Constructing implication functions from fuzzy negations

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Abstract

A class of implication functions is constructed from fuzzy negations. The interest of this new class lies in its simplicity and in the fact that when N is Id-symmetrical, the corresponding implication agrees with the residuum of a commutative semicopula.

Keywords: Fuzzy implication, Fuzzy negation, Residual implication

1. Introduction

Implication functions are probably one of the main operations in fuzzy logic, having a similar role to the one that classical implication plays in crisp logic. It is usually required of any fuzzy concept to generalizes the corresponding crisp one, and consequently fuzzy implications restricted to $\{0,1\}^2$ must coincide with the classical implication. Additionally, some restrictions are imposed to binary functions in order to be fuzzy implications, mainly adequate monotonicities, but they are flexible enough to allow several classes of implications with different additional properties.

Implication functions are mainly used to perform any fuzzy "if-then" rule in fuzzy systems and also in inference processes, through Modus Ponens and Modus Tollens (see [18]). So, depending on the context, and on the proper rule and its behaviour, different implications with different properties could be adequate ([28]). This is also true in other fields where fuzzy implications play an important role, such as fuzzy mathematical morphology ([17]) or fuzzy DI-subsethood measures and image processing ([7, 8]), among many others.

The logical consequence of this fact is the proposal of different classes of implications. Among the most used ones, we can highlight two different strategies to generate most of these classes.

- The first strategy relates to the use of t-norms and t-conorms obtaining the class of (S,N)-implications ([2]), the class of residual or R-implications ([13]), and the classes of QL-operations and D-operations ([21]), all of them collected in the book [4] and the survey [23]. From these classes, some generalizations based on considering different kinds of aggregation functions instead of t-norms and t-conorms appear. This is the case of implication functions derived from uninorms ([3, 11, 22]), copulas and quasi-copulas ([12]) and even from aggregations in general ([26]), allowing new classes of implications with interesting properties.
- Another approach in order to obtain implication functions is based on the direct use of additive generating functions. In this way, Yager's f- and g-generated fuzzy implications [29] can be seen as implications generated from continuous additive generators of continuous Archimedean t-norms or t-conorms, respectively. Analogously, Balasubramaniam's h-generated implications [5, 6] can be seen as implications generated from multiplicative generators of t-conorms, and h and (h,e)-implications are generated from additive generators of representable uninorms. Most of these classes can be found in [4, 16] and [24].

There are also other systems of generating implication functions that have been recently collected in [25]. Some of these systems are based on the use of fuzzy negations like in [27] or in [25]. Following in this line we want to present in this communication a new method of obtaining implication functions from fuzzy negations. The interest of this new method lies in its simplicity and in the fact that many of the obtained implications can be viewed as residual implications of some kind of aggregation functions.

2. Preliminaries

In this section we give some basic results that will be used along the paper.

Definition 1 *([13]) A function $N:[0,1]\to[0,1]$ is said to be a* fuzzy negation *if it is decreasing with $N(0)=1$ and $N(1)=0$. A fuzzy negation N is said to be*

- strict *when it is strictly decreasing and continuous.*
- strong *when it is an involution, i.e., $N(N(x))=x$ for all $x\in[0,1]$.*

Studies on symmetry of subsets of $[0,1]$ were initially done in [19] and [20]. See also [1] for the following adapted definitions.

Definition 2 *([1]) Let $N : [0,1] \to [0,1]$ be any fuzzy negation and let G be the graph of N, that is*

$$G = \{(x, N(x)) \mid x \in [0,1]\}.$$

For any point of discontinuity s of N, let s^- be the limit from left and s^+ be the limit from right, with the convention $s^- = 1$ when $s = 0$ and $s^+ = 0$ when $s = 1$. Then, we define the completed graph *of N, denoted by $G(N)$, as the set obtained from G by adding the vertical segments from s^- to s^+ in any discontinuity point s.*

Definition 3 *([1]) A subset S of $[0,1]^2$ is said to be* Id-symmetrical *if for all $(x,y) \in [0,1]^2$ it holds that*

$$(x,y) \in S \iff (y,x) \in S\,.$$

Definition 4 *([1]) A fuzzy negation $N : [0,1] \to [0,1]$ is called* Id-symmetrical *if its completed graph $G(N)$ is Id-symmetrical.*

The following theorem gives a mathematical description of Id-symmetrical fuzzy negations.

Theorem 5 *([1]) Let $N : [0,1] \to [0,1]$ be a fuzzy negation. The following items are equivalent:*

i) N is Id-symmetrical

ii) N satisfies the following two conditions:

Condition (A) *For all $x \in [0,1]$ it is*

$$\inf\{y \in [0,1] \mid N(y) = N(x)\} \leq N(N(x)) \leq \sup\{y \in [0,1] \mid N(y) = N(x)\}$$

Condition (B) *N is constant, say $N(x) = s$ in the interval $]p, q[$ with $p < q$, where*

$$p = \inf\{y \in [0,1] \mid N(y) = s\}$$

and

$$q = \sup\{y \in [0,1] \mid N(y) = s\},$$

if and only if, $s \in]0,1[$ is a point of discontinuity of N and it is satisfied that

$$p = s^+ \quad \text{and} \quad q = s^-\,.$$

Definition 6 *([9], [14]) A binary function $F : [0,1] \times [0,1] \to [0,1]$ will be called an* aggregation function *when it is non-decreasing in each place, $F(0,0) = 0$ and $F(1,1) = 1$. F is said to be a conjunctor when $F(1,0) = F(0,1) = 0$ for all $x \in [0,1]$.*

Let N be any fuzzy negation and let us define the function F_N in the following way:

$$F_N(x,y) = \max(0, (x \wedge y) - N(x \vee y)) \qquad (1)$$

for all $x, y \in [0,1]$.

Next proposition gives a list of properties of this function (see [1] for details).

Proposition 7 *Let N be a fuzzy negation and let F_N be given by equation (1). Then*

i) F_N is a commutative and conjunctive aggregation function.

ii) F_N has 0 as annihilator element and 1 as neutral element.

iii) F_N is always a semicopula.

Let

$$a_N = \inf\{x \in [0,1] \mid N(x) \leq x\} \qquad (2)$$

In [1] it is proved that there always exists an Id-symmetrical fuzzy negation $\mathbf{N}$ such that $F_{\mathbf{N}} = F_N$, $a_{\mathbf{N}} = a_N$, and the Id-symmetrical fuzzy negation $\mathbf{N}$ can be taken such that $\mathbf{N}(a_{\mathbf{N}}) \leq a_{\mathbf{N}}$. Moreover, if $F_{\mathbf{N}}$ is left-continuous, then $\mathbf{N}(a_{\mathbf{N}}) = a_{\mathbf{N}}$. In that paper, the following proposition is also proved:

Proposition 8 *Let N be an Id-symmetrical fuzzy negation with $N(a_N) = a_N$. Then the following items are equivalent:*

i) N is left-continuous (continuous) in $[a_N, 1]$.

ii) F_N is left-continuous (continuous).

Definition 9 *([13], [4]) A binary operator $I : [0,1] \times [0,1] \to [0,1]$ is said to be an implication operator, or an implication, if it satisfies:*

I1) I is decreasing in the first variable and increasing in the second one, that is, for all $x, x_1, x_2, y, y_1, y_2 \in [0,1]$,

$$\text{if } x_1 \leq x_2, \text{ then } I(x_1, y) \geq I(x_2, y)$$

and

$$\text{if } y_1 \leq y_2, \text{ then } I(x, y_1) \leq I(x, y_2)$$

I2) $I(0,0) = I(1,1) = 1$ and $I(1,0) = 0$.

Note that, from the definition, it follows that $I(0,x) = 1$ and $I(x,1) = 1$ for all $x \in [0,1]$ whereas the symmetrical values $I(x,0)$ and $I(1,x)$ are not derived from the definition.

Among many other properties usually required for fuzzy implications we recall here some of the most important ones.

i) Contraposition with respect to a fuzzy negation N, $(CP(N))$:

$$I(x,y) = I(N(y), N(x)), \quad \text{for all} \quad x, y \in [0,1].$$

ii) Exchange Principle, (EP):

$$I(x, I(y,z)) = I(y, I(x,z)), \text{ for all } x, y, z \in [0,1].$$

iii) (Left) Neutrality Property, (NP):

$$I(1,y) = y \qquad \text{for all} \quad y \in [0,1].$$

iv) Ordering Property, (OP):

$$I(x,y)=1 \iff x \le y \quad \text{for all} \quad x,y\in[0,1].$$

v) Strong Negation Principle, (SN):

$$I(x,0) \quad \text{is a strong negation for all} \quad x\in[0,1].$$

vi) Identity Principle, (IP):

$$I(x,x)=1 \quad \text{for all} \quad x\in[0,1].$$

The residuum or the R-implication derived from a t-norm T has been extensively studied, specially when T is left-continuous (see [4]). Moreover, many generalizations are introduced obtaining residuum from uninorms, copulas and in fact, from any binary operator ([4], [12], [26]).

Definition 10 *([12],[26]) Let F be a conjunctor. The R-implication defined from F is the binary operation on $[0,1]$ given by*

$$I_F(x,y)=\sup\{z\in[0,1] \mid F(x,z)\le y\} \tag{3}$$

for all $x,y\in[0,1]$.

Since F is in particular a conjunctor, the expression above always gives an implication in the sense of Definition 9.

3. R-implications defined from fuzzy negations

We want to deal in this section with residual implications obtained from semicopulas F_N defined as in (1), with N an Id-symmetrical fuzzy negation such that $N(a_N)=a_N$.

First of all, we prove that for Id-symmetrical fuzzy negations, left-continuity and super-involutivity are equivalent conditions.

Proposition 11 *Let N be an Id-symmetrical fuzzy negation. Then N is left-continuous if, and only, if $N^2(x)\ge x\ \forall x\in[0,1]$.*

Proof: It is already known that if a decreasing unary operator N is super-involutive, then it is left-continuous (see [10]). Let us prove the converse.

- If N is strictly decreasing in x, then since N is Id-symmetrical we have that $N^2(x)=x$.
- For any interval where N is constant, let us prove that $N^2(x)\ge x$ holds for any x in this interval. Let us suppose that N is constant, say $N(x)=s$ in the interval $]p,q[$ with $p<q$, where

  $$p=\inf\{y\in[0,1] \mid N(y)=s\}$$

  and

  $$q=\sup\{y\in[0,1] \mid N(y)=s\},$$

  then by Theorem 5, $s\in]0,1[$ is a point of discontinuity of N and it is satisfied that

  $$p=s^+ \quad \text{and} \quad q=s^- .$$

  Now, since N is left-continuous, $N(s)=s^-=q$, that is, $N^2(x)=N(s)=q\ge x$.
- Finally, if x is a point of discontinuity of N, let $p=x^+$ and $q=x^-$. Since N is left-continuous, $N(x)=q$. Thus by Theorem 5, $N(y)=x$ for all $y\in]p,q[$, where

  $$p=\inf\{y\in[0,1] \mid N(y)=x\}$$

  and

  $$q=\sup\{y\in[0,1] \mid N(y)=x\}$$

  Then, since N is left-continuous, $N(q)=x$ and thus $N^2(x)=N(N(x))=N(q)=x$. ∎

Let N be an Id-symmetrical, left-continuous fuzzy negation with $N(a_N)=a_N$. Let

$$\alpha=\inf\{x\in]0,1] \mid N(x)=0\} \tag{4}$$

Let δ be the diagonal section of the semicopula F_N defined in Equation (1), that is,

$$\delta(x)=F_N(x,x)=\begin{cases} 0 & \text{if } x<a_N \\ x-N(x) & \text{if } x\ge a_N \end{cases}$$

Then $\delta(x)=0$ for all $x\le a_N$, $\delta(1)=1$, $\delta(x)<x$ for all $0<x<\alpha$ and the restriction of δ to the interval $[a_N,1]$ is an increasing function from $[a_N,1]$ to $[0,1]$.

The following theorem gives the expression of the R-implication I_N derived from F_N.

Theorem 12 *Let N be an Id-symmetrical, left-continuous fuzzy negation and δ the diagonal section of F_N. Then the R-implication I_N derived from F_N is given by*

$$I_N(x,y)=\begin{cases} 1 & \text{if } x\le y \\ N(x)+y & \text{if } x>a_N \text{ and } y<\delta(x) \\ N(x-y) & \text{otherwise} \end{cases} \tag{5}$$

Proof: Let us consider first the case when $x\le y$. In this case, since F_N is non-decreasing with $F_N(x,1)=x$, we clearly have

$$I_N(x,y)=\sup\{z\in[0,1] \mid F_N(x,z)\le y\}=1.$$

On the other hand, we divide the case when $x>y$ in two parts by considering the following two regions:

$$R_1=\{(x,y)\in[0,1]^2 \mid \ y<x\le a_N \quad \text{or} \quad x>a_N \text{ with } \delta(x)\le y<x\},$$

and

$$R_2=\{(x,y)\in[0,1]^2 \mid \ x>a_N \quad \text{and} \quad y<\delta(x)\}.$$

- If $(x,y) \in R_1$, we have $\delta(x) \leq y$ which implies that

$$I_N(x,y) = \sup\{z \in [0,1] \mid F_N(x,z) \leq y\} \geq x.$$

Thus, to reach the value of $I_F(x,y)$ we must look for $z \geq x$ such that $F_N(x,z) \leq y$. But for these values we have

$$\begin{aligned} F_N(x,z) \leq y &\iff \max(0, x - N(z)) \leq y \\ &\iff x - N(z) \leq y \\ &\iff N(z) \geq x - y \\ &\iff z \leq N(x-y) \end{aligned}$$

since N is super-involutive (Proposition 11). Thus $I_N(x,y) = N(x-y)$.

- If $(x,y) \in R_2$ we have $x > a_N$ and $x - N(x) > y$. This implies that $I_N(x,y) \leq x$ and in this case, to reach the value of $I_N(x,y)$ we must look for $z < x$ such that $F_N(x,z) \leq y$. But for these values we have

$$\begin{aligned} F_N(x,z) \leq y &\iff \max(0, z - N(x)) \leq y \\ &\iff z - N(x) \leq y \\ &\iff z \leq N(x) + y \end{aligned}$$

Thus $I_N(x,y) = N(x) + y$. ∎

Observe that the expression (5) gives implications in more general cases, even when N is not Id-symmetrical.

Proposition 13 *Let N be a fuzzy negation. Then the function I_N defined from N through the expression (5) is an implication if, and only if,*

$$N^2(x) \geq x \ \forall x \in [a_N, \alpha] \tag{6}$$

Proof: The border conditions follow immediately from the expression of I_N. To prove the decreasingness with respect to the first component and the increasingness with respect to the second component, observe that, in R_1, $I_N(x,y) = N(x-y)$, which is decreasing in x and increasing in y, and in R_2, $I_N(x,y) = N(x) + y$, also decreasing in x and increasing in y. Then we only have to prove that when $0 < y = \delta(x) < x, N(x-y) \geq N(x) + y$. But since $\delta(x) = x - N(x)$, we have to prove that $N(x - (x - N(x))) \geq N(x) + x - N(x)$, that is, $N(N(x)) \geq x$ and this is true if, and only if, N is super-involutive in $[a_N, \alpha]$. ∎

Remark 14 *The expression (5) defines a function I_N directly from a fuzzy negation N and we have seen that it is an implication if, and only if, the negation is super-involutive in $[a_N, \alpha]$. Observe that this implication I_N does not need to be the residuation of any F.*

Next proposition gives a list of properties satisfied by the function I_N given by (5).

Proposition 15 *Let N be a fuzzy negation and I_N the function given by (5).*

1) The negation induced by I_N is $N_{I_N}(x) = I_N(x,0) = N(x)$ and, thus, it satisfies (SN) if, and only if, N is a strong negation.
2) I_N satisfies (NP).
3) $I_N(x,y) \geq y$ for all $x,y \in [0,1]$.
4) I_N satisfies (IP).
5) I_N satisfies (OP) if, and only if, $N(x) < 1 \ \forall x > 0$.
6) I_N is continuous if, and only if, N is continuous, $N(x) > 0 \ \forall x < 1$, and $N^2(x) = x \ \forall x \in [a_N, 1]$.

Proof: The first four properties come directly from the expression of I_N.

To prove 5), observe that from the expression (5), we have that $I_N(x,y) = 1$ for all $x \leq y$. Now, if $(x,y) \in R_1$ is such that $x > y$, then $I_N(x,y) = N(x-y) < 1$ if, and only if, $N(x) < 1 \ \forall x > 0$. Finally, the decreasingness of I_N with respect to the first component gives $I_N(x,y) < 1$ in R_2.

Let us now prove 6). Suppose first that N satisfies the conditions stated in 6). Since N is continuous, we only have to prove the continuity of I_N at points of the form $(x, \delta(x))$ and (x,x). The continuity at $(x, \delta(x))$ is equivalent to the involutivity of N, since $N(x - \delta(x)) = N(x - (x - N(x))) = N^2(x)$ and $N(x) + \delta(x) = N(x) + (x - N(x)) = x$. Finally, observe that the continuity at (x,x) is obvious in R_1 and it never holds in R_2; thus I_N is continuous at (x,x) if, and only if, R_2 has not contact with the diagonal, that is, if, and only if, $N(x) > 0 \ \forall x < 1$. Finally, the converse follows trivially. ∎

Proposition 16 *Let N be a fuzzy negation and I_N the function given by (5). If N is not a strong negation, then I_N does not satisfy (CP) with respect to any fuzzy negation.*

Proof: Since I_N satisfies (NP), the result is an immediate consequence of Corollary 1.5.5 in [4]. ∎

On the other hand, if N is strong, the corresponding implication I_N may or may not satisfy (CP) with respect to N (see examples 18-*i*) and 18-*iv*)).

The following result is proved in [26].

Proposition 17 *Let F be nondecreasing with respect to both arguments. If F is left-continuous, commutative and associative, then I_F satisfies (EP).*

It is known that when N is Id-symmetrical, the corresponding F_N is in fact a t-norm in many cases (see Theorem 28 in [1]), and it is left-continuous when so is N. In all of theses cases I_N clearly satisfies (EP) (see examples 18-*i*) and 18-*ii*))

Example 18

i) Let us consider the classical negation $N(x) = 1 - x$*, which is Id-symmetrical. Then the corresponding semicopula* $F_N = T_{\mathbf{L}}$ *is the Łukasiewicz t-norm and then* I_N *is the residuum of* $T_{\mathbf{L}}$*, that is, the well known Łukasiewicz implication*

$$I_{\mathbf{L}}(x,y) = \min\{1, 1-x+y\}\,,\ \ x,y \in [0,1].$$

ii) Let us consider the weakest fuzzy negation

$$N_{\mathbf{wt}}(x) = \begin{cases} 0 & \text{if } x > 0 \\ 1 & \text{if } x = 0. \end{cases}$$

Note that $N_{\mathbf{wt}}$ *is again Id-symmetrical. In this case the corresponding semicopula* $F_N = T_{\mathbf{M}}$ *is the Minimum t-norm and so* I_N *is the residuum of the Minimum, that is, the Gödel implication:*

$$I_{GD}(x,y) = \begin{cases} 1 & \text{if } x \le y \\ y & \text{if } y < x \end{cases}$$

iii) Let $a \in]0,1[$ *and let* N *be the fuzzy negation defined by*

$$N(x) = \begin{cases} 1 & \text{if } x \le a \\ a & \text{if } x = a \\ 0 & \text{if } x > a. \end{cases}$$

N *is not Id-symmetrical, but it satisfies that* $N^2(x) \ge x\ \forall x \in [a_N, \alpha]$ *since in this case,* $a_N = \alpha = a$ *and* $N(a) = a$*. Thus, by applying Proposition 13 we have that* I_N *is an implication, which is represented in Figure 1 and is given by*

$$I_N(x,y) = \begin{cases} a & \text{if } x = a \text{ and } y = 0 \\ y & \text{if } x > a \text{ and } x > y \\ 1 & \text{otherwise.} \end{cases}$$

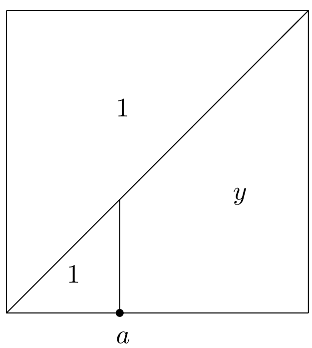

Figure 1: The implication I_N of Example 18-*iii*).

This implication does not satisfy (OP) *since* $N(x) = 1$ *for all* $0 < x < a$ *(using Proposition 15), nor* (CP) *with respect to any fuzzy negation (using Proposition 16 since* N *is not a strong negation), but it satisfies* (EP) *as it can be proved by direct computation.*

iv) Let $a \in]0,1/2[$ *and let us consider the following Id-symmetrical fuzzy negation:*

$$N(x) = \begin{cases} \dfrac{a-1}{a}x + 1 & \text{if } x \in [0,a] \\[2ex] \dfrac{a}{a-1}(x-1) & \text{if } x \in]a,1] \end{cases}$$

Note that in this case the corresponding semicopula F_N *coincides with the so-called singular conic copulas introduced by Jwaid et al. (see Example 8 in [15]) and its residuum is the implication* I_N *given by*

$$I_N(x,y) = \begin{cases} 1 & \text{if } x \le y \\[1ex] \dfrac{a}{a-1}(x-1) + y & \text{if } x > a \text{ and } y < \dfrac{x-a}{1-a} \\[2ex] \dfrac{a-1}{a}(x-y) + 1 & \text{otherwise.} \end{cases}$$

The structure of this implication I_N *can be viewed in Figure 2.*

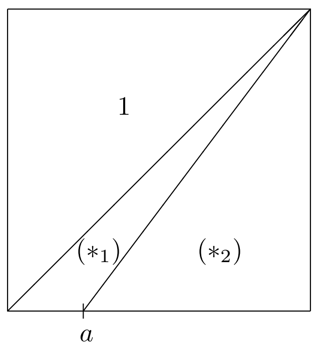

Figure 2: The implication of Example 18-*iv*) (where $(*_1)$ stands for $\frac{a-1}{a}(x-y)+1$ and $(*_2)$ stands for $\frac{a}{a-1}(x-1)+y$).

Note that this implication satisfies (OP) *since* $N(x) < 1$ *for all* $x > 0$*, and it is continuous since* N *is strong. Moreover, although* N *is strong,* I_N *does not satisfy* (CP) *with respect to* N *(just take, for instance,* $x = a$ *and* $y = a/2$*) and thus it does not satisfy* (CP) *with respect to any fuzzy negation (see Lemma 1.5.4 in [4]). Consequently,* I_N *can not satisfy* (EP) *by Lemma 1.5.6 in [4].*

v) Let $a \in]0,1[$ *and let us consider the following fuzzy negation:*

$$N(x) = \begin{cases} 1 - \dfrac{2-a}{2a}x & \text{if } x \in [0,a] \\[2ex] 0 & \text{if } x \in]a,1] \end{cases}$$

Then $a_N = \frac{2a}{a+2}, \alpha = a$ *and the corresponding function* I_N *is given by*

$$I_N(x,y) = \begin{cases} 1 & \text{if } x \le y \\ y & \text{if } (x,y) \in R_2 \\ 1 - \dfrac{2-a}{2a}(x-y) & \text{otherwise,} \end{cases}$$

where R_2 is the region delimited by the points $(x, y) \in [0, 1]^2$ such that

$$x > \frac{2a}{a+2} \quad and \quad y < \delta(x),$$

being δ the diagonal section given by

$$\delta(x) = \begin{cases} 0 & if\ x \leq \frac{2a}{a+2} \\ \frac{a+2}{2a}x - 1 & if\ \frac{2a}{a+2} < x \leq a \\ x & if\ x > a. \end{cases}$$

The fuzzy negation N and the corresponding implication I_N can be viewed in Figures 3 and 4, respectively.

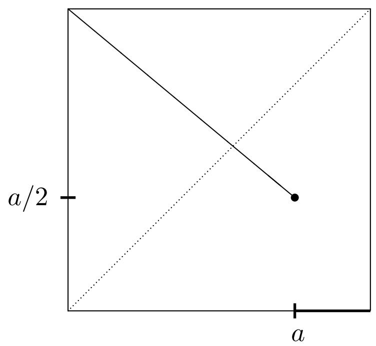

Figure 3: The fuzzy negation of Example 18-v).

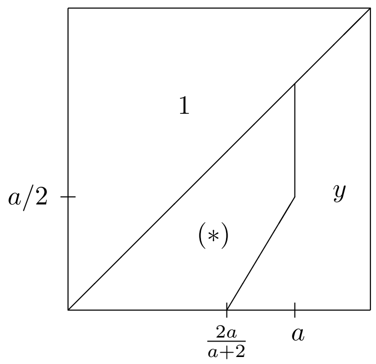

Figure 4: The implication of Example 18-v) (where $(*)$ stands for $1 - \frac{2-a}{2a}(x - y)$).

A straightforward computation proves that $N^2(x) \geq x\ \forall x \in \left[\frac{2a}{a+2}, a\right]$ and so I_N is an implication. This implication satisfies (OP) since $N(x) < 1\ \forall x > 0$, but it does not satisfy neither (CP) with respect to any fuzzy negation (again by applying Proposition 16) nor (EP).

All the previous examples present a wide range of implication functions derived from fuzzy negations with different properties and lead to some open questions to deal with. For instance,

- Characterize those negations N for which the corresponding I_N satisfies (CP) with respect to N (of course by Proposition 16, such an N should be strong),
- Characterize those negations N for which the corresponding I_N satisfies (EP). Note that in this case the result should include all Id-symmetrical fuzzy negations for which the corresponding associated semicopula F_N is in fact a left-continuous t-norm (see Theorem 28 in [1]).

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