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The Nonabelian Liouville-Arnold Integrability by Quadratures Problem: a Symplectic Approach

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Abstract

A symplectic theory approach is devised for solving the problem of algebraic-analytical construction of integral submanifold imbeddings for integrable (via the nonabelian Liouville-Arnold theorem) Hamiltonian systems on canonically symplectic phase spaces.

0. Introduction

0.1. As is well known [1, 4], the integrability by quadratures of a differential equation in space $\mathbb{R}^n$ is a method of seeking its solutions by means of finite number of algebraic operations (together with inversion of functions) and "quadratures" – calculations of integrals of known functions.

Assume that our differential equation is given as a Hamiltonian dynamical system on some appropriate symplectic manifold $\left(M^{2n},\omega^{(2)}\right)$, $n\in\mathbb{Z}_+$, in the form

$$du/dt = \{H, u\}, \tag{0.1}$$

where $u\in M^{2n}$, $H: M^{2n}\to\mathbb{R}$ is a sufficiently smooth Hamiltonian function [1, 4] with respect to the Poisson bracket $\{\cdot,\cdot\}$ on $\mathcal{D}(M^{2n})$, dual to the symplectic structure $\omega^{(2)}\in\Lambda^2(M^{2n})$, and $t\in\mathbb{R}$ is the evolution parameter.

More than one hundred and fifty years ago French mathematicians and physicists, first E. Bour and next J. Liouville, proved the first "integrability by quadratures" theorem which in modern terms [33] can be formulated as follows.

Theorem 0.1. *Let $M^{2n} \simeq T^*(\mathbb{R}^n)$ be a canonically symplectic phase space and there be given a dynamical system (0.1) with a Hamiltonian function $H\colon M^{2n}\times\mathbb{R}_t \to \mathbb{R}$, possessing a Poissonian Lie algebra $\mathcal{G}$ of $n \in \mathbb{Z}_+$ invariants $H_j : M^{2n} \times \mathbb{R}_t \to \mathbb{R}$, $j = \overline{1,n}$, such that*

$$\{H_i, H_j\} = \sum_{s=1}^{n} c_{ij}^s H_s, \tag{0.2}$$

and for all $i, j, k = \overline{1,n}$ the $c_{ij}^s \in \mathbb{R}$ are constants on $M^{2n} \times \mathbb{R}_t$. Suppose further that

$$M_h^{n+1} := \left\{(u,t) \in M \times \mathbb{R}_t : h(H_j) = h_j,\ j = \overline{1,n},\ h \in \mathcal{G}^*\right\}, \tag{0.3}$$

the integral submanifold of the set $\mathcal{G}$ of invariants at a regular element $h \in \mathcal{G}^*$, is a well defined connected submanifold of $M \times \mathbb{R}_t$. Then, if:

i) all functions of $\mathcal{G}$ are functionally independent on M_h^{n+1};

ii) $\sum\limits_{s=1}^{n} c_{ij}^s h_s = 0$ for all $i, j = \overline{1,n}$;

iii) the Lie algebra $\mathcal{G} = \operatorname{span}_{\mathbb{R}}\left\{H_j : M^{2n} \times \mathbb{R}_t \to \mathbb{R} : j = \overline{1,n}\right\}$ is solvable, the Hamiltonian system (0.1) on M^{2n} is integrable by quadratures.

As a simple corollary of the Bour-Liouville theorem one gets the following:

Corollary 0.2. *If a Hamiltonian system on $M^{2n} = T^*(\mathbb{R}^n)$ possesses just $n \in \mathbb{Z}_+$ functionally independent invariants in involution, that is a Lie algebra $\mathcal{G}$ is abelian, then it is integrable by quadratures.*

In the autonomous case when a Hamiltonian $H = H_1$, and invariants $H_j : M^{2n} \to \mathbb{R}$, $j = \overline{1,n}$, are independent of the evolution parameter $t \in \mathbb{R}$, the involutivity condition $\{H_i, H_j\} = 0$, $i, j = \overline{1,n}$, can be replaced by the weaker one $\{H, H_j\} = c_j H$ for some constants $c_j \in \mathbb{R}$, $j = \overline{1,n}$.

The first proof of Theorem 0.1. was based on a result of S. Lie, which can be formulated as follows.

Theorem 0.3 (S. Lie). *Let vector fields $K_j \in \Gamma(M^{2n})$, $j = \overline{1,n}$, be independent in some open neighborhood $U_h \in M^{2n}$, generate a solvable Lie algebra $\mathcal{G}$ with respect to the usual commutator $[\cdot,\cdot]$ on $\Gamma(M^{2n})$ and $[K_j, K] = c_j K$ for all $j = \overline{1,n}$, where $c_j \in \mathbb{R}$, $j = \overline{1,n}$, are constants. Then the dynamical system*

$$du/dt = K(u), \tag{0.1$'$}$$

where $u \in U_h \subset M^{2n}$, is integrable by quadratures.

Example 0.4. ***Motion of three particles on line $\mathbb{R}$ under uniform potential field.*** The motion of three particles on the axis $\mathbb{R}$ pairwise interacting via a uniform potential field $Q(\|\cdot\|)$ is described as a Hamiltonian system on the canonically symplectic phase space $M = T^*(\mathbb{R}^3)$ with the following Lie algebra $\mathcal{G}$ of invariants on M^{2n}:

$$H = H_1 = \sum_{j=1}^{3} p_j^2/2m_j + \sum_{i<j=1}^{3} Q(\|q_i - q_j\|), \quad H_2 = \sum_{j=1}^{3} q_j p_j, \quad H_3 = \sum_{j=1}^{3} p_j, \tag{0.4}$$

where $(q_j, p_j) \in T^*(\mathbb{R})$, $j = \overline{1,3}$, are coordinates and momenta of particles on the axis $\mathbb{R}$. The commutation relations for the Lie algebra $\mathcal{G}$ are

$$\{H_1, H_3\} = 0, \qquad \{H_2, H_3\} = H_3, \qquad \{H_1, H_2\} = 2H_1, \tag{0.5}$$

hence it clearly solvable. Taking a regular element $h \in \mathcal{G}^*$, such that $h(H_j) = h_j = 0$, for $j = 1$ and 3, and $h(H_2) = h_2 \in \mathbb{R}$ being arbitrary, one obtains the integrability of the problem above in quadratures.

0.2. In 1974 V. Arnold proved [4] the following important result known as the commutative (abelian) Liouville-Arnold theorem.

Theorem 0.5 (J. Liouville – V. Arnold). *Suppose a set $\mathcal{G}$ of functions $H_j : M^{2n} \to \mathbb{R}$, $j = \overline{1,n}$, on a symplectic manifold M^{2n} is abelian, that is*

$$\{H_i, H_j\} = 0 \tag{0.6}$$

for all $i, j = \overline{1,n}$. If on the compact and connected integral submanifold $M_h^n = \{u \in M^{2n} : h(H_j) = h_j \in \mathbb{R},\ j = \overline{1,n},\ h \in \mathcal{G}^*\}$ with $h \in \mathcal{G}$ being regular, all functions $H : M^{2n} \in \mathbb{R}$, $j = \overline{1,n}$, are functionally independent, then M_h^n is diffeomorphic to the n-dimensional torus $\mathbb{T}^n \simeq M^{2n}$, and the motion on it with respect to the Hamiltonian $H = H_1 \in \mathcal{G}$ is a quasi-periodic function of the evolution parameter $t \in \mathbb{R}$.

A dynamical system satisfying the hypotheses of Theorem 0.5 is called completely integrable.

In 1978 Mishchenko and Fomenko [2] proved the following generalization of the Liouville-Arnold Theorem 0.5:

Theorem 0.6 (A. Mishchenko – A. Fomenko). *Assume that on a symplectic manifold $(M^{2n}, \omega^{(2)})$ there is a nonabelian Lie algebra $\mathcal{G}$ of invariants $H_j : M \in \mathbb{R}$, $j = \overline{1,k}$, with respect to the dual Poisson bracket on M^{2n}, that is*

$$\{H_i, H_j\} = \sum_{s=1}^{k} c_{ij}^s H_s, \tag{0.7}$$

where all values $c_{ij}^s \in \mathbb{R}$, $i, j, s = \overline{1,k}$, are constants, and the following conditions are satisfied:

i) the integral submanifold $M_h^r := \{u \in M^{2n} : h(H_j) = h_j \in \mathbb{R},\ j = \overline{1,k},\ h \in \mathcal{G}^*\}$ is compact and connected at a regular element $h \in \mathcal{G}^*$;

ii) all functions $H_j : M^{2n} \to \mathbb{R}$, $j = \overline{1,k}$, are functionally independent on M^{2n};

iii) the Lie algebra $\mathcal{G}$ of invariants satisfies the following relationship:

$$\dim \mathcal{G} + \operatorname{rank} \mathcal{G} = \dim M^{2n}, \tag{0.8}$$

where $\operatorname{rank} \mathcal{G} = \dim \mathcal{G}_h$ is the dimension of a Cartan subalgebra $\mathcal{G}_h \subset \mathcal{G}$. Then the submanifold $M_h^r \subset M^{2n}$ is $r = \operatorname{rank} \mathcal{G}$-dimensional, invariant with respect each vector field $K \in \Gamma(M^{2n})$, generated by an element $H \in \mathcal{G}_h$, and diffeomorphic to the r-dimensional torus $\mathbb{T}^r \simeq M_h^r$, on which the motion is a quasiperiodic function of the evolution parameter $t \in \mathbb{R}$.

0.3. The simplest proof of the Mishchenko-Fomenko Theorem 0.6 can be obtained from the well known [3, 16] classical Lie-Cartan theorem.

Theorem 0.7 (S. Lie – E. Cartan). *Suppose that a point $h \in \mathcal{G}^*$ for a given Lie algebra $\mathcal{G}$ of invariants $H_j : M^{2n} \to \mathbb{R}$, $j = \overline{1,k}$, is not critical, and the $\operatorname{rank} ||\{H_i, H_j\} : i, j = \overline{1,k}|| = 2(n - r)$ is constant in an open neighborhood $U_h \in \mathbb{R}^n$ of the point*

$\{h(H_j) = h_j \in \mathbb{R} : j = \overline{1,k}\} \subset \mathbb{R}^k$. *Then in the neighborhood* $(h \circ H^{-1} : U_h \subset M^{2n})$ *there exist* $k \in \mathbb{Z}_+$ *independent functions* $f_s : \mathcal{G} \to \mathbb{R}$, $s = \overline{1,k}$, *such that the functions* $F_s := (f_s \circ H) : M^{2n} \in \mathbb{R}$, $s = \overline{1,k}$, *satisfy the following relationships:*

$$\{F_1, F_2\} = \{F_3, F_4\} = \cdots = \{F_{2(n-r)-1}, F_{2(n-r)}\} = 1, \tag{0.9}$$

with all other brackets $\{F_i, F_j\} = 0$, *where* $(i,j) \neq (2s-1, 2s)$, $s = \overline{1, n-r}$. *In particular,* $(k + r - n) \in \mathbb{Z}_+$ *functions* $F_j : M^{2n} \to \mathbb{R}$, $j = \overline{1, n-r}$, *and* $F_s : M^{2n} \to \mathbb{R}$, $s = \overline{1, k-2(n-r)}$, *compose an abelian algebra* $\mathcal{G}_\tau$ *of new invariants on* M^{2n}, *independent on* $(h \circ H)^{-1}(U_h) \subset M^{2n}$.

As a simple corollary of the Lie-Cartan Theorem 0.7 one obtains the following: in the case of the Mishchenko-Fomenko theorem when $\operatorname{rank} \mathcal{G} + \dim \mathcal{G} = \dim M^{2n}$, that is $r + k = 2n$, the abelian algebra $\mathcal{G}_\tau$ (it is not a subalgebra of $\mathcal{G}$!) of invariants on M^{2n} is just $n = 1/2 \dim M^{2n}$-dimensional, giving rise to its local complete integrability in $(h \circ H)^{-1}(U_h) \subset M^{2n}$ via the abelian Liouville-Arnold Theorem 0.5. It is also evident that the Mishchenko-Fomenko nonabelian integrability Theorem 0.6 reduces to the commutative (abelian) Liouville-Arnold case when a Lie algebra $\mathcal{G}$ of invariants is just abelian, since then $\operatorname{rank} \mathcal{G} = \dim \mathcal{G} = 1/2 \dim M^{2n} = n \in \mathbf{Z}_+$ – the standard complete integrability condition.

All the cases of integrability by quadratures described above pose the following fundamental question: How can one effectively construct by means of algebraic-analytical methods the corresponding integral submanifold imbedding

$$\pi_h : M_h^r \to M^{2n}, \tag{0.10}$$

where $r = \dim \operatorname{rank} \mathcal{G}$, thereby making it possible to express the solutions of an integrable flow on M_h^r as some exact quasi-periodic functions on the torus $\mathbb{T}^r \simeq M_h^r$.

Below we shall describe an algebraic-analytical algorithm for resolving this question for the case when a symplectic manifold M^{2n} is diffeomorphic to the canonically symplectic cotangent phase space $T^*(\mathbb{R}) \simeq M^{2n}$.

1 General setting

1.1. Our main object of study will be differential systems of vector fields on the cotangent phase space $M^{2n} = T^*(\mathbb{R}^n)$, $n \in \mathbb{Z}_+$, endowed with the canonical symplectic structure $\omega^{(2)} \in \Lambda^2(M^{2n})$, where by $\omega^{(2)} = d(\mathrm{pr}^* \alpha^{(1)})$, and

$$\alpha^{(1)} := \langle p, dq \rangle = \sum_{j=1}^{n} p_j dq_j, \tag{1.1}$$

is the canonical 1-form on the base space $\mathbb{R}^n$, lifted naturally to the space $\Lambda^1(M^{2n})$, $(q,p) \in M^{2n}$ are canonical coordinates on $T^*(\mathbb{R}^n)$, $\mathrm{pr} : T^*(\mathbb{R}^n) \to \mathbb{R}$ is the canonical projection, and $\langle \cdot, \cdot \rangle$ is the usual scalar product in $\mathbb{R}^n$.

Assume further that there is also given a Lie subgroup G (not necessarily compact), acting symplectically via the mapping $\varphi : G \times M^{2n} \to M^{2n}$ on M^{2n}, generating a Lie algebra homomorphism $\varphi_* : T(\mathcal{G}) \to \Gamma(M^{2n})$ via the diagram

$$\begin{array}{ccccc} \mathcal{G} & \simeq & T(\mathcal{G}) & \overset{\varphi_*(u)}{\rightarrow} & T(M^{2n}) \\ & & \downarrow & & \downarrow \\ & & \mathcal{G} & \overset{\varphi(u)}{\rightarrow} & M^{2n} \end{array} \tag{1.2}$$

where $u \in M^{2n}$. Thus, for any $a \in \mathcal{G}$ one can define a vector field $K_a \in \Gamma(M^{2n})$ as follows:

$$K_a = \varphi_* \cdot a. \tag{1.3}$$

Since the manifold M^{2n} is symplectic, one can naturally define for any $a \in \mathcal{G}$ a function $H_a \in \mathcal{D}(M^{2n})$ as follows:

$$-i_{K_a}\omega^{(2)} = dH_a, \tag{1.4}$$

whose existence follows from the invariance property

$$L_{K_a}\omega^{(2)} = 0 \tag{1.5}$$

for all $a \in \mathcal{G}$. The following lemma [1] is useful in applications.

Lemma 1.1. *If the first homology group $H_1(\mathcal{G};\mathbb{R})$ of the Lie algebra $\mathcal{G}$ vanishes, then the mapping $\Phi : \mathcal{G} \to \mathcal{D}(M^{2n})$ defined as*

$$\Phi(a) := H_a \tag{1.6}$$

for any $a \in \mathcal{G}$, is a Lie algebra homomorphism of $\mathcal{G}$ and $\mathcal{D}(M^{2n})$ (endowed with the Lie structure induced by the symplectic structure $\omega^{(2)} \in \Lambda^2(M^{2n})$). In this case $\mathcal{G}$ is said to be Poissonian.

As the mapping $\Phi : \mathcal{G} \to \mathcal{D}(M^{2n})$ is evidently linear in $\mathcal{G}$, the expression (1.6) naturally defines a momentum mapping $l : M^{2n} \to \mathcal{G}^*$ as follows: for any $u \in M^{2n}$ and all $a \in \mathcal{G}$

$$(l(u), a)_{\mathcal{G}} := H_a(u), \tag{1.7}$$

where $(\cdot,\cdot)_{\mathcal{G}}$ is the standard scalar product on the dual pair $\mathcal{G}^* \times \mathcal{G}$. The following characteristic equivariance [1] lemma holds.

Lemma 1.2. *The diagram*

$$\begin{array}{ccc} M^{2n} & \overset{l}{\rightarrow} & \mathcal{G}^* \\ \varphi_g \big\downarrow & & \big\downarrow \mathrm{Ad}^*_{g^{-1}} \\ M^{2n} & \overset{l}{\rightarrow} & \mathcal{G}^* \end{array} \tag{1.8}$$

commutes for all $g \in G$, where $\mathrm{Ad}^*_{g^{-1}} : \mathcal{G}^* \to \mathcal{G}^*$ is the corresponding co-adjoint action of the Lie group G on the dual space $\mathcal{G}^*$.

Take now any vector $h \in \mathcal{G}^*$ and consider a subspace $\mathcal{G}_h \subset \mathcal{G}$, consisting of elements $a \in \mathcal{G}$, such that $\mathrm{ad}^*_a h = 0$, where $\mathrm{ad}^*_a : \mathcal{G}^* \to \mathcal{G}^*$ is the corresponding Lie algebra $\mathcal{G}$ representation in the dual space $\mathcal{G}^*$.

The following lemmas hold.

Lemma 1.3. *The subspace $\mathcal{G}_h \subset \mathcal{G}$ is a Lie subalgebra of $\mathcal{G}$, called here a Cartan subalgebra.*

Lemma 1.4. *Assume a vector $h \in \mathcal{G}^*$ is chosen in such a way that $r = \dim \mathcal{G}_h$ is minimal. Then the Cartan Lie subalgebra $\mathcal{G}_h \subset \mathcal{G}$ is abelian.*

In Lemma 1.4 the corresponding element $h \in \mathcal{G}^*$ is called regular and the number $r = \dim \mathcal{G}_h$ is called the rank $\mathcal{G}$ of the Lie algebra $\mathcal{G}$.

1.2. Some twenty years ago Mishchenko and Fomenko [2] proved the following important noncommutative (nonabelian) Liouville-Arnold theorem.

Theorem 1.5. *On a symplectic space $\left(M^{2n}, \omega^{(2)}\right)$ let there be given a set of smooth functions $H_j \in \mathcal{D}(M^{2n})$, $j = \overline{1,k}$, whose linear span over $\mathbb{R}$ comprises a Lie algebra $\mathcal{G}$ with respect to the corresponding Poisson bracket on M^{2n}. Suppose also that the set*

$$M_h^{2n-k} := \left\{u \in M^{2n} : h(H_j) = h_j \in \mathbb{R},\ j = \overline{1,k},\ h \in \mathcal{G}^*\right\}$$

with $h \in \mathcal{G}^*$ regular, is a submanifold of M^{2n}, and on M_h^{2n-k} all the functions $H_j \in \mathcal{D}(M^{2n})$, $j = \overline{1,k}$, are functionally independent. Assume also that the Lie algebra $\mathcal{G}$ satisfies the following condition:

$$\dim \mathcal{G} + \operatorname{rank} \mathcal{G} = \dim M^{2n}. \tag{1.9}$$

Then the submanifold $M_h^r := M_h^{2n-k}$ is $\operatorname{rank} \mathcal{G} = r$-dimensional and invariant with respect to each vector field $K_{\overline{a}} \in \Gamma(M^{2n})$ with $\overline{a} \in \mathcal{G}_h \subset \mathcal{G}$. Given a vector field $K = K_{\overline{a}} \in \Gamma(M^{2n})$ with $\overline{a} \in \mathcal{G}_h$ or $K \in \Gamma(M^{2n})$ such that $[K, K_a] = 0$ for all $a \in \mathcal{G}$, then, if the submanifold M_h^r is connected and compact, it is diffeomorphic to the r-dimensional torus $\mathbb{T}^r \simeq M_h^r$ and the motion of the vector field $K \in \Gamma(M^{2n})$ on it is a quasiperiodic function of the evolution parameter $t \in \mathbb{R}$.

The easiest proof of this result can be obtained from the well known [3] classical Lie-Cartan theorem, mentioned in the Introduction. Below we shall only sketch the original Mishchenko-Fomenko proof which is heavily based on symplectic theory techniques, some of which have been discussed above.

◀ ***Sketch of the proof.*** Define a Lie group G naturally as $G = \exp \mathcal{G}$, where $\mathcal{G}$ is the Lie algebra of functions $H_j \in \mathcal{D}(M^{2n})$, $j = \overline{1,k}$, in the theorem, with respect to the Poisson bracket $\{\cdot,\cdot\}$ on M^{2n}. Then for an element $h \in \mathcal{G}^*$ and any $a = \sum\limits_{j=1}^{k} c_j H_j \in \mathcal{G}$, where $c_j \in \mathbb{R}$, $j = \overline{1,k}$, the following equality

$$(h, a)_{\mathcal{G}} := \sum_{j=1}^{k} c_j h(H_j) = \sum_{j=1}^{k} c_j h_j \tag{1.10}$$

holds. Since all functions $H_j \in \mathcal{D}(M^{2n})$, $j = \overline{1,k}$, are independent on the level submanifold $M_h^r \subset M^{2n}$, this evidently means that the element $h \in \mathcal{G}^*$ is regular for the Lie algebra $\mathcal{G}$. Consequently, the Cartan Lie subalgebra $\mathcal{G}_h \subset \mathcal{G}$ is abelian. The latter is proved by means of simple straightforward calculations. Moreover, the corresponding momentum mapping $l : M^{2n} \to \mathcal{G}^*$ is constant on M_h^r and satisfies the following relation:

$$l(M_h^r) = h \in \mathcal{G}^*. \tag{1.11}$$

From this it can be shown that all vector fields $K_{\overline{a}} \in \Gamma(M^{2n})$, $\overline{a} \in \mathcal{G}_h$, are tangent to the submanifold $M_h^r \subset M^{2n}$. Thus the corresponding Lie subgroup $G_h := \exp \mathcal{G}_h$ acts naturally and invariantly on M_h^r. If the submanifold $M_h^r \subset M^{2n}$ is connected and compact, it follows from (1.9) that $\dim M_h^r = \dim M^{2n} - \dim \mathcal{G} = \operatorname{rank} \mathcal{G} = r$, and one obtains via the Arnold theorem [4], that $M_h^r \simeq \mathbb{T}^r$ and the motion of the vector field $K \in \Gamma(M^{2n})$ is a quasiperiodic function of the evolution parameter $t \in \mathbb{R}$; thus proving the theorem. ▶

As a nontrivial consequence of the Lie-Cartan theorem mentioned before and of the Theorem 1.5, one can prove the following dual theorem about abelian Liouville-Arnold integrability.

Theorem 1.6. *Let a vector field $K \in \Gamma(M^{2n})$ be completely integrable via the nonabelian scheme of Theorem 1.5. Then it is also Liouville-Arnold integrable on M^{2n} and possesses, under some additional conditions, yet another abelian Lie algebra $\mathcal{G}_h$ of functionally independent invariants on M^{2n}, for which $\dim \mathcal{G}_h = n = 1/2 \dim M^{2n}$.*

The available proof of the theorem above is quite complicated, and we shall comment on it in detail later on. We mention here only that some analogs of the reduction Theorem 1.5 for the case where $M^{2n} \simeq \mathcal{G}^*$, so that an arbitrary Lie group G acts symplectically on the manifold, were proved also in [6–10, 34]. Notice here, that in case when the equality (1.9) is not satisfied, one can then construct in the usual way the reduced manifold $\overline{M}_h^{2n-k-r} := M_h^{2n-k}/G_h$ on which there exists a symplectic structure $\overline{\omega}_h^{(2)} \in \Lambda^2\left(\overline{M}_h^{2n-k-r}\right)$, defined as

$$r_h^* \overline{\omega}_h^{(2)} = \pi_h^* \omega^{(2)} \tag{1.12}$$

with respect to the following compatible reduction-imbedding diagram:

$$\overline{M}_h^{2n-k-r} \overset{r_h}{\longleftarrow} M_h^{2n-k} \overset{\pi_h}{\rightarrow} M^{2n}, \tag{1.13}$$

where $r_h : M_h^{2n-k} \to \overline{M}_h^{2n-k-r}$ and $\pi_h : M_h^{2n-k} \to M^{2n}$ are, respectively, the corresponding reductions and imbedding mappings. The nondegeneracy of the 2-form $\overline{\omega}_h^{(2)} \in \Lambda^2(\overline{M}_h)$ defined by (1.12), follows simply from the expression

$$\begin{aligned} \ker\left(\pi_h^* \omega^{(2)}(u)\right) &= T_u\left(M_h^{2n-k}\right) \cap T_u^{\perp}\left(M_h^{2n-k}\right) \\ &= \operatorname{span}_{\mathbb{R}}\left\{K_{\overline{a}}(u) \in T_u\left(\overline{M}_h^{2n-k-r} := M_h^{2n-k}/G_h\right) : \overline{a} \in \mathcal{G}_h\right\} \end{aligned} \tag{1.14}$$

for any $u \in M_h^{2n-k}$, since all vector fields $K_{\overline{a}} \in \Gamma(M^{2n})$, $\overline{a} \in \mathcal{G}_h$, are tangent to $\overline{M}_h^{2n-k-r} := M_h^{2n-k}/G_h$. Thus, the reduced space $\overline{M}_h^{2n-k-r} := M_h^{2n-k}/G_h$ with respect to the orbits of the Lie subgroup G_h action on M_h^{2n-k} will be a $(2n-k-r)$ -dimensional symplectic manifold. The latter evidently means that the number $2n-k-r = 2s \in \mathbb{Z}_+$ is even as there is no symplectic structure on odd-dimensional manifolds. This obviously is closely connected with the problem of existence of a symplectic group action of a Lie group G on a given symplectic manifold $\left(M^{2n}, \omega^{(2)}\right)$ with a symplectic structure $\omega^{(2)} \in \Lambda^{(2)}(M^{2n})$ being apriori fixed. From this point of view one can consider the inverse problem of constructing symplectic structures on a manifold M^{2n}, admitting a Lie group G action. Namely, owing to the equivariance property (1.8) of the momentum mapping $l : M^{2n} \to \mathcal{G}^*$, one can obtain the induced symplectic structure $l^* \Omega_h^{(2)} \in \Lambda^2\left(\overline{M}_h^{2n-k-r}\right)$ on $\overline{M}_h^{2n-k-r}$ from the canonical symplectic structure $\Omega_h^{(2)} \in \Lambda^{(2)}(Or(h;G))$ on the orbit $Or(h;G) \subset \mathcal{G}^*$ of a regular element $h \in \mathcal{G}^*$. Since the symplectic structure $l^*\Omega_h^{(2)} \in \Lambda^2(\overline{M}_h)$ can be naturally lifted to the 2-form $\widetilde{\omega}^{(2)} = (r_h^* \circ l^*)\Omega_h^{(2)} \in \Lambda^2\left(M_h^{2n-k}\right)$, the latter being degenerate on M_h^{2n-k} can apparently be nonuniquely extended on the whole manifold M^{2n} to a symplectic structure $\omega^{(2)} \in \Lambda^2(M^{2n})$, for which the action of the Lie group G is apriori symplectic. Thus, many properties of a given dynamical system with a Lie algebra $\mathcal{G}$ of invariants on M^{2n} are deeply connected with the symplectic structure $\omega^{(2)} \in \Lambda^2(M^{2n})$ the manifold M^{2n} is endowed with, and in particular, with the corresponding integral submanifold imbedding mapping $\pi_h : M_h^{2n-k} \to M^{2n}$ at a regular element $h \in \mathcal{G}^*$. The problem of direct algebraic-analytical construction of this mapping was in part solved in [11] in the case where $n = 2$ for an abelian algebra $\mathcal{G}$ on the manifold $M^4 = T^*(\mathbb{R}^2)$. The treatment of this problem in [11] has been extensively based both on the classical Cartan studies of integral submanifolds of ideals in Grassmann algebras and on the modern Galisau-Reeb-Francoise results for a symplectic manifold $\left(M^{2n}, \omega^{(2)}\right)$ structure, on which there exists an involutive set $\mathcal{G}$ of functionally independent invariants $H_j \in \mathcal{D}(M^{2n})$, $j = \overline{1,n}$. In what follows below we generalize the Galisau-Reeb-Francoise results to the case of a nonabelian set of functionally independent functions $H_j \in \mathcal{D}(M^{2n})$, $j = \overline{1,k}$, comprising a Lie algebra $\mathcal{G}$ and satisfying the Mishchenko-Fomenko condition (1.9): $\dim \mathcal{G} + \operatorname{rank} \mathcal{G} = \dim M^{2n}$. This makes it possible to devise an effective algebraic-analytical method of constructing the corresponding integral submanifold imbedding and reduction mappings, giving rise to a wide class of exact, integrable by quadratures solutions of a given integrable vector field on M^{2n}.

2 Integral submanifold imbedding problem for an abelian Lie algebra of invariants

2.1. We shall consider here only a set $\mathcal{G}$ of commuting polynomial functions $H_j \in \mathcal{D}(M^{2n})$, $j = \overline{1,n}$, on the canonically symplectic phase space $M^{2n} = T^*(\mathbb{R}^n)$. Due to the Liouville-Arnold theorem [4], any dynamical system $K \in \Gamma(M^{2n})$ commuting with corresponding Hamiltonian vector fields K_a for all $a \in \mathcal{G}$, will be integrable by quadratures in case of a regular element $h \in \mathcal{G}^*$, which defines the corresponding integral submanifold $M_h^n := \left\{u \in M^{2n} : h(H_j) = h_j \in \mathbb{R},\ j = \overline{1,n}\right\}$ which is diffeomorphic (when compact and connected) to the n-dimensional torus $\mathbb{T}^n \simeq M_h^n$. This in particular means that there exists some algebraic-analytical expression for the integral submanifold imbedding mapping $\pi_h : M_h^n \to M^{2n}$ into the ambient phase space M^{2n}, which one should find in order to properly demonstrate integrability by quadratures.

The problem formulated above was posed and in part solved (as was mentioned above) for $n = 2$ in [11] and in [13] for a Henon-Heiles dynamical system which had previously been integrated [14, 15] using other tools. Her we generalize the approach of [11] for the general case $n \in \mathbb{Z}_+$ and proceed further in Chapter 3 to solve this problem in the case of a nonabelian Lie algebra $\mathcal{G}$ of polynomial invariants on $M^{2n} = T^*(\mathbb{R}^n)$, satisfying all the conditions of Mishchenko-Fomenko Theorem 1.5.

2.2. Define now the basic vector fields $K_j \in \Gamma(M^{2n})$, $j = \overline{1,n}$, generated by basic elements $H_j \in \mathcal{G}$ of an abelian Lie algebra $\mathcal{G}$ of invariants on M^{2n}, as follows:

$$-i_{K_j}\omega^{(2)} = dH_j \tag{2.1}$$

for all $j = \overline{1,n}$. It is easy to see that the condition $\{H_j, H_i\} = 0$ for all $i, j = \overline{1,n}$, yields also $[K_i, Kj] = 0$ for all $i, j = \overline{1,n}$. Taking into account that $\dim M^{2n} = 2n$ one obtains the equality $(\omega^{(2)})^n = 0$ identically on M^{2n}. This makes it possible to formulate the following Galisau-Reeb result.

Theorem 2.1. *Assume that an element $h \in \mathcal{G}^*$ is chosen to be regular and a Lie algebra $\mathcal{G}$ of invariants on M^{2n} is abelian. Then there exist differential 1-forms $h_j^{(1)} \in \Lambda^1(U(M_h^n))$, $j = \overline{1,n}$, where $U(M_h^n)$ is some open neighborhood of the integral submanifold $M_h^n \subset M^{2n}$, satisfying the following properties:*

i) $\omega^{(2)}\big|_{U(M_h^n)} = \sum_{j=1}^{n} dH_j \wedge h_j^{(1)}$;

ii) the exterior differentials $dh_j^{(1)} \in \Lambda^2(U(M_h^n))$ belong to the ideal $\mathcal{I}(\mathcal{G})$ in the Grassmann algebra $\Lambda(U(M_h^n))$, generated by 1-forms $dH_j \in \Lambda^1(U(M_h^n))$, $j = \overline{1,n}$.

◀ ***Proof.*** Consider the following identity on M^{2n}:

$$(\otimes_{j=1}^n i_{K_j})\left(\omega^{(2)}\right)^{n+1} = 0 = \pm(n+1)!\left(\wedge_{j=1}^n dH_j\right) \wedge \omega^{(2)}, \tag{2.2}$$

which implies that the 2-form $\omega^{(2)} \in \mathcal{I}(\mathcal{G})$. Whence, one can find 1-forms $h_j^{(1)} \in \Lambda^1(U(M_h^n))$, $j = \overline{1,n}$, satisfying the condition

$$\omega^{(2)}\Big|_{U(M_h^n)} = \sum_{j=1}^{n} dH_j \wedge h_j^{(1)}. \tag{2.3}$$

Since $\omega^{(2)} \in \Lambda^2(U(M_h^n))$ is nondegenerate on M^{2n}, it follows that all 1-forms $h_j^{(1)}$, $j = \overline{1,n}$, in (2.3) are independent on $U(M_h^n)$, proving part $i)$ of the theorem. As $d\omega^{(2)} = 0$ on M^{2n}, from (2.3) one gets that

$$\sum_{j=1}^{n} dH_j \wedge dh_j^{(1)} = 0 \tag{2.4}$$

on $U(M_h^n)$, hence it is obvious that $dh_j^{(1)} \in \mathcal{I}(\mathcal{G}) \subset \Lambda(U(M_h^n))$ for all $j = \overline{1,n}$, proving part $ii)$ of the theorem. ▶

Now we proceed to study properties of the integral submanifold $M_h^n \subset M^{2n}$ of the ideal $\mathcal{I}(\mathcal{G})$ in the Grassmann algebra $\Lambda(U(M_h^n))$. In general, the integral submanifold M_h^n is completely described [16] by means of the imbedding

$$\pi_h : M_h^n \to M^{2n} \tag{2.5}$$

and using this, one can reduce all vector fields $K_j \in \Gamma(M^{2n})$, $j = \overline{1,n}$, on the submanifold $M_h^n \subset M^{2n}$, since they are all evidently in its tangent space. If $\overline{K}_j \in \Gamma(M_h^n)$, $j = \overline{1,n}$, are the corresponding pulled-back vector fields $K_j \in \Gamma(M^{2n})$, $j = \overline{1,n}$, then by definition, the equality

$$\pi_{h*} \circ \overline{K}_j = K_j \circ \pi_h \tag{2.6}$$

holds for all $j = \overline{1,n}$. Similarly one can construct 1-forms $\overline{h}_j^{(1)} := \pi_h^* \circ h_j^{(1)} \in \Lambda^1(M_h^n)$, $j = \overline{1,n}$, which are characterized by the following Cartan-Jost [16] theorem.

Theorem 2.2. *The following assertions are true:*

i) the 1-forms $\overline{h}_j^{(1)} \in \Lambda^1(M_h^n)$, $j = \overline{1,n}$, *are independent on* M_h^n;

ii) the 1-forms $\overline{h}_j^{(1)} \in \Lambda^1(M_h^n)$, $j = \overline{1,n}$, *are exact on* M_h^n *and satisfy* $\overline{h}_j^{(1)}(\overline{K}_j) = \delta_{ij}$, $i, j = \overline{1,n}$.

◀ ***Proof.*** As the ideal $\mathcal{I}(\mathcal{G})$ is by definition vanishing on $M_h^n \subset M^{2n}$ and closed on $U(M_h^n)$, the integral submanifold M_h^n is well defined in the case of a regular element $h \in \mathcal{G}^*$. This implies that the imbedding (2.5) is nondegenerate on $M_h^n \subset M^{2n}$, or the 1-forms $\overline{h}_j^{(1)} := \pi_h^* \circ h_j^{(1)}$, $j = \overline{1,n}$, will persist in being independent if they are 1-forms $h_j^{(1)} \in \Lambda^1(U(M_h^n))$, $j = \overline{1,n}$, proving part *i)* of the theorem. Using property *ii)* of Theorem 2.1, one sees that on the integral submanifold $M_h^n \subset M^{2n}$ all 2-forms $d\overline{h}_j^{(1)} = 0$, $j = \overline{1,n}$. Consequently, owing to the Poincaré lemma [1, 16], the 1-forms $\overline{h}_j^{(1)} = d\overline{t}_j \in \Lambda^1(M_h^n)$, $j = \overline{1,n}$, for some mappings $\overline{t}_j : M_h^n \to \mathbb{R}$, $j = \overline{1,n}$, defining global coordinates on an appropriate universal covering of M_h^n. Consider now the following identity based on the representation (2.3):

$$i_{K_j}\omega^{(2)}\Big|_{U(M_h^n)} = -\sum_{i=1}^{n} h_i^{(1)}(K_j)dH_i := -dH_j, \tag{2.7}$$

which holds for any $j = \overline{1,n}$. As all $dH_j \in \Lambda^1(U(M_h^n))$, $j = \overline{1,n}$, are independent, from (2.7) one infers that $h_i^{(1)}(K_j) = \delta_{ij}$ for all $i, j = \overline{1,n}$. Recalling now that for any $i = \overline{1,n}$, $K_i \circ \pi_h = \pi_{h*} \circ K_i$, one readily computes that $\overline{h}_i^{(1)}(\overline{K}_j) = \pi_h^* h_i^{(1)}(\overline{K}_j) := h_i^{(1)}(\pi_{h*} \circ K_j) := h_i^{(1)}(K_j \circ \pi_h) = \delta_{ij}$ for all $i, j = \overline{1,n}$, proving part *ii)* of the theorem. ▶

The following is a simple consequence of Theorem 2.2:

Corollary 2.3. *Suppose that the vector fields* $K_j \in \Gamma(M^{2n})$, $j = \overline{1,n}$, *are parametrized globally along their trajectories by means of the corresponding parameters* $t_j : M^{2n} \to \mathbb{R}$, $j = \overline{1,n}$, *that is on the phase space* M^{2n}

$$d/dt_j := K_j \tag{2.8}$$

for all $j = \overline{1,n}$. *Then the following important equalities hold (up to constant normalizations) on the integral submanifold* $M_h^n \subset M^{2n}$:

$$t_j|_{M_h^n} = \overline{t}_j, \tag{2.9}$$

where $1 \leq j \leq n$.

2.3. To proceed with our investigation of the algebraic-analytical properties of the imbedding (2.5), we shall summarize some useful facts about the canonical transformations [1, 4] of symplectic manifolds and their generating functions.

Suppose that a symplectomorphism $\Phi : M^{2n} \to \tilde{M}^{2n}$ satisfies the condition

$$\Phi^*\tilde{\omega}^{(2)} = \omega^{(2)}, \tag{2.10}$$

where $\tilde{\omega}^{(2)} \in \Lambda^2(\tilde{M}^{2n})$ is a symplectic structure on $\tilde{M}^{2n} = T^*(\tilde{M}^n)$. Since by assumption $\omega^{(2)} = d(\mathrm{pr}^*\alpha^{(1)})$, where $\alpha^{(1)} \in \Lambda^1(\mathbb{R}^n)$ is defined by (1.1), and there exists a local 1-form $\tilde{\alpha}^{(1)} \in \Lambda^1(\tilde{M}^n)$ such that

$$\mathrm{pr}^*\alpha^{(1)} - \mathrm{pr}^*\tilde{\alpha}^{(1)} + d < \widetilde{p}, \widetilde{q} >= dS, \tag{2.11}$$

where locally, $S : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is a differentiable mapping called a generating function. Defining on $\tilde{M}^n$

$$\tilde{\alpha}^{(1)} = \sum_{j=1}^{n} \tilde{p}_j d\tilde{q}_j, \tag{2.12}$$

where $\tilde{p} \in T^*(\tilde{M}^n)$ are canonical local (momentum) coordinates, from (2.11), (2.12) and (1.1) one readily finds that

$$p_j = \partial S(q, \tilde{p})/\partial q_j, \qquad \tilde{q}_j = \partial S(q, \tilde{p})/\partial \tilde{p}_j \tag{2.13}$$

for any $j = \overline{1, n}$. The mapping $\Phi : M^{2n} \to \tilde{M}^{2n}$ should in local coordinates satisfy the following condition:

$$\det(\partial \widetilde{p}(q, p)/\partial p) \neq 0 \tag{2.14}$$

almost everywhere (a.e.) on M^{2n}. Since due to (2.14) one can define a.e. the mapping $p : \mathbb{R}^n \times \tilde{M}^n \to \mathbb{R}^n$, the equations (2.13) determine the generating function $S : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ up to some constant. And conversely [4], if there is given a generating function $S : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, satisfying a.e. the condition

$$\det(\partial^2 S(q, \widetilde{p})/\partial q \partial p) \neq 0 \tag{2.15}$$

on $\mathbb{R}^n \times \mathbb{R}^n$, then one can determine a canonical transformation $\Phi : M^{2n} \to \tilde{M}^{2n}$ of symplectic manifolds M^{2n} and $\tilde{M}^{2n}$, satisfying a.e. the condition (2.14).

Assume now additionally that the submanifold $\tilde{M}^n \subset \tilde{M}^{2n}$ coincides up to a diffeomorphism with the integral submanifold M_h^n of the ideal $\mathcal{I}(G)$ considered above. Then the corresponding symplectic manifold $\tilde{M}^{2n}_{(\mathcal{G}^*)} := \cup_{h \in \mathcal{G}^*} T^*(M_h^n)$ is the usual topological sum of cotangent spaces, giving rise to a natural fibration of the symplectic manifold $M^{2n} = T^*(\mathbb{R}^n)$:

$$M^{2n} = T^*(\mathbb{R}^n) \simeq \cup_{h \in \mathcal{G}^*} T^*(M_h^n) = \tilde{M}^{2n}_{(\mathcal{G}^*)}. \tag{2.16}$$

The representation (2.16) appears to be very useful for treating the imbedding (2.5). Namely, assume further that the integral submanifold $M_h^n \subset M^{2n}$ admits a.e. on M_h^n coordinate charts from the base space $\mathbb{R}^n$ of the entire phase space $M^{2n} = T^*(\mathbb{R}^n)$. This evidently means that the set of annihilating 1-forms $dH_j \in \Lambda^1(M^{2n})$, $j = \overline{1, n}$, on M_h^n must be solvable with respect to the cotangent differentials $dp_j \in \Lambda^1(M_h^n)$, $j = \overline{1, n}$:

$$\left\{ dH_j|_{M_h^n} = 0,\ j = \overline{1, n} \right\} \Rightarrow \left\{ dp_j = \sum_{k=1}^{n} Q_{jk}(q, p) dq_j : (q, p) \in T^*(M_h^n) \right\}, \tag{2.17}$$

where $Q : T^*(M_h^n) \to \mathrm{Hom}(\mathbb{R}^n)$ is an invertible a.e. mapping. The implication (2.17) and a result of Arnold [4] imply the existence on M_h^n of special cyclic coordinates realizing the isomorphisms $M_h^n \simeq \mathbb{T}^n \simeq \otimes_{j=1}^n \mathbb{S}_j^1$ (when it is compact and connected as will be assumed in the sequel). It follows directly from (2.16) that the phase space can be represented up to a symplectomorphism as

$$M^{2n} = T^*(\mathbb{R}^n) \simeq \cup_{h \in \mathcal{G}^*} T^* \left(\otimes_{j=1}^n \mathbb{S}_j^1 \right) \simeq \tilde{M}^{2n}_{(\mathcal{G}^*)} \tag{2.18}$$

and the integral submanifold $M_h^n \subset M^{2n}$ can be covered by charts with images in the base space $\mathbb{R}^n \subset T^*(\mathbb{R}^n)$ of the phase space M^{2n}. On the integral submanifold M_h^n there is induced a canonical 1-form $\alpha_h^{(1)} \in \Lambda^1(M_h^n)$ with respect to the imbedding (2.5), projected upon the base space $\mathbb{R}^n$:

$$\alpha_h^{(1)} := \pi_h^* \circ \mathrm{pr}^* \alpha^{(1)}. \tag{2.19}$$

This means that the following diagram is commutative:

$$\begin{array}{ccccccc}
T^*(T^*(\mathbb{R}^n)) & \overset{\pi_h^*}{\rightarrow} & T^*(M_h^n) & \simeq|_{\mathrm{loc}} & T^*(\mathbb{R}^n) & \overset{\mathrm{pr}^*}{\rightarrow} & T^*(T^*(\mathbb{R}^n)) \\
\mathrm{pr}' \downarrow & & \downarrow \mathrm{pr}_h & & \downarrow \mathrm{pr} & & \downarrow \mathrm{pr}' \\
M^{2n} = T^*(\mathbb{R}^n) & \overset{\pi_h}{\longleftarrow} & M_h^n & \simeq|_{\mathrm{loc}} & \mathbb{R}^n & \overset{\mathrm{pr}}{\longleftarrow} & M^{2n} = T^*(\mathbb{R}^n)
\end{array},$$

where $\mathrm{pr}' : T^*(T^*(\mathbb{R}^n)) \rightarrow T^*(\mathbb{R}^n)$ is the standard projection mapping. The representation (2.19) together with the isomorphism $M_h^n \simeq \otimes_{j=1}^n \mathbb{S}_j^1$ implies, in particular, that in circle-like coordinates $\mu \in \otimes_{j=1}^n \mathbb{S}_j^1$ the canonical 1-form $\alpha_h^{(1)} \in \Lambda^1(U(M_h^n))$ can be written as

$$\alpha_h^{(1)} = \sum_{j=1}^n w_j d\mu_j = \pi_h^* \circ \mathrm{pr}^* \alpha^{(1)}, \tag{2.20}$$

which is naturally lifted to a canonical 1-form $\mathrm{pr}_h^* \circ \alpha_h^{(1)} \in \Lambda^1(\, T^*(U(M_h^n)))$, where $\mathrm{pr}_h : T^*(M_h^n) \rightarrow M_h^n$ is the natural projection upon the base M_h^n, and $w : \otimes_{j=1}^n \mathbb{S}_j^1 \rightarrow \mathbb{R}^n$ is an smooth a.e. mapping parametrized by $h \in \mathcal{G}^*$. Thus, in local coordinates on $T^*(\otimes_{j=1}^n \mathbb{S}_j^1)$ the imbedding (2.5) parametrized by regular elements $h \in \mathcal{G}^*$ has the following form:

$$q_j = q_j(\mu; h), \qquad p_j = p_j(\mu; h), \tag{2.21}$$

where in virtue of (2.20) for any $j = \overline{1, n}$

$$p_j = \sum_{i=1}^n w_i(\mu; h) \partial \mu_i(q; h) / \partial q_j \Big|_{M_h^n}, \tag{2.22}$$

with the mapping $q := \mathrm{pr} \circ \pi_h : M_h^n \rightarrow \mathbb{R}^n$ being invertible owing to the implication (2.17), $\mu : \mathbb{R}^n \rightarrow \otimes_{j=1}^n \mathbb{S}_j^1$ is its inverse, and the mapping $w : (\otimes_{j=1}^n \mathbb{S}_j^1) \times \mathcal{G}^* \rightarrow \mathbb{R}^n$ is as yet not defined. The above analysis of the imbedding problem for the integral submanifold $M_h^n \subset M^{2n}$ in the case when the implication (2.17) is solvable, tells us that the Liouville-Arnold foliation (2.18) can be described effectively by choosing a special parametrization by elements $h \in \mathcal{G}^*$, for which the mapping $w : (\otimes_{j=1}^n \mathbb{S}_j^1) \times \mathcal{G}^* \rightarrow \mathbb{R}^n$ is separable in the variables $\mu \in \otimes_{j=1}^n \mathbb{S}_j^1$, that is on M_h^n the expressions

$$w_j = w_j(\mu_j; h) \tag{2.23}$$

should hold for all $j = \overline{1,n}$. Such a case is called [1, 4] the Hamilton-Jacobi separation of variables method which can now be applied very naturally to our problem of finding the embedding (2.5) by algebraic-analytical means.

To proceed we first apply first to $\tilde{M}^{2n}$ a canonical transformation into a new fibration $\tilde{M}^{2n}_{(\mathbb{R}^n)}$ of the phase space $M^{2n} = T^*(\mathbb{R}^n)$ as defined in (2.18), defined as follows:

$$M^{2n} = T^*(\mathbb{R}^n) \simeq \cup_{\tau \in \mathbb{R}^n} T^*(M^n_\tau) := \tilde{M}^{2n}_{(\mathbb{R}^n)}, \tag{2.24}$$

where M^n_τ is an integral submanifold of a dual integrable ideal $\mathcal{I}(h^{(1)}) \subset \Lambda(U(M^n_h))/\mathcal{I}(\mathcal{G}^*)$, which is generated by closed 1-forms $\tilde{h}^{(1)}_j \in \Lambda^1(U(M^n_h))/\mathcal{I}(\mathcal{G}^*)$, $j = \overline{1,n}$, obtained via an extension (preserving closedness) of the closed 1-forms $\bar{h}^{(1)}_j \in \Lambda^1(M^n_h)$, $j = \overline{1,n}$. This obviously means that the constructed ideal $\mathcal{I}(h^{(1)})$ is also integrable on $U(M^n_h) \subset M^{2n}$, and possesses integral submanifolds $M^n_\tau \subset U(M^n_h)$ parametrized by a constant vector parameter $\tau := (t_1, t_2, \ldots, t_n) \in \mathbb{R}^n$, composed of evolution parameters $t_j \in \mathbb{R}$, $j = \overline{1,n}$, of corresponding vector fields $K_j = d/dt_j$, $j = \overline{1,n}$, on the neighborhood $U(M^n_h) \subset M^{2n}$. It follows that the two fibrations $\tilde{M}^{2n}_{(\mathcal{G}^*)}$ and $\tilde{M}^{2n}_{(\mathbb{R}^n)}$ are locally diffeomorphic in an open neighborhood $U(M^n_h)$ of the integral submanifold $M^n_h \subset M^{2n}$. As they are symplectomorphic, one finds that

$$\alpha^{(1)}_\tau = -\sum_{j=1}^n t_j dh_j = \pi^*_\tau \mathrm{pr}^* \alpha^{(1)}, \tag{2.25}$$

holds on the integral submanifold M^n_τ for the canonical 1-form $\alpha^{(1)}_\tau \in \Lambda^1(M^n_\tau)$, where $\pi_\tau : M^n_\tau \to M^{2n}$ is the corresponding imbedding mapping. Recall now that each symplectomorphism of the manifold M^{2n} is described [1, 4] by means of the corresponding generating function $S : M^n_h \times \mathbb{R}^n \to \mathbb{R}$, where by definition,

$$\alpha^{(1)}_h = \alpha^{(1)}_\tau + dS. \tag{2.26}$$

Making use now of the expressions (2.20) and (2.25), one obtains the local relationship

$$\sum_{j=1}^n w_j d\mu_j + \sum_{j=1}^n t_j dh_j = dS(\mu; h) \tag{2.27}$$

on some open neighborhood $M^{2n}_{h,\tau} \subset U(M^n_h) \cap U(M^n_\tau)$, where $U(M^n_\tau) \subset M^{2n}$ is an open neighborhood of the integral submanifold $M^n_\tau \subset M^{2n}$.

Using the Hamilton-Jacobi separability condition (2.23) in (2.27), one gets

$$S(\mu; h) = \sum_{j=1}^n \int_{\mu^0_j}^{\mu_j} w_j(\lambda; h) d\lambda \tag{2.28}$$

for all $(\mu; h) \in M^{2n}_{h,\tau} \subset M^{2n}$, where $\mu^0 \in M^n_h \cap M^{2n}_{h,\tau}$ is some fixed point. Whence one obtains the desired expression for a vector parameter $\tau := (t_1, t_2, \ldots, t_n) \in \mathbb{R}^n$:

$$t_j = \partial S(\mu; h)/\partial h_j, \tag{2.29}$$

where $j = \overline{1,n}$ and $(\mu;h) \in M^{2n}_{h,\tau}$. On the other hand, in virtue of (2.9) the set of parameters (2.29) can be represented dually as differential forms

$$\bar{h}_j^{(1)} = dt_j = \partial dS\left(\mu;h\right)/\partial h_j = \sum_{i=1}^{n} \left(\partial w_i(\mu_i;h)/\partial h_j\right) d\mu_i \tag{2.30}$$

for all $j = \overline{1,n}$ on $M^{2n}_{h,\tau}$. Since the 1-forms $\bar{h}_j^{(1)} \in \Lambda^1(M^{2n}_{h,\tau})$, $j = \overline{1,n}$, are assumed here to be known explicitly from the characteristic equation (2.3), we can write them as

$$\bar{h}_j^{(1)} = \sum_{i=1}^{n} \bar{h}_{ji}(q,p) dq_i, \tag{2.31}$$

where $\bar{h}_{ji} : M^{2n}_{h,\tau} \to \mathbb{R}$, $i,j = \overline{1,n}$ are some algebraic expressions. Making use of the representations (2.21) and (2.28) and equation (2.31), the set of 1-forms (2.30) is reduced to the following purely differential-algebraic relationships on $M^{2n}_{h,\tau}$:

$$\partial w_i(\mu_i;h)/\partial h_j = \mathbf{P}_{ji}(\mu,w;h), \tag{2.32}$$

generalizing similar ones of [31, 18], where the characteristic functions $\mathbf{P}_{ji} : T^*(M^n_h) \to \mathbb{R}$, $i,j = \overline{1,n}$, are defined as follows:

$$\mathbf{P}_{ji}(\mu,w;h) := \sum_{i=1}^{n} \bar{h}_{js}(q(\mu;h), w\partial\mu/\partial q(\mu;h))\partial q_s/\partial\mu_i. \tag{2.33}$$

A simple analysis of the relationships (2.32) and (2.33) tells us that for all $j = \overline{1,n}$ and $i \neq s = \overline{1,n}$ the following algebraic relations

$$\partial\mathbf{P}_{ji}(\mu;w;h)/\partial w_s = 0 \tag{2.34}$$

must be satisfied identically if $i \neq s$. It is clear that the above set of purely differential-algebraic relationships (2.33) and (2.34) makes it possible to write down explicitly some first order compatible differential-algebraic equations, whose solution yields the first half of the desired imbedding (2.5) for the integral submanifold $M^n_h \subset M^{2n}$ in an open neighborhood $M^{2n}_{h,\tau} \subset M^{2n}$.

Let a mapping $q = \bar{q}(\mu;h)$, $(\mu;h) \in M^{2n}_{h,\tau}$, be an appropriate algebraic solution to equations (2.34). Substituting this into the characteristic equations (2.32), one obtains, owing to (2.34), the following characteristic Picard-Fuchs type [18, 19] equations on $T^*(M^n_h)$:

$$\partial w_i(\mu_i;h)/\partial h_j = \bar{\mathbf{P}}_{ji}(\mu_i,w_i;h) \tag{2.35}$$

for all $i,j = \overline{1,n}$, where by definition,

$$\begin{aligned}\bar{\mathbf{P}}_{ji}(\mu_i,w_i;h) &= \mathbf{P}_{ji}(\mu;w;h)|_{M^n_h \simeq \otimes_j^n \mathbb{S}^1_j} \\ &= \sum_{i=1}^{n} \bar{h}_{js}(q(\mu;h), w\partial\mu/\partial q(\mu;h))\partial q_s/\partial\mu_i.\end{aligned} \tag{2.36}$$

As a result of the above computations one can formulate the following main theorem.

Theorem 2.4. *The imbedding (2.5) for the integral submanifold $M^n_h \subset M^{2n}$ (compact and connected), parametrized by a regular parameter $h \in \mathcal{G}^*$, is an algebraic solution* *(up to diffeomorphism) to the set of characteristic Picard-Fuchs type equations (2.35) on $T^*(M_h^n)$, and can be represented in general case [19] in the following algebraic-geometric form:*

$$w_j^{n_j} + \sum_{s=1}^{n} c_{js}(\lambda; h) w_j^{n_j - s} = 0, \tag{2.37}$$

where $c_{js} : \mathbb{R} \times \mathcal{G}^* \to \mathbb{R}$, $s, j = \overline{1,n}$ are algebraic expressions, depending only on the functional structure of the original abelian Lie algebra $\mathcal{G}$ of invariants on M^{2n}. In particular, if the right-hand side of the characteristic equations (3.5) is independent of $h \in \mathcal{G}^*$, then this dependence will be linear in $h \in \mathcal{G}^*$.

It should be noted here that some ten years ago an attempt was made in [18, 19] to describe the explicit algebraic form of the Picard-Fuchs type equations (2.35) by means of straightforward calculations for the well known completely integrable Kowalewskaya top Hamiltonian system. The idea suggested in [18, 19] was in some aspects very close to that devised independently and thoroughly analyzed in [11] which did not consider the explicit form of the algebraic curves (2.37) starting from an abelian Lie algebra $\mathcal{G}$ of invariants on a canonically symplectic phase space M^{2n}.

As is well-known, a set of algebraic curves (2.31), prescribed via the above algorithm, to a given apriori abelian Lie algebra $\mathcal{G}$ of invariants on the canonically symplectic phase space $M^{2n} = T^*(\mathbb{R}^n)$ can be realized by means of a set of n_j-sheeted Riemannian surfaces $\Gamma_h^{n_j}$, $j = \overline{1,n}$, covering the corresponding real-valued cycles $\mathbb{S}_j^1$, $j = \overline{1,n}$, which generate the corresponding homology group $H_1(\mathbb{T}^n; \mathbb{Z})$ of the Arnold torus $\mathbb{T}^n \simeq \otimes_{j=1}^n \mathbb{S}_j^1$ diffeomorphic to the integral submanifold $M_h^n \subset M^{2n}$.

Thus, upon solving the set of algebraic equations (2.37) with respect to functions $w_j : \mathbb{S}_j^1 \times \mathcal{G}^* \to \mathbb{R}$, $j = \overline{1,n}$, from (2.29) one obtains a vector parameter $\tau = (t_1, \ldots, t_n) \in \mathbb{R}^n$ on M_h^n explicitly described by means of the following Abelian type equations:

$$t_j = \sum_{s=1}^{n} \int_{\mu_s^0}^{\mu_s} d\lambda \partial w_s(\lambda; h)/\partial h_j = \sum_{s=1}^{n} \int_{\mu_s^0}^{\mu_s} d\lambda \bar{\mathbf{P}}_{js}(\lambda, w_s; h), \tag{2.38}$$

where $j = \overline{1,n}$, $(\mu^0; h) \in \otimes_{j=1}^n \Gamma_h^{n_j} \times \mathcal{G}^*$. Using the expression (2.28) and recalling that the generating function $S : M_h^n \times \mathbb{R}^n \to \mathbb{R}$ is a one-valued mapping on an appropriate covering space $\left(\bar{M}_h^n; H_1(M_h^n; \mathbb{Z})\right)$, one can construct via the method of Arnold [4] the so called action-angle coordinates on M_h^n. Denote the basic oriented cycles on M_h^n by $\sigma_j \subset M_h^n$, $j = \overline{1,n}$. These cycles together with their duals generate homology group $H_1(M_h^n; \mathbb{Z}) \simeq H_1(\mathbb{T}^n; \mathbb{Z}) = \oplus_{j=1}^n \mathbb{Z}_j$. In virtue of the diffeomorphism $M_h^n \simeq \otimes_{j=1}^n \mathbb{S}_j^1$ described above, there is a one-to-one correspondence between the basic cycles of $H_1(M_h^n; \mathbb{Z})$ and those on the algebraic curves $\Gamma_h^{n_j}$, $j = \overline{1,n}$, given by (2.37):

$$\rho : H_1(M_h^n; \mathbb{Z}) \to \oplus_{j=1}^n \mathbb{Z}_j \sigma_{h,j}, \tag{2.39}$$

where $\sigma_{h,j} \subset \Gamma_h^{n_j}$, $j = \overline{1,n}$ are the corresponding real-valued cycles on the Riemann surfaces $\Gamma_h^{n_j}$, $j = \overline{1,n}$.

Assume that the following meanings of the mapping (2.39) are prescribed:

$$\rho(\sigma_i) := \oplus_{j=1}^n n_{ij} \sigma_{h,j} \tag{2.40}$$

for each $i = \overline{1,n}$, where $n_{ij} \in \mathbb{Z}$, $i, j = \overline{1,n}$ some fixed integers. Then following the Arnold construction [4, 18], one obtains the set of so called action-variables on $M_h^n \subset M^{2n}$:

$$\gamma_{j:} := \frac{1}{2\pi} \oint_{\sigma_j} dS = \sum_{s=1}^{n} n_{js} \oint_{\sigma_{h,s}} d\lambda w_s(\lambda; h), \tag{2.41}$$

where $j = \overline{1,n}$. It is easy to show [4, 16], that expressions (2.41) naturally define an a.e. differentiable invertible mapping

$$\xi : \mathcal{G}^* \to \mathbb{R}^n, \tag{2.42}$$

which enables one to treat the integral submanifold M_h^n as a submanifold $M_\gamma^n \subset M^{2n}$, where

$$M_\gamma^n := \left\{ u \in M^{2n} : \xi(h) = \gamma \in \mathbb{R}^n \right\}. \tag{2.43}$$

But, as was demonstrated in [18, 32], the functions (2.43) do not in general generate a global foliation of the phase space M^{2n}, as they are connected with both topological and analytical constraints. Since the functions (2.41) are evidently also commuting invariants on M^{2n}, one can define a further canonical transformation of the phase space M^{2n}, generated by the following relationship on $M_{h,\tau}^{2n}$:

$$\sum_{j=1}^{n} w_j d\mu_j + \sum_{j=1}^{n} \varphi_j d\gamma_j = dS(\mu; \gamma), \tag{2.44}$$

where $\varphi = (\varphi_1, \ldots, \varphi_n) \in \mathbb{T}^n$ are the so called angle-variables on the torus $\mathbb{T}^n \simeq M_h^n$ and $S : M_\gamma^n \times \mathbb{R}^n \to \mathbb{R}$ is the corresponding generating function. Whence it follows easily from (2.28) and (2.38) that

$$\varphi_j := \frac{\partial S(\mu; \gamma)}{\partial \gamma_j} = \sum_{s=1}^{n} \frac{\partial S(\mu; \gamma(h))}{\partial h_s} \frac{\partial h_s}{\partial \gamma_j} = \sum_{s=1}^{n} t_s \omega_{sj}(\gamma), \qquad \frac{1}{2\pi} \oint_{\sigma_j} d\varphi_k = \delta_{jk}, \tag{2.45}$$

where $\Omega := \left\{ \omega_{sj} : \mathbb{R}^n \to \mathbb{R},\ s, j = \overline{1,n} \right\}$ is the so called [4] frequency matrix, which is a.e. invertible on the integral submanifold $M_\gamma^n \subset M^{2n}$. As an evident result of (2.45), we claim that the evolution of any vector field $K_a \in \Gamma(M^{2n})$ for $a \in \mathcal{G}$ on the integral submanifold $M_\gamma^n \subset M^{2n}$ is quasiperiodic with a set of frequencies generated by the matrix $\Omega \overset{\text{a.e.}}{\in} \mathrm{Aut}(\mathbb{R}^n)$, defined above. As examples showing the effectiveness of the above method of construction of integral submanifold imbeddings for abelian integrable Hamiltonian systems, one can verify the Liouville-Arnold integrability of all Henon-Heiles and Neumann type systems described in detail in [21, 22]; however, we shall not dwell on this here.

3 Integral submanifold imbedding problem for a nonabelian Lie algebra of invariants

3.1. We shall assume below that there is given a Hamiltonian vector field $K \in \Gamma(M^{2n})$ on the canonically symplectic phase space $M^{2n} = T^*(\mathbb{R}^n)$, $n \in \mathbb{Z}_+$, which is endowed with a nonabelian Lie algebra $\mathcal{G}$ of invariants, satisfying all the conditions of the Mishchenko-Fomenko Theorem 1.5, that is

$$\dim \mathcal{G} + \operatorname{rank} \mathcal{G} = \dim M^{2n}. \tag{3.1}$$

Then, as was proved above, an integral submanifold $M_h^r \subset M^{2n}$ at a regular element $h \in \mathcal{G}^*$ is rank $\mathcal{G} = r$-dimensional and diffeomorphic (when compact and connected) to the standard r-dimensional torus $\mathbb{T}^r \simeq \otimes_{j=1}^r \mathbb{S}_j^1$. It is natural to ask the following question: How does one construct the corresponding integral submanifold imbedding

$$\pi_h : M_h^r \to M^{2n}, \tag{3.2}$$

which characterizes all possible orbits of the dynamical system $K \in \Gamma(M^{2n})$?

Having gained some experience in constructing the imbedding (3.2) in the case of the abelian Liouville-Arnold theorem on integrability by quadratures, we proceed below to study the integral submanifold $M_h^r \subset M^{2n}$ by means of Cartan's theory [3, 12, 16, 22] of the integrable ideals in the Grassmann algebra $\Lambda(M^{2n})$. Let $\mathcal{I}(G^*)$ be an ideal in $\Lambda(M^{2n})$, generated by independent differentials $dH_j \in \Lambda^1(M^{2n})$, $j = \overline{1,k}$, on an open neighborhood $U(M_h^r)$, where by definition, $r = \dim \mathcal{G}$. The ideal $\mathcal{I}(G^*)$ is obviously Cartan integrable [23, 16] with the integral submanifold $M_h^r \subset M^{2n}$ (at a regular element $h \in \mathcal{G}^*$), on which it vanishes, that is $\pi_h^* \mathcal{I}(G^*) = 0$. The dimension $\dim M_h^r = \dim M^{2n} - \dim \mathcal{G} = r = \operatorname{rank} \mathcal{G}$ due to the condition (3.1) imposed on the Lie algebra $\mathcal{G}$. It is useful to note here that owing to the inequality $r \leq k$ for the rank $\mathcal{G}$, one readily obtains from (3.1) that $\dim \mathcal{G} = k \geq n$. Since each base element $H_j \in \mathcal{G}$, $j = \overline{1,k}$, generates a symplectically dual vector field $K_j \in \Gamma(M^{2n})$, $j = \overline{1,k}$, one can try to study the corresponding differential system $K(\mathcal{G})$ which is also Cartan integrable on the entire open neighborhood $U(M_h^r) \subset M^{2n}$. Denote the corresponding dimension of the integral submanifold by $\dim M_h^k = \dim K(\mathcal{G}) = k$. Consider now an abelian differential system $K(\mathcal{G}_h) \subset K(\mathcal{G})$, generated by the Cartan subalgebra $\mathcal{G}_h \subset \mathcal{G}$ and its integral submanifold $\bar{M}_h^r \subset U(M_h^r)$. Since the Lie subgroup $G_h = \exp \mathcal{G}_h$ acts on the integral submanifold M_h^r invariantly (see Chapter 1) and $\dim \bar{M}_h^r = \operatorname{rank} \mathcal{G} = r$, it follows that $\bar{M}_h^r = M_h^r$. On the other hand, the system $K(\mathcal{G}_h) \subset K(\mathcal{G})$ by definition, meaning that the integral submanifold M_h^r is an invariant part of the integral submanifold $M_h^k \subset U(M_h^r)$ with respect to the Lie group $G = \exp \mathcal{G}$ – action on M_h^k. In this case one has the following result.

Lemma 3.1. *There exist just $(n-r) \in \mathbb{Z}_+$ vector fields $\tilde{F}_j \in K(\mathcal{G})/K(\mathcal{G}_h)$, $j = \overline{1, n-r}$, for which*

$$\omega^{(2)}(\tilde{F}_i, \tilde{F}_j) = 0 \tag{3.3}$$

on $U(M_h^r)$ for all $i, j = \overline{1, n-r}$.

◄ ***Proof.*** It is obvious that the matrix $\omega(\tilde{K}) := \left\{ \omega^{(2)}(\tilde{K}_i, \tilde{K}_j) : \ i, j = \overline{1,k} \right\}$ has on $U(M_h^r)$ the rank $\omega(\tilde{K}) = k - r$, since $\dim_{\mathbb{R}} \ker(\pi_h^* \omega^{(2)}) = \dim_{\mathbb{R}}(\pi_{h*} K(\mathcal{G}_h)) = r$ on M_h^r at the regular element $h \in \mathcal{G}^*$. Let us now complexify the tangent space $T(U(M_h^r))$ using its even dimensionality. Whence one can easily deduce that on $U(M_h^r)$ there exist just $(n-r) \in \mathbb{Z}_+$ vectors (not vector fields!) $\tilde{K}_j^{\mathbb{C}} \in K^{\mathbb{C}}(\mathcal{G})/K^{\mathbb{C}}(\mathcal{G}_h)$, $j = \overline{1, n-r}$, from the complexified [24] factor space $K^{\mathbb{C}}(\mathcal{G})/K^{\mathbb{C}}(\mathcal{G}_h)$. To show this, let us reduce the skew-symmetric matrix $\omega(\tilde{K}) \in \operatorname{Hom}(\mathbb{R}^{k-r})$ to its selfadjoint equivalent $\omega(\tilde{K}^{\mathbb{C}}) \in \operatorname{Hom}(\mathbb{C}^{n-r})$, having taken into account that $\dim_{\mathbb{R}} \mathbb{R}^{k-r} = \dim_{\mathbb{R}} \mathbb{R}^{k+r-2r} = \dim_{\mathbb{R}} \mathbb{R}^{2(n-r)} = \dim_{\mathbb{C}} \mathbb{C}^{n-r}$. Let now $f_j^{\mathbb{C}} \in \mathbb{C}^{n-r}$, $j = \overline{1, n-r}$, be eigenvectors of the nondegenerate selfadjoint matrix $\omega(\tilde{K}^{\mathbb{C}}) \in \mathrm{Hom}(\mathbb{C}^{n-r})$, that is

$$\omega\left(\tilde{K}^{\mathbb{C}}\right) f_j^{\mathbb{C}} = \tilde{\lambda}_j f_j^{\mathbb{C}}, \tag{3.4}$$

where $\tilde{\lambda}_j \in \mathbb{R}$, $j = \overline{1, n-r}$, and for all $i, j = \overline{1, n-r}$, $\langle f_i^{\mathbb{C}}, f_j^{\mathbb{C}} \rangle = \delta_{i,j}$. The above obviously means that in the basis $\left\{ f_j^{\mathbb{C}} \in K^{\mathbb{C}}(\mathcal{G})/K^{\mathbb{C}}(\mathcal{G}_h) : j = \overline{1, n-r} \right\}$ the matrix $\omega(\tilde{K}^{\mathbb{C}}) \in \mathrm{Hom}(\mathbb{C}^{n-r})$ is strictly diagonal and representable as

$$\omega(\tilde{K}^{\mathbb{C}}) = \sum_{j=1}^{n-r} \tilde{\lambda}_j f_j^{\mathbb{C}} \otimes_{\mathbb{C}} f_j^{\mathbb{C}}, \tag{3.5}$$

where $\otimes_{\mathbb{C}}$ is the usual Kronecker tensor product of vectors from $\mathbb{C}^{n-r}$. Owing to the construction of the complexified matrix $\omega\left(\tilde{K}^{\mathbb{C}}\right) \in \mathrm{Hom}(\mathbb{C}^{n-r})$, one sees that the space $K^{\mathbb{C}}(\mathcal{G})/K^{\mathbb{C}}(\mathcal{G}_h) \simeq \mathbb{C}^{n-r}$ carries a Kähler structure [24] with respect to which the following expressions

$$\omega\left(\tilde{K}\right) = \mathrm{Im}\, \omega(\tilde{K}), \qquad \langle \cdot, \cdot \rangle_{\mathbb{R}} = \mathrm{Re}\, \langle \cdot, \cdot \rangle \tag{3.6}$$

hold. Making use now of the representation (3.5) and expressions (3.6), one can find vector fields $\tilde{F}_j \in K(\mathcal{G})/K(\mathcal{G}_h)$, $j = \overline{1, n-r}$, such that

$$\omega\left(\tilde{F}\right) = \mathrm{Im}\, \omega\left(\tilde{F}^{\mathbb{C}}\right) = J, \tag{3.7}$$

holds on $U(M_h^r)$, where $J \in \mathrm{Sp}(\mathbb{C}^{n-r})$ is the standard symplectic matrix, satisfying the complex structure [24] identity $J^2 = -I$. In virtue of the normalization conditions $\langle f_j^{\mathbb{C}}, f_j^{\mathbb{C}} \rangle = \delta_{i,j}$, for all $i, j = \overline{1, n-r}$, one easily infers from (3.7) that $\omega^{(2)}\left(\tilde{F}_i, \tilde{F}_j\right) = 0$ for all $i, j = \overline{1, n-r}$, where by definition

$$\tilde{F}_j := \mathrm{Re}\, \tilde{F}_j^{\mathbb{C}} \tag{3.8}$$

for all $j = \overline{1, n-r}$, and this proves the lemma. ▶

Assume now that the Lie algebra $\mathcal{G}$ of invariants on M^{2n} has been split into a direct sum of subspaces as

$$\mathcal{G} = \mathcal{G}_h \oplus \widetilde{\mathcal{G}}_h, \tag{3.9}$$

where $\mathcal{G}_h$ is the Cartan subalgebra at a regular element $h \in \mathcal{G}^*$ (being abelian) and $\widetilde{\mathcal{G}}_h \simeq \mathcal{G}/\mathcal{G}_h$ is the corresponding complement to $\mathcal{G}_h$. Denote a basis of $\mathcal{G}_h$ as $\{\bar{H}_i \in \mathcal{G}_h : i = \overline{1, r}\}$, where $\dim \mathcal{G}_h = \mathrm{rank}\, \mathcal{G} = k \in \mathbb{Z}_+$, and correspondingly, a basis of $\widetilde{\mathcal{G}}_h$ as $\left\{ \tilde{H}_j \in \widetilde{\mathcal{G}}_h \simeq \mathcal{G}/\mathcal{G}_h : j = \overline{1, k-r} \right\}$. Then, owing to the results of Chapter 1, the following relationships hold:

$$\{\bar{H}_i, \bar{H}_j\} = 0, \qquad h(\{\bar{H}_i, \tilde{H}_s\}) = 0 \tag{3.10}$$

on the open neighborhood $U(M_h^r) \subset M^{2n}$ for all $i, j = \overline{1, r}$ and $s = \overline{1, k-r}$. We have as yet had nothing to say of expressions $h(\{\tilde{H}_s, \tilde{H}_m\})$ for $s, m = \overline{1, k-r}$. Making use of the representation (3.8) for our vector fields (if they exist) $\tilde{F}_j \in K(\mathcal{G})/K(\mathcal{G}_h)$, $j = \overline{1, n-r}$, one can write down the following expansion:

$$\tilde{F}_i = \sum_{j=1}^{k-r} c_{ji}(h)\tilde{K}_j, \tag{3.11}$$

where $i_{\tilde{K}_j}\omega^{(2)} := -d\tilde{H}_j$, $c_{ji} : \mathcal{G}^* \to \mathbb{R}$, $i = \overline{1, n-r}$, $j = \overline{1, k-r}$, are real-valued functions on $\mathcal{G}^*$, being defined uniquely as a result of (3.11). Whence it clearly follows that there exist invariants $\tilde{f}_s : U(M_h^r) \to \mathbb{R}$, $s = \overline{1, n-r}$, such that

$$-i_{\tilde{F}_s}\omega^{(2)} = \sum_{j=1}^{k-r} c_{js}(h)d\tilde{H}_j := d\tilde{f}_s, \tag{3.12}$$

where $\tilde{f}_s = \sum_{j=1}^{k-r} c_{js}(h)\tilde{H}_j$, $s = \overline{1, n-r}$, holds on $U(M_h^r)$.

3.2. To proceed further, let us look at the following identity which is similar to (2.2):

$$\left(\otimes_{j=1}^{r} i_{\bar{K}_j}\right)\left(\otimes_{s=1}^{n-r} i_{\tilde{F}_s}\right)\left(\omega^{(2)}\right)^{n+1} = 0 = \pm(n+1)!\left(\wedge_{j=1}^{r} d\bar{H}_j\right)\left(\wedge_{s=1}^{n-r} d\tilde{f}_s\right)\wedge\omega^{(2)}, \tag{3.13}$$

on $U(M_h^r)$. Whence, the following result is easily obtained using Cartan theory [3, 16]:

Lemma 3.2. *The symplectic structure $\omega^{(2)} \in \Lambda^2(U(M_h^r))$ has the following canonical representation:*

$$\omega^{(2)}\Big|_{U(M_h^r)} = \sum_{j=1}^{r} d\bar{H}_j \wedge \bar{h}_j^{(1)} + \sum_{s=1}^{n-r} d\tilde{f}_s \wedge \tilde{h}_s^{(1)}, \tag{3.14}$$

where $\bar{h}_j^{(1)}$, $\tilde{h}_s^{(1)} \in \Lambda^1(U(M_h^r))$, $j = \overline{1, r}$, $s = \overline{1, n-r}$.

The expression (3.14) obviously means, that on $U(M_h^r) \subset M^{2n}$ the differential 1-forms $\bar{h}_j^{(1)}$, $\tilde{h}_s^{(1)} \in \Lambda^1(U(M_h^r))$, $j = \overline{1, r}$, $s = \overline{1, n-r}$, are independent together with exact 1-forms $d\bar{H}_j$, $j = \overline{1, r}$, and $d\tilde{f}_s$, $s = \overline{1, n-r}$. Since $d\omega^{(2)} = 0$ on M^{2n} identically, from (3.14) one obtains that the differentials $d\bar{h}_j^{(1)}$, $d\tilde{h}_s^{(1)} \in \Lambda^2(U(M_h^r))$, $j = \overline{1, r}$, $s = \overline{1, n-r}$, belong to the ideal $\mathcal{I}(\widetilde{\mathcal{G}}_h) \subset \mathcal{I}(\mathcal{G}^*)$, generated by exact forms $d\tilde{f}_s$, $s = \overline{1, n-r}$, and $d\bar{H}_j$, $j = \overline{1, r}$, for all regular $h \in \mathcal{G}^*$. Consequently, one obtains the following analog of the Galisau-Reeb Theorem 2.1.

Theorem 3.3. *Let a Lie algebra $\mathcal{G}$ of invariants on the symplectic space M^{2n} be non-abelian and satisfy the Mishchenko-Fomenko condition (3.1). At a regular element $h \in \mathcal{G}^*$ on some open neighborhood $U(M_h^r)$ of the integral submanifold $M_h^r \subset M^{2n}$ there exist differential 1-forms $\bar{h}_j^{(1)}$, $j = \overline{1, n}$, and $\tilde{h}_s^{(1)}$, $s = \overline{1, n-r}$, satisfying the following properties:*

i) $\omega^{(2)}\big|_{U(M_h^r)} = \sum_{j=1}^{r} d\bar{H}_j \wedge \bar{h}_j^{(1)} + \sum_{s=1}^{n-r} d\tilde{f}_s \wedge \tilde{h}_s^{(1)}$, where $\bar{H}_j \in \mathcal{G}$, $j = \overline{1, r}$, is a basis of the Cartan subalgebra $\mathcal{G}_h \subset \mathcal{G}$ (being abelian), and $\tilde{f}_s \in \mathcal{G}$, $s = \overline{1, n-r}$, are invariants from the complementary space $\widetilde{\mathcal{G}}_h \simeq \mathcal{G}/\mathcal{G}_h$;

ii) 1-forms $\bar{h}_j^{(1)} \in \Lambda^1(U(M_h^r))$, $j = \overline{1,r}$, *and* $\tilde{h}_s^{(1)} \in \Lambda^1(U(M_h^r))$, $s = \overline{1,n-r}$, *are exact on* M_h^r *and satisfy the equations:* $\bar{h}_j^{(1)}(\bar{K}_i) = \delta_{i,j}$ *for all* $i,j = \overline{1,r}$, $\bar{h}_j^{(1)}(\tilde{F}_s) = 0$ *and* $\tilde{h}_s^{(1)}(\bar{K}_j) = 0$ *for all* $j = \overline{1,r}$, $s = \overline{1,n-r}$, *and* $\tilde{h}_s^{(1)}(\tilde{F}_m) = \delta_{s,m}$ *for all* $s,m = \overline{1,n-r}$.

◀ ***Proof.*** Obviously we need to prove only the last statement *ii)*. Making use of Theorem 3.3, one finds on the integral submanifold $M_h^r \subset M^{2n}$ the differential 2-forms $d\bar{h}_j^{(1)} \in \Lambda^2(U(M_h^r))$, $j = \overline{1,r}$, and $d\tilde{h}_s^{(1)} \in \Lambda^2(U(M_h^r))$, $s = \overline{1,n-r}$, are identically vanishing. This means in particular, owing to the classical Poincaré lemma [1, 4, 16], that there exist some exact 1-forms $d\bar{t}_{h,j} \in \Lambda^1(U(M_h^r))$, $j = \overline{1,r}$, and $d\tilde{t}_{h,s} \in \Lambda^1(U(M_h^r))$, $s = \overline{1,n-r}$, where $\bar{t}_{h,j} : M_h^r \to \mathbb{R}$, $j = \overline{1,r}$, and $\tilde{t}_{h,s} : M_h^r \to \mathbb{R}$, $s = \overline{1,n-r}$, are smooth independent a.e. functions on M_h^r; they are one-valued on an appropriate covering of the manifold $M_h^r \subset M^{2n}$ and supply global coordinates on the integral submanifold M_h^r. Using the representation (3.14), one can easily obtain that

$$-\, i_{\bar{K}_i}\omega^{(2)}\Big|_{U(M_h^r)} = \sum_{j=1}^{r} d\bar{H}_j\bar{h}_j^{(1)}(\bar{K}_i) + \sum_{s=1}^{n-r} d\tilde{f}_s\tilde{h}_s^{(1)}(\bar{K}_i) = d\bar{H}_i \tag{3.15}$$

for all $i = \overline{1,r}$ and

$$-\, i_{\tilde{F}_m}\omega^{(2)}\Big|_{U(M_h^r)} = \sum_{j=1}^{r} d\bar{H}_j\bar{h}_j^{(1)}(\tilde{F}_m) + \sum_{s=1}^{n-r} d\tilde{f}_s\tilde{h}_s^{(1)}(\tilde{F}_m) = d\tilde{f}_m \tag{3.16}$$

for all $m = \overline{1,n-r}$. Whence, from (3.15) it follows on that on $U(M_h^r)$,

$$\bar{h}_j^{(1)}(\bar{K}_i) = \delta_{i,j}, \qquad \tilde{h}_s^{(1)}(\bar{K}_i) = 0 \tag{3.17}$$

for all $i,j = \overline{1,r}$ and $s = \overline{1,n-r}$, and similarly, from (3.16) it follows that on $U(M_h^r)$,

$$\bar{h}_j^{(1)}(\tilde{F}_m) = 0, \qquad \tilde{h}_s^{(1)}(\tilde{F}_m) = 0 \tag{3.18}$$

for all $j = \overline{1,r}$ and $s,m = \overline{1,n-r}$. Thus the theorem is proved. ▶

Having now defined global evolution parameters $t_j : M^{2n} \to \mathbb{R}$, $j = \overline{1,r}$, of the corresponding vector fields $\bar{K}_j = d/dt_j$, $j = \overline{1,r}$, and local evolution parameters $\tilde{t}_s : M^{2n} \cap U(M_h^r) \to \mathbb{R}$, $s = \overline{1,n-r}$, of the corresponding vector fields $\tilde{F}_s\Big|_{U(M_h^r)} := d/d\tilde{t}_s$, $s = \overline{1,n-r}$, one can easily see from (3.18) that the equalities

$$t_j|_{U(M_h^r)} = \bar{t}_j, \qquad \tilde{t}_s|_{U(M_h^r)} = \tilde{t}_{h,s} \tag{3.19}$$

hold for all $j = \overline{1,r}$, $s = \overline{1,n-r}$, up to constant normalizations. Thereby, one can develop a new method, similar to that of Chapter 2, for studying the integral submanifold imbedding problem in the case of the nonabelian Liouville-Arnold integrability theorem.

Before starting, it is interesting to note that the system of invariants

$$\mathcal{G}_\tau := \mathcal{G}_h \oplus \mathrm{span}_{\mathbb{R}}\{\tilde{f}_s \in \mathcal{G}/\mathcal{G}_h : s = \overline{1,n-r}\}$$

constructed above, comprise a new involutive (abelian) complete algebra $\mathcal{G}_\tau$, to which one can apply the abelian Liouville-Arnold theorem on integrability by quadratures and the integral submanifold imbedding theory devised in Chapter 2, in order to obtain exact solutions by means of algebraic-analytical expressions. Namely, the following corollary holds.

Corollary 3.5. *Assume that a nonabelian Lie algebra $\mathcal{G}$ satisfies the Mishchenko-Fomenko condition (3.1) and $M_h^r \subset M^{2n}$ is its integral submanifold (compact and connected) at a regular element $h \in \mathcal{G}^*$, is diffeomorphic to the standard torus $\mathbb{T}^r \simeq M_{h,\tau}^n$. Assume also that the dual complete abelian algebra $\mathcal{G}_\tau$ $(\dim \mathcal{G}_\tau = n = 1/2 \dim M^{2n})$ of independent invariants constructed above is globally defined. Then its integral submanifold $M_{h,\tau}^n \subset M^{2n}$ is diffeomorphic to the standard torus $\mathbb{T}^n \simeq M_{h,\tau}^n$, and contains the torus $\mathbb{T}^r \simeq M_h^r$ as a direct product with some completely degenerate torus $\mathbb{T}^{n-r}$, that is $M_{h,\tau}^n \simeq M_h^r \times \mathbb{T}^{n-r}$.*

Thus, having successfully applied the algorithm of Chapter 2 to the algebraic-analytical characterization of integral submanifolds of a nonabelian Liouville-Arnold integrable Lie algebra $\mathcal{G}$ of invariants on the canonically symplectic manifold $M^{2n} \simeq T^*(\mathbb{R}^n)$, one can produce a wide class of exact solutions represented by quadratures – which is just what we set out to find. At this point it is necessary to note that up to now the (dual to $\mathcal{G}$) abelian complete algebra $\mathcal{G}_\tau$ of invariants at a regular $h \in \mathcal{G}^*$ was constructed only on some open neighborhood $U(M_h^r)$ of the integral submanifold $M_h^r \subset M^{2n}$. As was mentioned before, the global existence of the algebra $\mathcal{G}_\tau$ strongly depends on the possibility of extending these invariants to the entire manifold M^{2n}. This possibility is in 1–1 correspondence with the existence of a global complex structure [24] on the reduced integral submanifold $\tilde{M}_{h,\tau}^{2(n-r)} := M_h^k/G_h$, induced by the reduced symplectic structure $\pi_\tau^* \omega^{(2)} \in \Lambda^2(M_h^k/G_h)$, where $\pi_\tau : M_h^k \to M^{2n}$ is the imbedding for the integrable differential system $K(\mathcal{G}) \subset \Gamma(M^{2n})$, introduced above. If this is the case, the resulting complexified manifold ${}^{\mathbb{C}}\tilde{M}_{h,\tau}^{n-r} \simeq \tilde{M}_{h,\tau}^{2(n-r)}$ will be endowed with a Kählerian structure, which makes it possible to produce the dual abelian algebra $\mathcal{G}_\tau$ as a globally defined set of invariants on M^{2n}. This problem will be analyzed in more detail in Chapter 5.

4 Examples

4.1. Below we consider some examples of nonabelian Liouville-Arnold integrability by quadratures covered by Theorem 1.5.

Example 4.1. ***Point vortices in the plane.*** Consider $n \in \mathbb{Z}_+$ point vortices on the plane $\mathbb{R}^2$, described by the Hamiltonian function

$$H = -\frac{1}{2\pi} \sum_{i \neq j=1}^{n} \xi_i \xi_j \ln \|q_i - p_j\| \tag{4.1}$$

with respect to the following partially canonical symplectic structure on $M^{2n} \simeq T^*(\mathbb{R}^n)$:

$$\omega^{(2)} = \sum_{j=1}^{n} \xi_j dp_j \wedge dq_j, \tag{4.2}$$

where $(p_j, q_j) \in \mathbb{R}^2$, $j = \overline{1,n}$, are coordinates of the vortices in $\mathbb{R}^2$. There exist three additional invariants

$$P_1 = \sum_{j=1}^{n} \xi_j q_j, \qquad P_2 = \sum_{j=1}^{n} \xi_j p_j, \qquad P = \frac{1}{2}\sum_{j=1}^{n} \xi_j \left(q_j^2 + p_j^2\right), \tag{4.3}$$

satisfying the following Poisson bracket conditions:

$$\begin{aligned} &\{P_1, P_2\} = -\sum_{j=1}^{n} \xi_j, \qquad \{P_1, P\} = -P_2, \\ &\{P_2, P\} = P_1, \qquad \{P, H\} = 0 = \{P_j, H\}. \end{aligned} \tag{4.4}$$

It is evident, that invariants (4.1) and (4.3) comprise on $\sum\limits_{j=1}^{n} \xi_j = 0$ a four-dimensional Lie algebra $\mathcal{G}$, whose rank $\mathcal{G} = 2$. Indeed, assume a regular vector $h \in \mathcal{G}^*$ is chosen, and parametrized by real values $h_j \in \mathbb{R}$, $j = \overline{1,4}$, where

$$h(P_i) = h_i, \qquad h(P) = h_3, \qquad h(H) = h_4, \tag{4.5}$$

and $i = \overline{1,2}$. Then, one can easily verify that the element

$$Q_h = \left(\sum_{j=1}^{n} \xi_j\right) P - \sum_{i=1}^{n} h_i P_i \tag{4.6}$$

belongs to the Cartan Lie subalgebra $\mathcal{G}_h \subset \mathcal{G}$, that is

$$h(\{Q_h, P_i\}) = 0, \qquad h(\{Q_h, P\}) = 0. \tag{4.7}$$

Since $\{Q_h, H\} = 0$ for all values $h \in \mathcal{G}^*$, we claim that $\mathcal{G}_h = \mathrm{span}_{\mathbb{R}}\{H, Q_h\}$ – the Cartan subalgebra of $\mathcal{G}$. Thus, rank $\mathcal{G} = \dim \mathcal{G}_h = 2$, and one comes right away that the condition (3.1)

$$\dim M^{2n} = 2n = \mathrm{rank}\, \mathcal{G} + \dim \mathcal{G} = 6 \tag{4.8}$$

holds only if $n = 3$. Thereby, the following theorem is proved.

Theorem 4.1. *The three – vortex problem (4.1) on the plane $\mathbb{R}^2$ is nonabelian Liouville-Arnold integrable by quadratures on the phase space $M^6 \simeq T^*(\mathbb{R}^3)$ with the symplectic structure (4.2).*

As a result, the corresponding integral submanifold $M_h^2 \subset M^6$ is two-dimensional and diffeomorphic (when compact and connected) to the torus $\mathbb{T}^2 \simeq M_h^2$, on which the motions are quasiperiodic functions of the evolution parameter.

Concerning the Corollary 3.5, the dynamical system (4.1) is also abelian Liouville-Arnold integrable with an extended integral submanifold $M_{h,\tau}^3 \subset M^6$, which can be found via the scheme suggested in Chapter 3. Using simple calculations, one obtains an additional invariant $Q = \left(\sum\limits_{j=1}^{3} \xi_j\right) P - \sum\limits_{i=1}^{3} P_i^2 \notin \mathcal{G}$, which commutes with H and P of $\mathcal{G}_h$.

Therefore, there exists a new complete dual abelian algebra $\mathcal{G}_\tau = \mathrm{span}_{\mathbb{R}}\{Q, P, H\}$ of independent invariants on M^6 with $\dim \mathcal{G}_\tau = 3 = 1/2 \dim M^6$, whose integral submanifold $M^3_{h,\tau} \subset M^6$ (when compact and connected) is diffeomorphic to the torus $\mathbb{T}^3 \simeq M^2_h \times \mathbb{S}^1$.

Note also here, that the above additional invariant $Q \in \mathcal{G}_\tau$ can be naturally extended to the case of an arbitrary number $n \in \mathbb{Z}_+$ of vortices as follows: $Q = \left(\sum_{j=1}^n \xi_j\right) P - \sum_{i=1}^n P_i^2 \in \mathcal{G}_\tau$, which obviously also commutes with invariants (4.1) and (4.3) on the entire phase space M^{2n}.

Example 4.2. ***A material point motion in a central field.*** Consider the motion of a material point in the space $\mathbb{R}^3$ under a central potential field whose Hamiltonian

$$H = \frac{1}{2} \sum_{j=1}^{3} p_j^2 + Q(\|q\|), \tag{4.9}$$

contains a central field $Q : \mathbb{R}_+ \to \mathbb{R}$. The motion is takes place in the canonical phase space $M^6 = T^*(\mathbb{R}^3)$, and possesses three additional invariants:

$$P_1 = p_2 q_3 - pq, \qquad P_2 = p_3 q_1 - p_1 q_3, \qquad P_3 = p_1 q_2 - p_2 q_1, \tag{4.10}$$

satisfying the following Poisson bracket relations:

$$\{P_1, P_2\} = P_3, \qquad \{P_3, P_1\} = P_2, \qquad \{P_2, P_3\} = P_1. \tag{4.11}$$

Since $\{H, P_j\} = 0$ for all $j = \overline{1,3}$, one sees that the problem under consideration has a four-dimensional Lie algebra $\mathcal{G}$ of invariants, isomorphic to the classical rotation Lie algebra $so(3) \times \mathbb{R} \simeq \mathcal{G}$. Let us show that at a regular element $h \in \mathcal{G}^*$ the Cartan subalgebra $\mathcal{G}_h \subset \mathcal{G}$ has the dimension $\dim \mathcal{G}_h = 2 = \mathrm{rank}\, \mathcal{G}$. Indeed, one easily verifies that the invariant

$$P_h = \sum_{j=1}^{3} h_j P_j \tag{4.12}$$

belongs to the Cartan subalgebra $\mathcal{G}_h$, that is

$$\{H, P_h\} = 0, \qquad h(\{P_h, P_j\}) = 0 \tag{4.13}$$

for all $j = \overline{1,3}$. Thus, as the Cartan subalgebra $\mathcal{G}_h = \mathrm{span}_{\mathbb{R}}\{H \text{ and } P_h \subset \mathcal{G}\}$, one gets $\dim \mathcal{G}_h = 2 = \mathrm{rank}\, \mathcal{G}_h$, and the Mishchenko-Fomenko condition 3.1

$$\dim M^6 = 6 = \mathrm{rank}\, \mathcal{G} + \dim \mathcal{G} = 4 + 2 \tag{4.14}$$

holds. Hence one can prove its integrability by quadratures via the nonabelian Liouville Liouville-Arnold Theorem 1.5 and obtain the following theorem:

Theorem 4.3. *It follows from Theorem 1.5 that the free material point motion in $\mathbb{R}^3$ is a completely integrable by quadratures dynamical system on the canonical symplectic phase space $M^6 = T^*(\mathbb{R}^3)$. The corresponding integral submanifold $M_h^2 \subset M^6$ at a regular element $h \in \mathcal{G}^*$ (if compact and connected) is two-dimensional and diffeomorphic to the standard torus $\mathbb{T}^2 \simeq M_h^2$.*

Making use of the integration algorithm devised in Chapters 1 and 2, one can readily obtain the corresponding integral submanifold imbedding mapping $\pi_h : M_h^2 \to M^6$ by means of algebraic-analytical expressions and transformations.

There are clearly many other interesting nonabelian Liouville-Arnold integrable Hamiltonian systems on canonically symplectic phase spaces that arise in applications, which can similarly be integrated using algebraic-analytical means. We hope to study several of these systems in detail elsewhere.

5 Existence problem for a global set of invariants

5.1 It was proved in Chapter 3, that locally, in some open neighborhood $U(M_h^r) \subset M^{2n}$ of the integral submanifold $M_h^r \subset M^{2n}$ one can find by algebraic-analytical means just $n-r \in \mathbb{Z}_+$ independent vector fields $\tilde{F}_j \in K(\mathcal{G})/K(\mathcal{G}_h) \cap \Gamma(U(M_h^r))$, $j = \overline{1, n-r}$, satisfying the condition (3.3). Since each vector field $\tilde{F}_j \in K(\mathcal{G})/K(\mathcal{G}_h)$, $j = \overline{1, n-r}$, is generated by an invariant $\tilde{H}_j \in \mathcal{D}(U(M_h^r))$, $j = \overline{1, n-r}$, it follows readily from (3.3) that

$$\{\tilde{H}_i, \tilde{H}_j\} = 0 \tag{5.1}$$

for all $i, j = \overline{1, n-r}$. Thus, on an open neighborhood $U(M_h^r)$ there exist just $n-r$ invariants in addition to $\tilde{H}_j \in \mathcal{D}(U(M_h^r))$, $j = \overline{1, n-r}$, all of which are in involution. Denote as before this new set of invariants as $\mathcal{G}_\tau$, keeping in mind that $\dim \mathcal{G}_\tau = r + (n-r) = n \in \mathbb{Z}_+$. Whence, on an open neighborhood $U(M_h^r) \subset M^{2n}$ we have constructed the set $\mathcal{G}_\tau$ of just $n = 1/2 \dim M^{2n}$ invariants commuting with each other, thereby guaranteeing via the abelian Liouville-Arnold theorem its local complete integrability by quadratures. Consequently, there exists locally a mapping $\pi_\tau : M_{h,\tau}^k \to M^{2n}$, where $M_{h,\tau}^k := U(M_h^r) \cap M_\tau^k$ is the integral submanifold of the differential system $K(\mathcal{G})$, and one can therefore describe the behavior of integrable vector fields on the reduced manifold $\bar{M}_{h,\tau}^{2(n-r)} := M_{h,\tau}^{k-r}/G_h$. For global integrability properties of a given set $\mathcal{G}$ of invariants on $\left(M^{2n}, \omega^{(2)}\right)$, satisfying the Mishchenko-Fomenko condition (3.1), it is necessary to have the additional set of invariants $\tilde{H}_j \in \mathcal{D}(U(M_h^r))$, $j = \overline{1, n-r}$, extended from $U(M_h^r)$ to the entire phase space M^{2n}. This problem evidently depends on the existence of extensions of vector fields $\tilde{F}_j \in \Gamma(U(M_h^r))$, $j = \overline{1, n-r}$, from the neighborhood $U(M_h^r) \subset M^{2n}$ to the whole phase space M^{2n}. On the other hand, as stated before, the existence of such a continuation depends intimately on the properties of the complexified differential system $K^{\mathbb{C}}(\mathcal{G})/K^{\mathbb{C}}(\mathcal{G}_h)$, which has a nondegenerate complex metric $\omega\left(\tilde{K}^{\mathbb{C}}\right) : T\left(\bar{M}_{h,\tau}^{2(n-r)}\right)^{\mathbb{C}} \times T\left(\bar{M}_{h,\tau}^{2(n-r)}\right)^{\mathbb{C}} \to \mathbb{C}$, induced by the symplectic structure $\omega^{(2)} \in \Lambda^2(M^{2n})$. This point can be clarified more by using the notion [24–27] of a Kähler manifold and some of the associated constructions presented above. Namely, consider the local isomorphism $T\left(\bar{M}_{h,\tau}^{2(n-r)}\right)^{\mathbb{C}} \simeq T\left({}^{\mathbb{C}}\bar{M}_{h,\tau}^{n-r}\right)$, where ${}^{\mathbb{C}}\bar{M}_{h,\tau}^{n-r}$ is the complex $(n-r)$-dimensional local integral submanifold of the complexified differential system $K^{\mathbb{C}}(\mathcal{G})/K^{\mathbb{C}}(\mathcal{G}_h)$. This means that the space $T\left(\bar{M}_{h,\tau}^{2(n-r)}\right)$ is endowed with the standard almost complex structure

$$J : T\left(\bar{M}_{h,\tau}^{2(n-r)}\right) \to T\left(\bar{M}_{h,\tau}^{2(n-r)}\right), \qquad J^2 = -1, \tag{5.2}$$

such that the 2-form $\omega(\tilde{K}) := \mathrm{Im}\,\omega\left(\tilde{K}^{\mathbb{C}}\right) \in \Lambda^2\left(\bar{M}_{h,\tau}^{2(n-r)}\right)$ induced from the above metric on $T\left({}^{\mathbb{C}}\bar{M}_{h,\tau}^{n-r}\right)$ is closed, that is $d\omega(\tilde{K}) = 0$. If this is the case, the almost complex structure on the manifold $T\left(\bar{M}_{h,\tau}^{2(n-r)}\right)$ is said to be integrable. Define the proper complex manifold ${}^{\mathbb{C}}\bar{M}_{h,\tau}^{n-r}$, on which one can then define global vector fields $\tilde{F}_j \in K(\mathcal{G})/K(\mathcal{G}_h)$, $j = \overline{1, n-r}$, which are being sought for the involutive algebra $\mathcal{G}_\tau$ of invariants on M^{2n} to be integrable by quadratures via the abelian Liouville-Arnold theorem. Thus the following theorem can be obtained.

Theorem 5.1. *A nonabelian set $\mathcal{G}$ of invariants on the symplectic space $M^{2n} \simeq T^*(R^n)$, satisfying the Mishchenko-Fomenko condition 3.1, admits algebraic-analytical integration by quadratures for the integral submanifold imbedding $\pi_h : M_h^r \to M^{2n}$, if the corresponding complexified reduced manifold ${}^{\mathbb{C}}\bar{M}_{h,\tau}^{n-r} \simeq \bar{M}_{h,\tau}^{2(n-r)} = M_{h,\tau}^{k-r}/G_h$ of the differential system $K^{\mathbb{C}}(\mathcal{G})/K^{\mathbb{C}}(\mathcal{G}_h)$ is Kählerian with respect to the standard almost complex structure (5.1) and the nondegenerate complex metric $\omega\left(\tilde{K}^{\mathbb{C}}\right) : T\left(\bar{M}_{h,\tau}^{2(n-r)}\right)^{\mathbb{C}} \times T\left(\bar{M}_{h,\tau}^{2(n-r)}\right)^{\mathbb{C}} \to \mathbb{C}$, induced by the symplectic structure $\omega^{(2)} \in \Lambda^2(M^{2n})$ is integrable, that is $d\,\mathrm{Im}\,\omega(\tilde{K}^{\mathbb{C}}) = 0$.*

Theorem 5.1 shows, in particular, that nonabelian Liouville-Arnold integrability by quadratures does not in general imply integrability via the abelian Liouville-Arnold theorem; it actually depends on certain topological obstructions associated with the Lie algebra structure of invariants $\mathcal{G}$ on the phase space M^{2n}. We hope to explore this intriguing problem in another place.

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