enough. However, in the next Section we give the result of calculations for this bracket in the particular case $m = 2$.

# 17 Example: $\mathrm{RTL}_3^{(+)}(\alpha)$, the three-field analog of the relativistic Toda lattice

In order to provide the reader with an illustration of the bi-Hamiltonian structure of the system $\mathrm{RTL}_{m+1}^{(+)}(\alpha)$, we give here the corresponding formulas for the case $m = 2$, which is the next in complexity case after the usual relativistic Toda lattice $\mathrm{RTL}_+(\alpha)$. The equations of motion of the system $\mathrm{RTL}_3^{(+)}(\alpha)$ read:

$$\left\{\begin{array}{l} \dot{d}_k = (1+\alpha d_k)\left(c_k^{(1)} - c_{k-1}^{(1)}\right), \\ \dot{c}_k^{(1)} = c_k^{(j)}\left(d_{k+1} - d_k + \alpha c_{k+1}^{(1)} - \alpha c_{k-1}^{(1)}\right) + \left(c_k^{(2)} - c_{k-1}^{(2)}\right), \\ \dot{c}_k^{(2)} = c_k^{(2)}\left(d_{k+2} - d_k + \alpha c_{k+2}^{(1)} - \alpha c_{k-1}^{(1)}\right). \end{array}\right. \tag{17.1}$$

This is, of course, an $O(\alpha)$-perturbation of the system (10.1). The quadratic Poisson structure of Theorem 16.1 is charachterized by the following nonvanishing brackets:

$$
\begin{aligned}
&\left\{d_k, c_k^{(1)}\right\}_{2\alpha} = -c_k^{(1)}(1+\alpha d_k), && \left\{c_k^{(1)}, d_{k+1}\right\}_{2\alpha} = -c_k^{(1)}(1+\alpha d_{k+1}),\\
&\left\{d_k, c_k^{(2)}\right\}_{2\alpha} = -c_k^{(2)}(1+\alpha d_k), && \left\{c_k^{(2)}, d_{k+2}\right\}_{2\alpha} = -c_k^{(2)}(1+\alpha d_{k+2}),\\
&\left\{c_k^{(1)}, c_{k+1}^{(1)}\right\}_{2\alpha} = -c_k^{(2)} - \alpha c_k^{(1)} c_{k+1}^{(1)}, && \left\{c_k^{(1)}, c_{k+1}^{(2)}\right\}_{2\alpha} = -\alpha c_k^{(1)} c_{k+1}^{(2)},\\
&\left\{c_k^{(2)}, c_{k+2}^{(1)}\right\}_{2\alpha} = -\alpha c_k^{(2)} c_{k+2}^{(1)}, && \left\{c_k^{(2)}, c_{k+1}^{(2)}\right\}_{2\alpha} = -\alpha c_k^{(2)} c_{k+1}^{(2)},\\
&\left\{c_k^{(2)}, c_{k+2}^{(2)}\right\}_{2\alpha} = -\alpha c_k^{(2)} c_{k+2}^{(2)}.
\end{aligned}
\tag{17.2}
$$

This quadratic bracket is an $O(\alpha)$-perturbation of the *linear* invariant bracket (10.2) of the nonrelativistic system $\mathrm{TL}_3$. The cubic invariant Poisson bracket (see Proposition 16.3) is characterized by the following nonvanishing brackets:

$$
\begin{aligned}
&\{d_k, d_{k+1}\}_{3\alpha} = -\alpha c_k^{(1)} \left(1+\alpha d_k\right)\left(1+\alpha d_{k+1}\right),\\
&\left\{d_k, c_k^{(1)}\right\}_{3\alpha} = -c_k^{(1)} \left(1+\alpha d_k\right)\left(1+\alpha d_k + \alpha^2 c_k^{(1)}\right),\\
&\left\{c_k^{(1)}, d_{k+1}\right\}_{3\alpha} = -c_k^{(1)} \left(1+\alpha d_{k+1}\right)\left(1+\alpha d_{k+1} + \alpha^2 c_k^{(1)}\right),\\
&\left\{d_k, c_{k+1}^{(1)}\right\}_{3\alpha} = -\alpha\left(c_k^{(2)} + \alpha c_k^{(1)} c_{k+1}^{(1)}\right)\left(1+\alpha d_k\right),\\
&\left\{c_k^{(1)}, d_{k+2}\right\}_{3\alpha} = -\alpha\left(c_k^{(2)} + \alpha c_k^{(1)} c_{k+1}^{(1)}\right)\left(1+\alpha d_{k+2}\right),\\
&\left\{d_k, c_k^{(2)}\right\}_{3\alpha} = -c_k^{(2)} \left(1+\alpha d_k\right)\left(1+\alpha d_k + \alpha^2 c_k^{(1)}\right),\\
&\left\{c_k^{(2)}, d_{k+1}\right\}_{3\alpha} = -\alpha^2 c_k^{(2)} \left(1+\alpha d_{k+1}\right)\left(c_k^{(1)} - c_{k+1}^{(1)}\right),\\
&\left\{d_k, c_{k+1}^{(2)}\right\}_{3\alpha} = -\alpha^2 c_k^{(1)} c_{k+1}^{(2)} \left(1+\alpha d_k\right),\\
&\left\{c_k^{(2)}, d_{k+2}\right\}_{3\alpha} = -c_k^{(2)} \left(1+\alpha d_{k+2}\right)\left(1+\alpha d_{k+2} + \alpha^2 c_{k+1}^{(1)}\right),\\
&\left\{c_k^{(2)}, d_{k+3}\right\}_{3\alpha} = -\alpha^2 c_k^{(2)} c_{k+2}^{(1)} \left(1+\alpha d_{k+3}\right),\\
&\left\{c_k^{(1)}, c_{k+1}^{(1)}\right\}_{3\alpha} = -\left(1+\alpha d_{k+1}\right)\left(c_k^{(2)} + 2\alpha c_k^{(1)} c_{k+1}^{(1)}\right)\\
&\qquad -\alpha^2\left(c_k^{(1)} + c_{k+1}^{(1)}\right)\left(c_k^{(2)} + \alpha c_k^{(1)} c_{k+1}^{(1)}\right),\\
&\left\{c_k^{(1)}, c_{k+2}^{(1)}\right\}_{3\alpha} = -\alpha^2 c_k^{(2)} c_{k+2}^{(1)} - \alpha^2 c_k^{(1)} c_{k+1}^{(2)} - \alpha^3 c_k^{(1)} c_{k+1}^{(1)} c_{k+2}^{(1)},
\end{aligned}
\tag{17.3}
$$

$$\left\{c_k^{(1)}, c_k^{(2)}\right\}_{3\alpha} = -\alpha c_k^{(1)} c_k^{(2)} (1 + \alpha d_{k+1}) - \alpha^2 c_k^{(2)} \left(c_k^{(2)} + \alpha c_k^{(1)} c_{k+1}^{(1)}\right),$$

$$\left\{c_k^{(2)}, c_{k+1}^{(1)}\right\}_{3\alpha} = -\alpha c_k^{(2)} c_{k+1}^{(1)} (1 + \alpha d_{k+1}) - \alpha^2 c_k^{(2)} \left(c_k^{(2)} + \alpha c_k^{(1)} c_{k+1}^{(1)}\right),$$

$$\left\{c_k^{(1)}, c_{k+1}^{(2)}\right\}_{3\alpha} = -2\alpha c_k^{(1)} c_{k+1}^{(2)} (1 + \alpha d_{k+1}) - \alpha^2 c_{k+1}^{(2)} \left(c_k^{(2)} + \alpha c_k^{(1)} \left(c_k^{(1)} + c_{k+1}^{(1)}\right)\right),$$

$$\left\{c_k^{(2)}, c_{k+2}^{(1)}\right\}_{3\alpha} = -2\alpha c_k^{(2)} c_{k+2}^{(1)} (1 + \alpha d_{k+2}) - \alpha^2 c_k^{(2)} \left(c_{k+1}^{(2)} + \alpha c_{k+2}^{(1)} \left(c_{k+1}^{(1)} + c_{k+2}^{(1)}\right)\right),$$

$$\left\{c_k^{(1)}, c_{k+2}^{(2)}\right\}_{3\alpha} = -\alpha^2 c_{k+2}^{(2)} \left(c_k^{(2)} + \alpha c_k^{(1)} c_{k+1}^{(1)}\right),$$

$$\left\{c_k^{(2)}, c_{k+3}^{(1)}\right\}_{3\alpha} = -\alpha^2 c_k^{(2)} \left(c_{k+2}^{(2)} + \alpha c_{k+2}^{(1)} c_{k+3}^{(1)}\right),$$

$$\left\{c_k^{(2)}, c_{k+1}^{(2)}\right\}_{3\alpha} = -\alpha c_k^{(2)} c_{k+1}^{(2)} \left(2 + \alpha d_{k+1} + \alpha d_{k+2} + \alpha^2 c_k^{(1)} + \alpha^2 c_{k+2}^{(1)}\right),$$

$$\left\{c_k^{(2)}, c_{k+2}^{(2)}\right\}_{3\alpha} = -\alpha c_k^{(2)} c_{k+2}^{(2)} \left(2 + 2\alpha d_{k+2} + \alpha^2 c_{k+1}^{(1)} + \alpha^2 c_{k+2}^{(1)}\right),$$

$$\left\{c_k^{(2)}, c_{k+3}^{(2)}\right\}_{3\alpha} = -\alpha^3 c_k^{(2)} c_{k+2}^{(1)} c_{k+3}^{(2)},$$

This cubic bracket is again an $O(\alpha)$-perturbation of the *linear* invariant bracket (10.2) of the nonrelativistic system $\mathrm{TL}_3$. However, the linear combination

$$\alpha^{-1} \left(\{\cdot, \cdot\}_{3\alpha} - \{\cdot, \cdot\}_{2\alpha}\right)$$

of the both compatible Poisson brackets above leads, in the limit $\alpha \to 0$, to the *quadratic* invariant bracket (10.3) of $\mathrm{TL}_3$.

# 18 Conclusions

The results of this paper confirm once more that the $r$-matrix theory, with its various generalizations, provides very convenient means for studying the Hamiltonian aspects of integrable lattice systems. The relativistic lattice KP, which seemed for a long time to lie outside of the applicability area of the $r$-matrix theory, actually fits very nicely into this framework. Moreover, it allowed us to find for the first time the quadratic invariant bracket of the relativistic lattice KP, and to establish an amasing fact: that this attribute remains undeformed by the relativistic deformation. By the way, this gives also an answer to a problem left open in [17], concerning the Hamiltonian nature of the "relativistic Bogoyavlensky lattices". Namely, these relativistic systems are Hamiltonian with respect to the invariant quadratic brackets of their nonrelativistic counterparts. A detailed account of this latter result will be given elsewhere.

## Acknowledgement

I thank B.A. Kupershmidt for a very stimulating correspondence.

# References


[1] Adler M., and van Moerbecke P., Completely Integrable Systems, Kac–Moody Algebras and Curves, *Adv. Math.*, 1980, V.38, 267–317.

[2] Deift P., Li L.C., Nanda T. and Tomei C., The Toda Flow on a Generic Orbit is Integrable, *Commun. Pure Appl. Math.*, 1986, V.39, 183–232.

[3] Ercolani N., Flaschka H. and Singer S., The Geometry of the Full Kostant–Toda Lattice, in: Integrable Systems, Progress in Mathematics 115, Birkhäuser, 1993, 181–226.

[4] Kupershmidt B.A., Discrete Lax Equations and Differential-Difference Calculus, *Asterisque*, 1985, V.123.

[5] Kupershmidt B.A., Quasirelativistic Analogs of Lax Equations, in: Nonlinear Processes in Physics, Eds. A.S. Fokas, D.J. Kaup, A.C. Newell, V.E. Zakharov, Springer, 1993, 114–117.

[6] Kupershmidt B.A., KP or mKP: Noncommutative Mathematics of Lagrangian, Hamiltonian, and Integrable Systems, *Amer. Math. Soc.* (to appear).

[7] Gibbons J. and Kupershmidt B.A., Relativistic Analogs of Basic Integrable Systems, in: Integrable and Superintegrable Systems, Ed. B.A. Kupershmidt, World Scientific, 1990, 207–231.

[8] Li L.-C. and Parmentier S., Nonlinear Poisson Structures and $r$-Matrices, *Commun. Math. Phys.*, 1989, V.125, 545–563.

[9] Oevel W., Poisson Brackets for Integrable Lattice Systems, in: Algebraic Aspects of Integrable Systems, I. Dorfman Memorial Volume, Eds. A.S. Fokas, I.M. Gelfand, Birkhäuser, 1997, 261–283.

[10] Oevel W. and Ragnisco O., $R$-Matrices and Higher Poisson Brackets for Integrable Systems, *Physica A*, 1989, V.161, 181–220.

[11] Reyman A.G. and Semenov-Tian-Shansky M.A., Group Theoretical Methods in the Theory of Finite Dimensional Integrable Systems, in: Encyclopaedia of Mathematical Science, V.16: Dynamical Systems VII, Springer, 1994, 116–225.

[12] Ruijsenaars S.N.M., Relativistic Toda Systems, *Commun. Math. Phys.*, 1990, V.133, 217–247.

[13] Semenov-Tian-Shansky M.A., What is a Classical $r$-Matrix? *Funct. Anal. Appl.*, 1983, V.17, 259–272; Classical $r$-Matrices, Lax Equations, Poisson Lie Groups and Dressing Transformations, *Lecture Notes Phys.*, 1987, V.280, 174–214.

[14] Semenov-Tian-Shansky M.A. and Sevostyanov A., Classical and Quantum Nonultralocal Systems on the Lattice, in: Algebraic Aspects of Integrable Systems, I. Dorfman Memorial Volume, Eds. A.S. Fokas, I.M. Gelfand, Birkhäuser, 1997, 323–350.

[15] Suris Yu.B., On the Bi-Hamiltonian Structure of Toda and Relativistic Toda Lattices, *Phys. Lett. A*, 1993, V.180, 419–429.

[16] Suris Yu.B., Nonlocal Quadratic Poisson Algebras, Monodromy Map, and Bogoyavlensky Lattices, *J. Math. Phys.*, 199, V.38, 4179–4201.

[17] Suris Yu.B. and Ragnisco O., What is a Relativistic Volterra Lattice? *Commun. Math. Phys.*, 1999, V.200, 445–485.