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On Exact Solution of a Classical 3D Integrable Model

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Abstract

We investigate some classical evolution model in the discrete $2+1$ space-time. A map, giving an one-step time evolution, may be derived as the compatibility condition for some systems of linear equations for a set of auxiliary linear variables. Dynamical variables for the evolution model are the coefficients of these systems of linear equations. Determinant of any system of linear equations is a polynomial of two numerical quasimomenta of the auxiliary linear variables. For one, this determinant is the generating functions of all integrals of motion for the evolution, and on the other hand it defines a high genus algebraic curve. The dependence of the dynamical variables on the space-time point (exact solution) may be expressed in terms of theta functions on the jacobian of this curve. This is the main result of our paper.

1 Introduction

In this paper we give an exact solution of a classical evolution model in discrete $2+1$ space-time. This model was formulated in [1]. The map of dynamical variables, governing the one-step evolution, was derived as the compatibility condition for two sets of linear relations, associated with the usual graphical representation of left and right hand sides of the famous Yang – Baxter equation. Such form of the zero curvature condition generalizes the usual concept of the local Yang – Baxter equation as the zero curvature condition for 3d lattice models. Linear variables may be assigned either to the vertices or to the sites of the two dimensional auxiliary graphs (these two assignments are the dual ones), and the coefficients of the linear relations – nothing but the dynamical variables – are to be assigned to the vertices. Main feature of the map of the dynamical variables is that it is canonical with respect to local Poisson brackets and thus may be easily quantized [2], so that the auxiliary linear systems exist even in the quantum case [4].

Evolution models arise when one considers flat regular graphs, formed by straight lines, on a two dimensional torus. The only demand is that the graph must contain Yang-Baxter triangles, so that a simultaneous "bypass of some lines through appropriate vertices" will restore the geometry of the graph. Thus, with the map associated with the Yang – Baxter type triangles, there appears the map of the dynamical variables, assigned to the vertices of the primary lattice, into the set of the dynamical variables, assigned in fact to the same lattice. This gives the one-step evolution of the discrete $2+1$ evolution model.

Simple square lattice does not fit the demands, because of it does not contain the triangles. The simplest two dimensional graph with the properties demanded is the so called kagome lattice [1] (see the figures below). This is not something strange, the 2d kagome lattice is nothing but a section of a regular 3d cubic lattice by an inclined plane.

The governing map was derived for the open linear system (i.e. the number of the equations is less then the number of auxiliary linear variables), but nevertheless the linear systems may be written for the 2d lattice with the toroidal boundary conditions. Also, in general, dealing with the linear equations, one may impose on the linear variables quasiperiodical boundary conditions in both directions of the torus with two numerical quasimomenta. In this case the number of the auxiliary linear variables coincides with the number of the linear equations and one may ask for the admissibility of the linear system, i.e. for the zero value of corresponding determinant. Because of the evolution arises as a simple compatibility of two similar linear systems, the admissibility of the primary system provides the admissibility of the evaluated one. Therefore the determinant is at least an ideal of the evolution. Moreover, being normalized appropriately, formal determinant $J(A,B)$ of the linear system as a polynomial of the quasimomenta A and B is conserved by one-step evolution map, i.e. is the generating function of the integrals of motion. All these remains valid and in the quantum case, where $J(A,B)$ is an operator-valued functional.

In our classical case and finite spatial size of the two dimensional lattice, equation $J(A,B)=0$ gives a finite genus g algebraic curve Γ, so that the integrals of motion are interpreted as moduli of Γ. With Γ and a bit of additional information concerning the initial state given, the system of the auxiliary linear variables may be easily parametrized as the meromorphic functions on Γ in terms of the theta functions on Jac Γ. Doing this, we get at once the parametrization of the dynamical variables in terms of theta functions on Jac Γ and obtain the exact solution.

Perhaps it would be expedient to discuss several $3d$ discrete integrable models from the point of view of their linear systems and indicate the place of the model being considered among them. Spatial nature of any $3d$ integrable model means that geometrically linear variables are assigned to several elements of $3d$ cubic lattice. This assignment gives a type of the linear system. Consider three main scenarios corresponding to three relative integrable models.

The first, most simple type [5, 6, 7, 8]: let the vertices of the cubic lattice Z^3 have the coordinates $\mathsf{p}=(a,b,c)$, $a,b,c\in Z$. Consider the even sublattice of it, Z^3_{even}: $\mathsf{p}=(a,b,c)$, $a+b+c=$ even. Points Z^3_{even} triangulize the three dimensional Euclidean space into the following convex bodies: set of octahedra and sets of two types of tetrahedra (up to regular translations). Assign the auxiliary linear variables to the vertices of Z^3_{even}. Linear equations are assigned to the triangles of the triangulation described. Primary set of the linear equations maybe chosen for the triangles – sides of one of the tetrahedra. System of coefficients of the linear equations gives the tau function for Hirota's discrete bilinear equation (the "*octahedron equation*" from this naive geometrical point of view).

The second type: assign the auxiliary linear variables to the facets of Z^3. Linear

[1] "Kagome" is not a name, it is a kind of Japanese mats.

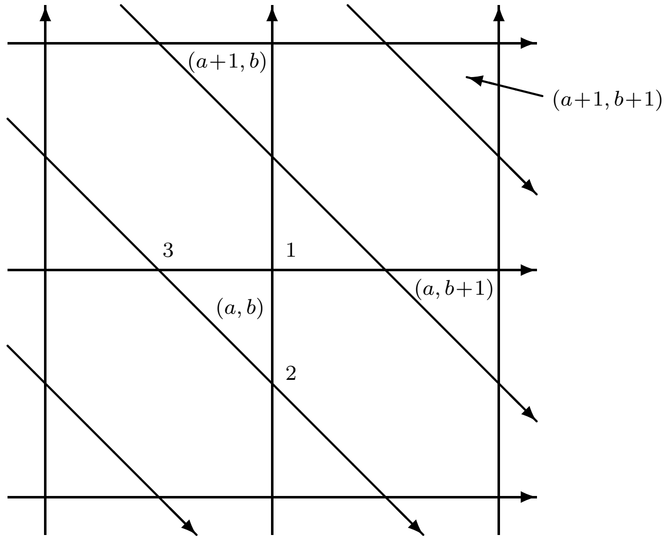

Figure 1. Labelling of the kagome lattice.

relation are to be written for each edge surrounded by four facets. This corresponds to Korepanov's block-matrix models [9]. Hirota and Hirota-Miwa' equations are the simplest compatibility conditions for Korepanov's linear problem [11, 10, 12].

The third type: dealing with the cubic lattice, most obvious scenario is to assign the linear variables to all the vertices of it (or, correspondingly, to the sites of the dual lattice). Each linear equation corresponds to a square (any face of the elementary cube). This is our case. Independent Lagrangian – type variables are a triplet of "tau – functions", obeying the system of the "*cube equations*". Taking a section of the cubic lattice by an inclined plane, we at first get the kagome lattice geometrically, and the linear system for it as a part of whole linear relations secondly. Details may be found in [1, 2, 3]. The advantages of this approach (Poisson structure, quantization etc.) were mentioned at the beginning of this introduction.

Few remarks concerning the section by the inclined plane and the evolution. For the cubic lattice $\mathsf{p} = (a, b, c)$, $a, b, c \in Z$, the sections mentioned are the planes $a+b+c = t =$ const. Equations of motion normally may be solved so that all the dynamical variables for $a + b + c = t + 1$ are expressed from the dynamical variables for $a + b + c = t$. This gives the natural notion of the evolution as the map from t to $t+1$. As it was mentioned, geometrically the section $a + b + c = t$ is the kagome lattice.

This paper is organized as follows. First, we recall the formulation of the model, describing the dynamical system and defining the evolution. Second, introducing the linear system, we define the generating function for the integrals of motion. All these are based on the results of [1, 2], and we rather enumerate the facts. In the fourth section we analyse the curve, parametrize the auxiliary linear variables and derive the expressions for the dynamical variables. In the final section we discuss possible applications of the results obtained.

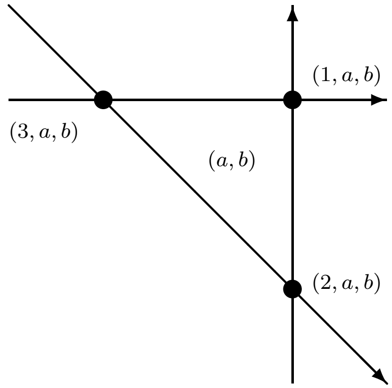

Figure 2. (a, b) – triangle.

2 Discrete evolution

System of the dynamical variables assigned to a finite $M \times M$ kagome lattice (see Figure 1) is a set of $3\,M^2$ pairs

$$[\,\mathsf{u}_{j,a,b}\,,\ \mathsf{w}_{j,a,b}\,]\,,\quad j\ =\ 1,2,3,\quad a,b\ \in\ Z_M\ . \tag{2.1}$$

Arrangement of the triangles of the kagome lattice is shown in Figure 1. Consideration of the lattice on a torus implies the periodical boundary conditions for the dynamical variables

$$\mathsf{u}_{j,a+M,b}\ =\ \mathsf{u}_{j,a,b+M}\ =\ \mathsf{u}_{j,a,b}\ ,\quad \mathsf{w}_{j,a+M,b}\ =\ \mathsf{w}_{j,a,b+M}\ =\ \mathsf{w}_{j,a,b}\ . \tag{2.2}$$

Geometrically the indices (j, a, b) are assigned to the vertices of the kagome lattice, so that for a, b given (j, a, b), $j = 1, 2, 3$ mark three vertices of a definite triangle, as it is shown in Figure 2. Whole kagome lattice may be obtained as the up-down and left-right translations of the triangle (a, b). It is supposed that a increases to up and b increases to right.

Impose the following Poisson structure on the set of the dynamical variables:

$$\{\,\mathsf{u}_{j,a,b}\,,\ \mathsf{w}_{j,a,b}\,\}\ =\ \mathsf{u}_{j,a,b}\,\mathsf{w}_{j,a,b}\ , \tag{2.3}$$

and any other Poisson bracket is zero. Remarkable is the locality of the dynamical variables.

Evolution of the system is governed by the fundamental map $\mathbf{R}$. Consider one isolated triangle (a, b). Define map

$$\mathbf{R}\ :\ [\mathsf{u}_j, \mathsf{w}_j]\ \mapsto\ [\mathsf{u}'_j, \mathsf{w}'_j]\quad j\ =\ 1,2,3\,,\quad (a,b)\ \text{ implied to be fixed,} \tag{2.4}$$

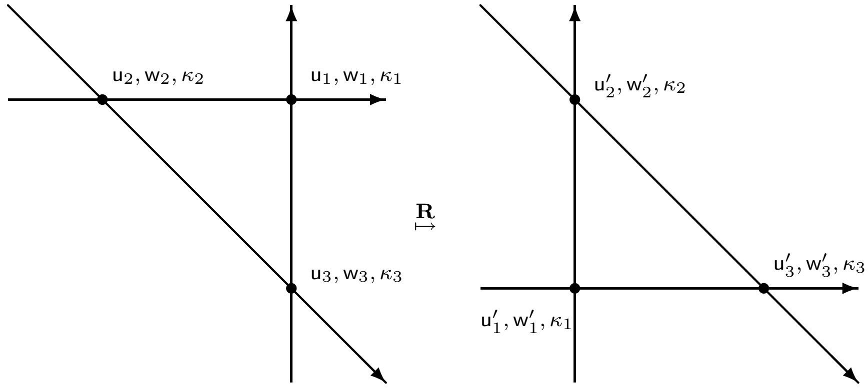

Figure 3. Geometrical representation of the map **R**.

by the following relations:

$$
(i)\quad \left\{ \begin{aligned} u'_1 &= \frac{\kappa_2\, u_1\, u_2\, w_2}{\kappa_1\, u_1\, w_2 \;+\; \kappa_3\, u_2\, w_3 \;+\; \kappa_1\, \kappa_3\, u_1\, w_3}\,, \\[2ex] w'_1 &= \frac{w_1\, w_2 \;+\; u_3\, w_2 \;+\; \kappa_3\, u_3\, w_3}{w_3}\,, \end{aligned} \right.
$$

$$
(ii)\quad \left\{ \begin{aligned} u'_2 &= \frac{u_1\, u_2\, u_3}{u_2\, u_3 \;+\; u_2\, w_1 \;+\; \kappa_1\, u_1\, w_1}\,, \\[2ex] w'_2 &= \frac{w_1\, w_2\, w_3}{w_1\, w_2 \;+\; u_3\, w_2 \;+\; \kappa_3\, u_3\, w_3}\,, \end{aligned} \right. \tag{2.5}
$$

$$
(iii)\quad \left\{ \begin{aligned} u'_3 &= \frac{u_2\, u_3 \;+\; u_2\, w_1 \;+\; \kappa_1\, u_1\, w_1}{u_1}\,, \\[2ex] w'_3 &= \frac{\kappa_2\, u_2\, w_2\, w_3}{\kappa_1\, u_1\, w_2 \;+\; \kappa_3\, u_2\, w_3 \;+\; \kappa_1\, \kappa_3\, u_1\, w_3}\,. \end{aligned} \right.
$$

$\kappa_{1,2,3}$ are arbitrary numbers (not the dynamical variables).

Proposition 1. *The map* **R**, *(2.5), conserves the local Poisson structure,*

$$
\{u_j, w_{j'}\} \;=\; \delta_{j,j'}\, u_j\, w_j \quad\Leftrightarrow\quad \{u'_j, w'_{j'}\} \;=\; \delta_{j,j'}\, u'_j\, w'_j\,, \tag{2.6}
$$

i.e. **R** *is the canonical map.*

Geometrically **R** may be interpreted as the map from one Yang – Baxter triangle to another, as it is shown in Figure 3, i.e. as a bypass of a line through the opposite vertex.

Turn now to the whole lattice. We will distinguish three types of lines with respect to their slopes by the letters x, y, z as it is shown in the Figure 4 and enumerate the lines, $x_\alpha, y_\beta, z_\gamma$, $\alpha, \beta, \gamma \in Z_M$. “Spectral parameters” $\kappa_{j,a,b}$ actually depend on numbers of the corresponding lines.

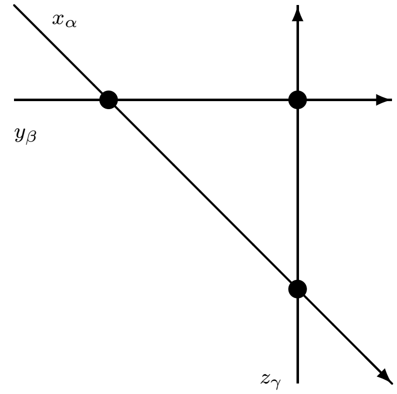

Figure 4. Accessories of the lattice: lines.

Evolution of the lattice is the simultaneous shift of all lines of one type in one direction. The result of such shift is the application of $\mathbf{R}$ to each (a,b) triangle and some re-numeration of the images of $\mathsf{u}'_{j,a,b}, \mathsf{w}'_{j,a,b}$. This re-numeration depends on what type of vertices we leave immovable. Choose the vertices of type $(1,a,b)$ motionless, i.e. we shift all x lines to the north-east direction. Define the functional map $\mathbf{U}$, acting on the space of functions of the dynamical variables $[\mathsf{u}_{j,a,b}, \mathsf{w}_{j,a,b}]$:

$$(\mathbf{U} \circ f)(\, \mathsf{u}_{j,a,b} \,,\, \mathsf{w}_{j,a,b} \,) \;=\; f(\, \mathbf{U}^* \circ \mathsf{u}_{j,a,b} \,,\, \mathbf{U}^* \circ \mathsf{w}_{j,a,b} \,)\;, \tag{2.7}$$

where

$$\begin{aligned} \mathbf{U}^* \circ \mathsf{u}_{1,a,b} &= \mathsf{u}'_{1,a,b}\,, & \mathbf{U}^* \circ \mathsf{w}_{1,a,b} &= \mathsf{w}'_{1,a,b}\,, \\ \mathbf{U}^* \circ \mathsf{u}_{2,a,b} &= \mathsf{u}'_{2,a+1,b}\,, & \mathbf{U}^* \circ \mathsf{w}_{2,a,b} &= \mathsf{w}'_{2,a+1,b}\,, \\ \mathbf{U}^* \circ \mathsf{u}_{3,a,b} &= \mathsf{u}'_{3,a,b+1}\,, & \mathbf{U}^* \circ \mathsf{w}_{3,a,b} &= \mathsf{w}'_{3,a,b+1}\,, \end{aligned} \tag{2.8}$$

where, for example, $\mathsf{u}'_{1,a,b}$ means that we take the expression for u'_1 from (2.5) and add the indices a,b to each u_j, w_j there. Indices $(a+1,b)$ and $(a,b+1)$ in the second and third lines of (2.8) are the re-enumeration mentioned above.

Obviously, $\mathbf{U}$ conserves the Poisson brackets, and thus it is the canonical map. $\mathbf{U}$ is identified with the one step discrete evolution, so that if

$$f(\, \mathsf{u}_{j,a,b} \,,\, \mathsf{w}_{j,a,b} \,) \;=\; f(t_0)\;, \tag{2.9}$$

then

$$f(t_0\,,\,t) \;=\; \Big(\mathbf{U}^t \circ f\Big)\,(t_0)\;. \tag{2.10}$$

3 Linear system and the integrals of motion

Map $\mathbf{R}$ (2.5) was "derived" in [4] as a zero curvature condition for a system of linear equations. There are at least two ways to define the linear system, and here we use the co-current form according to the terminology of [1, 2].

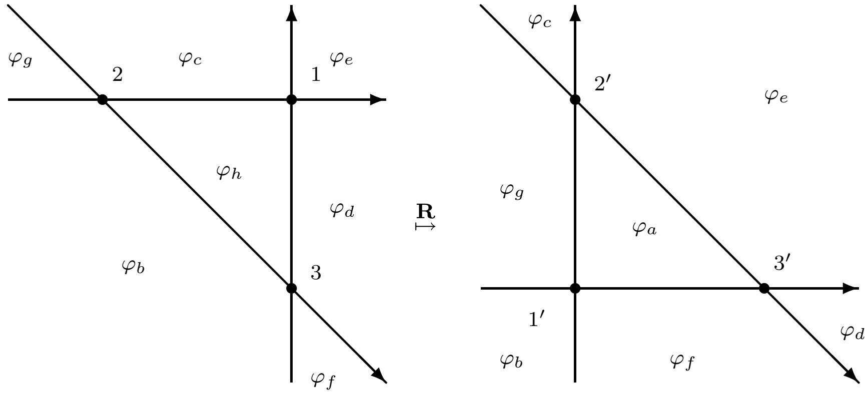

Figure 5. Linear variables of the map **R**.

We start from the linear system for isolated Yang – Baxter triangles and explain how the map **R** appears. First, in addition to the vertex variables $\mathsf{u}_j, \mathsf{w}_j, \kappa_j$ introduce eight auxiliary variables $\varphi_a ... \varphi_h$ living in the sites of the 2d graphs as it is shown in Figure 5.

Please note that the variables $\varphi_b ... \varphi_g$ are the same in the left and right hand sides of Figure 5 and belong to the equivalent open cells. Left and right hand side graphs differ by φ_h and φ_a. Consider now two sets of linear relations: for the left hand side graph

$$\begin{aligned} 0 &= f_1 \stackrel{def}{=} \varphi_c - \varphi_e \mathsf{u}_1 + \varphi_h \mathsf{w}_1 + \varphi_d \kappa_1 \mathsf{u}_1 \mathsf{w}_1 , \\ 0 &= f_2 \stackrel{def}{=} \varphi_h - \varphi_d \mathsf{u}_2 + \varphi_b \mathsf{w}_2 + \varphi_f \kappa_2 \mathsf{u}_2 \mathsf{w}_2 , \\ 0 &= f_3 \stackrel{def}{=} \varphi_g - \varphi_c \mathsf{u}_3 + \varphi_b \mathsf{w}_3 + \varphi_h \kappa_3 \mathsf{u}_3 \mathsf{w}_3 , \end{aligned} \tag{3.1}$$

and for the right hand side graph

$$\begin{aligned} 0 &= f'_1 \stackrel{def}{=} \varphi_g - \varphi_a \mathsf{u}'_1 + \varphi_b \mathsf{w}'_1 + \varphi_f \kappa_1 \mathsf{u}'_1 \mathsf{w}'_1 , \\ 0 &= f'_2 \stackrel{def}{=} \varphi_c - \varphi_e \mathsf{u}'_2 + \varphi_g \mathsf{w}'_2 + \varphi_a \kappa_2 \mathsf{u}'_2 \mathsf{w}'_2 , \\ 0 &= f'_3 \stackrel{def}{=} \varphi_a - \varphi_e \mathsf{u}'_3 + \varphi_f \mathsf{w}'_3 + \varphi_d \kappa_3 \mathsf{u}'_3 \mathsf{w}'_3 . \end{aligned} \tag{3.2}$$

Each linear expression f_j and f'_j is assigned to j-th vertex of the left and right graphs respectively. Into each f_j and f'_j there involved four linear variables from the sites surrounding the vertex. Coefficients in f_j and f'_j are the vertex variables, assigned with j-th vertex. Arrangement of the vertex variables and the site variables with respect to the arrows of the lines is the same for any vertex (just look attentively at the linear relations and the figure). Each expression f_j and f'_j becomes the equation, $f_j = 0$, $f'_j = 0$, and thus two sets of linear equations appear.

Theorem 1. *Two systems, (3.1) and (3.2) are linearly equivalent (after excluding extra linear variables φ_h and φ_a) iff $\mathsf{u}'_j, \mathsf{w}'_j$ are connected with $\mathsf{u}_j, \mathsf{w}_j$ via (2.5).*

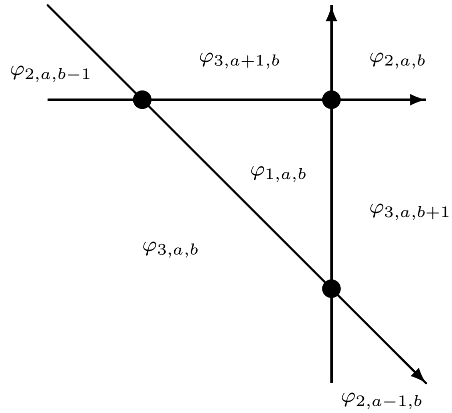

Figure 6. Accessories of the lattice: cocurents.

Turn now to the kagome lattice on the torus. Linear forms $f_{j,a,b}$ and and corresponding linear equations $f_{j,a,b} = 0$ are to be introduced for all vertices of the lattice.

$$\begin{aligned}
0 \;&=\; f_{1,a,b} \stackrel{\text{def}}{=} \varphi_{3,a+1,b} - \varphi_{2,a,b}\cdot \mathsf{u}_{1,a,b} + \varphi_{1,a,b}\cdot \mathsf{w}_{1,a,b} \\
&\qquad + \varphi_{3,a,b+1}\cdot \kappa_1\, \mathsf{u}_{1,a,b}\,\mathsf{w}_{1,a,b}\;, \\
0 \;&=\; f_{2,a,b} \stackrel{\text{def}}{=} \varphi_{1,a,b} - \varphi_{3,a,b+1}\cdot \mathsf{u}_{2,a,b} + \varphi_{3,a,b}\cdot \mathsf{w}_{2,a,b} \\
&\qquad + \varphi_{2,a-1,b}\cdot \kappa_2\, \mathsf{u}_{2,a,b}\,\mathsf{w}_{2,a,b}\;, \\
0 \;&=\; f_{3,a,b} \stackrel{\text{def}}{=} \varphi_{3,a+1,b} - \varphi_{1,a,b}\cdot \mathsf{u}_{3,a,b} + \varphi_{2,a,b-1}\cdot \mathsf{w}_{3,a,b} \\
&\qquad + \varphi_{3,a,b}\cdot \kappa_3\, \mathsf{u}_{3,a,b}\,\mathsf{w}_{3,a,b}\;.
\end{aligned} \tag{3.3}$$

What these writings mean. The linear objects $\varphi_{j,a,b}$, appeared in these relations, are assigned to the sites of the kagome lattice as it is shown in Figure 6. Each linear expression $f_{j,a,b}$ is assigned to (j,a,b)-th vertex of the kagome lattice, and into $f_{j,a,b}$ there involved four linear variables from the sites surrounding the vertex. Coefficients in $f_{j,a,b}$ are the vertex variables, assigned with (j,a,b)-th vertex. Arrangement of the vertex variables and the site variables with respect to the arrows of the lines is the same for any vertex. Each expression $f_{j,a,b}$ becomes the equation, $f_{j,a,b} = 0$, and thus the set of linear equations appears.

The linearity of $\varphi_{j,a,b}$ allows one to impose the quasiperiodical boundary conditions for them:

$$\varphi_{\alpha,a-M,b} = \varphi_{\alpha,a,b}\cdot A\;, \quad \varphi_{\alpha,a,b-M} = \varphi_{\alpha,a,b}\cdot B\;. \tag{3.4}$$

The coefficients of the linear system form a $3\,M^2 \times 3\,M^2$ matrix $\mathbf{L}$, depending on the dynamical variables and the quasimomenta A, B. The following set of propositions was proved in [1, 2]:

Proposition 2. *The determinant of* $\mathbf{L}$ *is a Laurent polynomial of* A, B*,*

$$\det \mathbf{L} = \sum_{\alpha,\beta\in\Pi} \widetilde{J}_{\alpha,\beta}\, A^{\alpha}\cdot B^{\beta}\;, \tag{3.5}$$

where domain Π *is described below. Being normalized in any way,*

$$J(A,B) \;=\; \frac{\det \mathbf{L}}{\widetilde{J}_{\alpha_0,\beta_0}}\;, \tag{3.6}$$

functional $J(A,B)$ *is the generating functions for the integrals of motion,*

$$J(A,B) \;=\; \sum_{\alpha,\beta\in\Pi} J_{\alpha,\beta}(\mathsf{u},\mathsf{w})\; A^{\alpha}\,\cdot\, B^{\beta}\;, \tag{3.7}$$

$$(U\,\circ\, J_{\alpha,\beta})(\mathsf{u},\mathsf{w}) \;=\; J_{\alpha,\beta}(\mathsf{u},\mathsf{w})\;. \tag{3.8}$$

Proposition 3. *Domain* Π *in the decomposition of* $J(A,B)$ *is the following hexagon:*

$$\Pi\;:\; -M\;\leq\;\alpha\;\leq\;M\;,\;\; -M\;\leq\;\beta\;\leq\;M\;,\;\; -M\;\leq\;\alpha+\beta\;\leq\;M\;, \tag{3.9}$$

where M *is the spatial size of the kagome lattice. According to the Riemann-Hurwitz theorem, the genus of the curve* $\Gamma\,:\,J(A,B)\;=\;0$ *is*

$$g\;=\;3\,M^2\;-\;3\,M\;+\;1\;. \tag{3.10}$$

Complete number of the integrals of motion is $3\,M^2+1$*, and one can choose exactly* $3\,M^2$ *involutive between them.*

Perimeter of the hexagon Π is formed by $6M$ points,

$$J_{M-n,n}\;,\; J_{-n,n-M}\;,\; J_{M,-n}\;,\; J_{-M,M-n}\;,\; J_{n-M,M}\;,\; J_{n,-M}\;, \tag{3.11}$$

where $n=0,...,M$. These perimeter integrals are not independent. Let

$$\begin{aligned} X_{\alpha} &= \prod_{\sigma} \mathsf{u}_{2,\alpha-\sigma,\sigma}^{-1}\,\mathsf{u}_{3,\alpha-\sigma,\sigma}^{-1}\;,\\ Y_{\beta} &= (-)^{M}\prod_{\sigma} \mathsf{u}_{1,\beta,\sigma}\,\mathsf{w}_{3,\beta,\sigma}^{-1}\;,\\ Z_{\gamma} &= \prod_{\sigma} \mathsf{w}_{1,\sigma,\gamma}\,\mathsf{w}_{2,\sigma,\gamma}\;. \end{aligned} \tag{3.12}$$

Each of these expressions corresponds naturally to its line, X_α to x_α etc., see Figure 4. The perimeter integrals (3.11) are some the symmetrical polynomials of X_α, or Y_β, or Z_γ. The Poisson brackets for $X_\alpha, Y_\beta, Z_\gamma$ $\forall \alpha,\beta,\gamma$ are

$$\{X_\alpha,Y_\beta\}\;=\;X_\alpha\,Y_\beta\;,\;\; \{Y_\beta,Z_\gamma\}\;=\;Y_\beta\,Z_\gamma\;,\;\; \{Z_\gamma,X_\alpha\}\;=\;Z_\gamma\,X_\alpha\;. \tag{3.13}$$

It is useful to extract common convolutive parts from $X_\alpha, Y_\beta, Z_\gamma$,

$$X_\alpha\;=\;X\,\cdot\,j(X_\alpha)\;,\;\; Y_\beta\;=\;Y\,\cdot\,j(Y_\beta)\;,\quad Z_\gamma\;=\;Z\,\cdot\,j(Z_\gamma)\;, \tag{3.14}$$

where

$$\{X,Y\}\;=\;X\,Y\;,\quad \text{etc.} \tag{3.15}$$

Now we can rewrite $J(A,B)$ in its final invariant form:

$$J \;=\; \sum_{\alpha,\beta,\gamma\,\in\,\Pi'} \; j_{\alpha,\beta,\gamma} \;\; (Z\,A)^{\alpha}\;(Y\,B^{-1})^{\beta}\;\left(X\,\frac{B}{A}\right)^{\gamma}\;, \tag{3.16}$$

where Π' is three squares of the cube:

$$\Pi' \;:\; 0\;\le\;\alpha,\beta,\gamma\;\le M\;,\quad \text{at least one of }\;\alpha,\beta,\gamma\;\text{ iz zero}\,. \tag{3.17}$$

Now we may describe the complete set of $3\,M^2$ involutive integrals: $g = 3M^2 - 3M + 1$ functionals $j_{\alpha,\beta,\gamma}$, corresponding to the inner points of Π', $3\,M-3$ independent projective $j(X_\alpha)$, $j(Y_\beta)$ and $j(Z_\gamma)$, one $X\cdot Y\cdot Z$ (this is the center for X,Y,Z), and any finally – any other single function of X, Y, Z.

Running ahead, for what purpose else one needs the perimeter integrals (3.11)? $J(A,B)=0$ is an algebraic curve Γ, and we will be interested in the divisors (A) and (B) of the algebraic functions $A=A(P)$ and $B=B(P)$, $P\in\Gamma$. In general the perimeter integrals describes the divisors (A) and (B), and after a bit cumbersome calculations we have obtained the following description:

$$\begin{aligned}
(A)_0 \;\;&:\;\; \begin{cases} A\;=\;0\,,\;\; \dfrac{A}{B}\;=\;X_\alpha\,,\quad \alpha\in Z_M\;, \\[3ex] A\;=\;0\,,\;\;B\;=\;Y_\beta\,,\quad \beta\in Z_M\;, \end{cases} \\[2ex]
(A)_\infty \;\;&:\;\; \begin{cases} A\;=\;\infty\,,\;\; \dfrac{A}{B}\;=\;\dfrac{X_\alpha}{(\kappa_2\kappa_3)^M}\;,\quad \alpha\in Z_M\;, \\[3ex] A\;=\;\infty\,,\;\;B\;=\;\left(\dfrac{\kappa_1}{\kappa_3}\right)^M Y_\beta\;,\quad \beta\in Z_M\;, \end{cases}
\end{aligned} \tag{3.18}$$

and

$$\begin{aligned}
(B)_0 \;\;&:\;\; \begin{cases} B\;=\;0\,,\;\; \dfrac{A}{B}\;=\;X_\alpha\,,\quad \alpha\in Z_M\;, \\[3ex] B\;=\;0\,,\;\;A\;=\;\dfrac{1}{(\kappa_1\kappa_2)^M Z_\gamma}\;,\quad \gamma\in Z_M\;, \end{cases} \\[2ex]
(B)_\infty \;\;&:\;\; \begin{cases} B\;=\;\infty\,,\;\; \dfrac{A}{B}\;=\;\dfrac{X_\alpha}{(\kappa_2\kappa_3)^M}\;,\quad \alpha\in Z_M\;, \\[3ex] B\;=\;\infty\,,\;\;A\;=\;\dfrac{1}{Z_\gamma}\;,\quad \gamma\in Z_M\;, \end{cases}
\end{aligned} \tag{3.19}$$

In these formulae it is supposed that the spectral parameters κ_j are the same for all lattice, but the structure of X_α, Y_β and Z_γ, associated with the lines x_α, y_β and z_γ makes obvious the situation when κ_j depend on the lines numbers.

4 The curve and solution

Turn at last to the algebraic geometry. We deal with the algebraic curve Γ, defined by $J(A,B) = 0$. A, B and $C = A/B$ are the meromorphic functions on Γ, their divisors are already obtained in the previous section, and (3.18,3.19) may be rewritten in decent notations as

$$\begin{aligned}(A) &= \sum_{\alpha} (P^+_{x_\alpha} - P^-_{x_\alpha}) + \sum_{\beta} \left(P^+_{y_\beta} - P^-_{y_\beta}\right) ,\\ (B) &= \sum_{\alpha} (P^+_{x_\alpha} - P^-_{x_\alpha}) - \sum_{\gamma} \left(P^+_{z_\gamma} - P^-_{z_\gamma}\right) ,\end{aligned} \tag{4.1}$$

where the notion of the points $P^\pm_{x_\alpha}$, $P^\pm_{y_\beta}$ and $P^\pm_{z_\gamma}$ comes from (3.18,3.19) transparently. Remarkable feature of these divisors is the natural correspondence between $P^\pm$ and the lines x_α, y_β, z_γ.

Return now to linear system (3.3). Solving it, one may put one of $\varphi_{j,a,b}$ to be unity, then all other linear variables become some meromorphic functions on Γ. Our next interest is a common pole divisor of all $\varphi_{j,a,b}$. Denote this divisor as $\widetilde{\mathcal{D}}$, its degree may be calculated:

$$\deg \widetilde{\mathcal{D}} = 3\,M^2 + 1\,. \tag{4.2}$$

This calculation is based directly on the dimension of linear system and needs no comments. The set $\varphi_{j,a,b}$ is the unique solution of the linear system, so due to the Riemann-Roch theorem the dimension of the linear space of the holomorphic functions with the pole divisor $\widetilde{\mathcal{D}}$ is

$$\dim L(\widetilde{\mathcal{D}}) = \deg \widetilde{\mathcal{D}} - g + 1 = 3\,M + 1\,. \tag{4.3}$$

This means that one may restore $\varphi_{j,a,b}$ as a function on Γ via just $3\,M$ points of positive $\mathcal{D}'_{j,a,b}$,

$$(\varphi_{j,a,b}) + \widetilde{\mathcal{D}} = \mathcal{D}^{(0)}_{j,a,b} + \mathcal{D}'_{j,a,b}\,, \quad \deg \mathcal{D}^{(0)}_{j,a,b} = g\,, \quad \deg \mathcal{D}'_{j,a,b} = 3\,M\,, \tag{4.4}$$

where positive $\mathcal{D}^{(0)}_{j,a,b}$ may be restored unambiguously. Obviously, the meaning of the quasimomenta A and B implies that we may choose the lines x_{α_0}, y_{β_0} and z_{γ_0}, where the quasimomenta appear, in any way. Therefore $\mathcal{D}'_{\mathsf{p}}$, $\mathsf{p} \equiv (j,a,b)$, are all governed by the same points as form the divisors (A) and (B).

The same is valid for $\widetilde{\mathcal{D}}$:

$$\widetilde{\mathcal{D}} = \widetilde{\mathcal{D}}^{(0)} + \widetilde{\mathcal{D}}'\,, \tag{4.5}$$

where $\widetilde{\mathcal{D}}'$, $\deg \widetilde{\mathcal{D}}' = 3\,M$ is also governed by the points of (A) and (B). Thus for any $\mathsf{p} = (j,a,b)$ the decomposition arises:

$$(\varphi_{\mathsf{p}}) = (\varphi_{\mathsf{p}})_0 - (\varphi_{\mathsf{p}})_\infty = \mathcal{D}^{(0)}_{\mathsf{p}} - \widetilde{\mathcal{D}}^{(0)} + \mathcal{D}'_{\mathsf{p}} - \widetilde{\mathcal{D}}'\,, \tag{4.6}$$

such that we can trace the following simple part of (φ_{p}):

$$\mathcal{D}_{\mathsf{p}} = \mathcal{D}'_{\mathsf{p}} - \widetilde{\mathcal{D}}'\,. \tag{4.7}$$

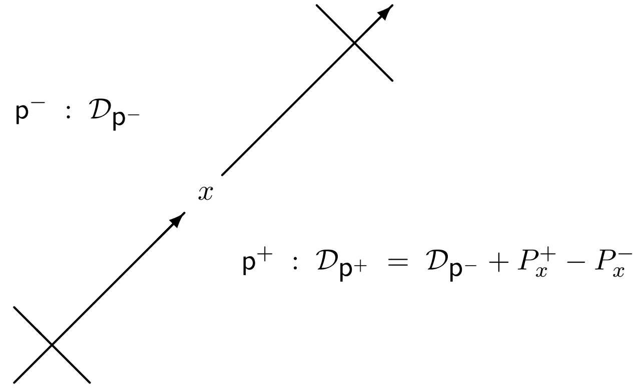

Figure 7. Sites separated by a segment of line x between two vertices and theirs divisors.

Before we give concrete expressions for φ_{p} recall a couple of notations of the algebraic geometry, see for example [13]. For a given curve Γ with normalized holomorphic one-forms ω it is defined its jacobian Jac Γ and the theta – functions on it. We'll use the conventional notations for $\Gamma \mapsto \text{Jac}\,\Gamma$: for $\mathcal{D}$, deg $\mathcal{D} = 0$, let

$$I(\mathcal{D}) \;=\; \int_{\Sigma:\mathcal{D}=\partial\Sigma} \omega \;. \tag{4.8}$$

Also for $P, Q \in \Gamma$ denote their prime form as $E(P,Q)$,

$$E(P,Q) \;\sim\; \Theta_\delta\left(\int_P^Q \omega\right) \;, \tag{4.9}$$

where the subscript δ means the nonsingular odd theta characteristic of the curve (see [13]). Let the same symbol E stands for a product of the prime forms

$$E(P,\mathcal{D}) \;=\; \prod_{Q\in\mathcal{D}} E(P,Q)\;, \quad \mathcal{D} \;>\; 0\;. \tag{4.10}$$

Theorem 2. *With the notations introduced, the expression for φ_{p} as the meromorphic function on Γ with the divisor given by (4.6,4.7) is*

$$\varphi_{\mathsf{p}} \;=\; \varphi_{\mathsf{p}}(P) \;=\; \varphi_{\mathsf{p}}^{(0)}\; \frac{\Theta(\mathbf{z}+I(P-P_0+\mathcal{D}_{\mathsf{p}}))}{\Theta(\mathbf{z}+I(P-P_0))}\; \frac{E(P,(\mathcal{D}_{\mathsf{p}})_0)}{E(P,(\mathcal{D}_{\mathsf{p}})_\infty)}\;, \tag{4.11}$$

where $P\in\Gamma$, vector $\mathbf{z}\in C^g$ and $P_0\in\Gamma$ are some auxiliary parameters in the parametrization (4.11), $\varphi_{\mathsf{p}}^{(0)}$ are constants (i.e. do not depend neither on P nor on $\mathbf{z}$), $\Theta(\mathbf{z}+I(P-P_0))$ corresponds to g poles from $\widetilde{\mathcal{D}}^{(0)}$, and so on.

Actually eq. (4.11) is the consequence of the Riemann-Roch theorem, see [9, 13].

Further it is simpler to cancel $\Theta(\mathbf{z}+I(P-P_0))$ from all φ_{p} and deal with the holomorphic with respect to $\mathbf{z}$ functions.

Describe now the explicit way of assigning the divisors $\mathcal{D}_{\mathsf{p}}$ (4.7) to the sites $\mathsf{p}=(j,a,b)$ of the kagome lattice.

To each line x_α, y_β and z_γ of the lattice the pair $P_{..}^+ - P_{..}^-$ is assigned, see (4.1) and the remarks at the beginning of this section. Let a segment of oriented line x separates two

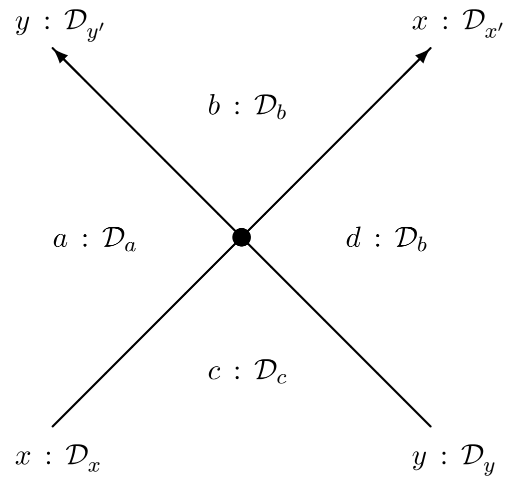

Figure 8. Divisor notations around a vertex.

sites, p^- on the left and p^+ on the right according to the orientation of x, see Figure 7. According to our previous considerations, the divisors $\mathcal{D}_{\mathsf{p}^+}$ and $\mathcal{D}_{\mathsf{p}^-}$, assigned to p^+ and p^- respectively, obey

$$\mathcal{D}_{\mathsf{p}^+} - \mathcal{D}_{\mathsf{p}^-} = P_x^+ - P_x^- , \tag{4.12}$$

where P_x^+ and P_x^- is the pair assigned to line x. Starting from any site p_0 on the lattice and using this procedure, we may define all $\mathcal{D}_{\mathsf{p}}$ up to the divisors of the algebraic functions (A) and (B). Such system of the divisors was introduces by I. Korepanov in [9], i.e. the divisor rules we've obtained (Figure 7, eq. (4.12)) coincide formally with that of the second type linear problem (see the introduction).

Introduce also a divisor of the edge of x separating p^+ and p^-:

$$\mathcal{D}_x = \mathcal{D}_{\mathsf{p}^-} + P_x^+ - P_0 = \mathcal{D}_{\mathsf{p}^+} + P_x^- - P_0 , \tag{4.13}$$

where P_0 is the same point as in (4.11). The meaning of $\mathcal{D}_x$ is following:

$$\frac{\varphi_{\mathsf{p}^-}}{\varphi_{\mathsf{p}^+}}(\mathbf{z} = \delta - I(\mathcal{D}_x)) = \frac{\varphi_{\mathsf{p}^-}^{(0)}}{\varphi_{\mathsf{p}^+}^{(0)}} . \tag{4.14}$$

Consider further a vertex formed by two lines x and y surrounded by the sites a, b, c, d as it is shown in Figure 8. The linear relation for it is

$$\varphi_a - \varphi_b \cdot \mathsf{u} + \varphi_c \cdot \mathsf{w} + \varphi_d \cdot \kappa\, \mathsf{u}\,\mathsf{w} = 0 . \tag{4.15}$$

With the parametrization (4.11) this relation must be the identity both in P and $\mathbf{z}$, so that neither u nor w depend on P, but $\mathsf{u} = \mathsf{u}(\mathbf{z})$, $\mathsf{w} = \mathsf{w}(\mathbf{z})$. This allows one to find $\mathsf{u}(\mathbf{z})$ and $\mathsf{w}(\mathbf{z})$ immediately.

Eight divisors correspond to eight elements of the vertex. For the site divisors we use obvious notations $\mathcal{D}_a$, $\mathcal{D}_b$, $\mathcal{D}_c$ and $\mathcal{D}_d$. Two lines involved, x and y, are divided by the intersection point into four edges with the divisors: $\mathcal{D}_x$ and $\mathcal{D}_{x'}$ for x line, and $\mathcal{D}_y$ and

$\mathcal{D}_{y'}$ for y line. According to divisor rules (4.12), the relations for all these eight divisors may be written in the following form:

$$\begin{aligned}
\mathcal{D}_a &= \mathcal{D}_x - P_x^+ + P_0 = \mathcal{D}_{y'} - P_y^+ + P_0 \,, \\
\mathcal{D}_b &= \mathcal{D}_{x'} - P_x^+ + P_0 = \mathcal{D}_{y'} - P_y^- + P_0 \,, \\
\mathcal{D}_c &= \mathcal{D}_x - P_x^- + P_0 = \mathcal{D}_y - P_y^+ + P_0 \,, \\
\mathcal{D}_d &= \mathcal{D}_{x'} - P_x^- + P_0 = \mathcal{D}_y - P_y^- + P_0 \,.
\end{aligned} \tag{4.16}$$

Testing (4.15) for the quasiperiodicity on $\mathbf{z}$ and taking zeros and poles of $\mathsf{u}(\mathbf{z})$ and $\mathsf{w}(\mathbf{z})$ (relation (4.14) is useful for it), we get unambiguously

$$\mathsf{u}(\mathbf{z}) = \mathsf{u}_0 \cdot \frac{\Theta(\mathbf{z} + I(\mathcal{D}_x))}{\Theta(\mathbf{z} + I(\mathcal{D}_{x'}))}\,, \quad \mathsf{w}(\mathbf{z}) = \mathsf{w}_0 \cdot \frac{\Theta(\mathbf{z} + I(\mathcal{D}_{y'}))}{\Theta(\mathbf{z} + I(\mathcal{D}_y))}\,. \tag{4.17}$$

Extra constant parameters obey

$$\begin{aligned}
\frac{\varphi_a^{(0)}}{\varphi_c^{(0)}} &= -\,\mathsf{w}_0 \cdot \frac{E(P_x^+, P_y^+)}{E(P_x^-, P_y^+)}\,, \quad & \frac{\varphi_b^{(0)}}{\varphi_d^{(0)}} &= \kappa\,\mathsf{w}_0 \cdot \frac{E(P_x^+, P_y^-)}{E(P_x^-, P_y^-)}\,, \\
\frac{\varphi_a^{(0)}}{\varphi_b^{(0)}} &= \mathsf{u}_0 \cdot \frac{E(P_x^+, P_y^+)}{E(P_x^+, P_y^-)}\,, \quad & \frac{\varphi_c^{(0)}}{\varphi_d^{(0)}} &= -\,\kappa\,\mathsf{u}_0 \cdot \frac{E(P_x^-, P_y^+)}{E(P_x^-, P_y^-)}\,.
\end{aligned} \tag{4.18}$$

These constants of the normalization, u_0 and w_0, are to be used to parametrize the "gauge" integrals of motion X_α, Y_β, Z_γ, and thus have no dynamical sense.

Actually (4.15), as an identity on $\mathbf{z}$, is a combination of two Fay's identities (see [13]). In the parametrization of u and w the theta functions are assigned naturally to the edges. These thetas are nothing but the triplet of tau-functions and thus solve a system of trilinear relations, see [3].

With the parametrization in terms of the divisors, the evolution becomes simple. It corresponds to the geometrical shift of the lines generating the change of the divisors of the sites. In general the lattice is heterogeneous and non-equidistant and thus there the formula for u,w is rather geometrical, because of "time" means a concrete geometrical configuration. But M-step evolution $\mathsf{S} = \mathbf{U}^M$ has the common form for all u,w:

$$[\mathsf{u},\mathsf{w}] = [\mathsf{u}(\mathbf{z}),\mathsf{w}(\mathbf{z})] \xrightarrow{\mathsf{S}} [\mathsf{u}(\mathbf{z}+\mathbf{T}),\mathsf{w}(\mathbf{z}+\mathbf{T})]\,, \tag{4.19}$$

where the vector of the periods

$$\mathbf{T} = I(\mathcal{T})\,, \tag{4.20}$$

is the same for all three "time" directions:

$$\mathcal{T} = \sum_\alpha P_{x_\alpha}^+ - P_{x_\alpha}^- = \sum_\alpha P_{y_\alpha}^- - P_{y_\alpha}^+ = \sum_\alpha P_{z_\alpha}^+ - P_{z_\alpha}^-\,, \tag{4.21}$$

where, surely, the equality signs mean the equivalence of the divisors.

5 Conclusion

In this conclusion we would like to discuss possible applications of all the considerations made in this paper. Usually the algebraic geometry gives rather formal results, which are hard to apply to calculate something "physical". Nevertheless one may use the considerations above in order to make some important conclusions for the quantum evolution model [2].

The quantum evolution model is based on the quantum analogue of the map $\mathbf{R}$ (2.5) for the local Weyl algebra replacing the local Poisson algebra (2.6),

$$\mathsf{u}_j \cdot \mathsf{w}_j = q \ \mathsf{w}_j \cdot \mathsf{u}_j \,, \tag{5.1}$$

where q is the commonly accepted parameter of the quantum deformation. The form of the map $\mathbf{R}$ and the linear system in the quantum case differ very slightly form (2.5) and (3.1,3.2), see [2] for the details. The determinant in the quantum case is well defined and also gives the complete set of the quantum integrals of motion. With a great effort we do not fall into speculations concerning a quantum jacobian.

Instead, turn to the case when q is N-th root of unity. For these q the Weyl algebra (5.1) has the finite dimensional representations, so that N-th powers u_j^N and w_j^N are centres, i.e. the parameters of the finite dimensional representation. The map for u_j^N and w_j^N coincides with the functional map (2.5) up to $\kappa_j \mapsto \kappa_j^N$, i.e. exactly the case considered. Actually the quantum map factorizes into a finite dimensional part (which is the vertex R matrix for a Zamolodchikov – Bazhanov – Baxter – type model, giving the Boltzmann weights for a statistical mechanics modelling) *times* the functional part for N-th powers. The evolution operator factorizes also in this way. The quantum S matrix is a high power of the quantum one-step evolution operator, and so in order to factorize it one has to carry out all the functional parts outside. Doing this, one changes the parameters of the finite dimensional one-step evolution operators. This situation corresponds to the consistent changes of the Boltzmann weights parameters from one time layer to the other. In two dimensions the simplest such models are known as the checkerboard models, and the simplest three dimensional chess model was described in [14, 15]. Note, the parameters $\mathsf{u}^M, \mathsf{w}^N, \kappa$ live on the lattice, i. e. u_V^N, w_V^N, κ_V are assigned to the vertex V as well as the operators $\mathsf{u}_V, \mathsf{w}_V$. Actually, for given spatial size of the kagome lattice M, one may consider another effective spatial size M' of the lattice of parameters, such that M is divisible by M'. Small M' correspond to the small heterogeneity of the lattice, and $M' = 1$ corresponds to the spatial homogeneity of the parameters.

In general the results of this paper would help one to parametrize the consistent evolution of the parameters of the finite dimensional one-step evolution operators or the transfer matrices.

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