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Real Forms of the Complex Twisted N=2 Supersymmetric Toda Chain Hierarchy in Real N=1 and Twisted N=2 Superspaces

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Abstract

Three nonequivalent real forms of the complex twisted N=2 supersymmetric Toda chain hierarchy (solv-int/9907021) in real N=1 superspace are presented. It is demonstrated that they possess a global twisted N=2 supersymmetry. We discuss a new superfield basis in which the supersymmetry transformations are local. Furthermore, a representation of this hierarchy is given in terms of two twisted chiral N=2 superfields. The relations to the s-Toda hierarchy by H. Aratyn, E. Nissimov and S. Pacheva (solv-int/9801021) as well as to the modified and derivative NLS hierarchies are established.

1 Introduction

Recently an $N{=}(1|1)$ supersymmetric generalization of the two-dimensional Darboux transformation was proposed in [1] in terms of $N = (1|1)$ superfields, and an infinite class of bosonic and fermionic solutions of its symmetry equation was constructed in [1] and [2], respectively. These solutions generate bosonic and fermionic flows of the complex $N = (1|1)$ supersymmetric Toda lattice hierarchy[1] which actually possesses a more rich symmetry, namely complex $N = (2|2)$ supersymmetry. Its one-dimensional reduction possessing complex $N = 4$ supersymmetry —the complex $N = 4$ Toda chain hierarchy— was discussed in [7]. There, the Lax pair representations of the bosonic and fermionic flows, the corresponding local and nonlocal Hamiltonians, finite and infinite discrete symmetries,

[1] A wide class of the complex Toda lattices connected with Lie superalgebras was first introduced in the pioneering papers [3, 4, 5] (see also recent papers [6] and references therein).

the first two Hamiltonian structures and the recursion operator were constructed. Furthermore, its nonequivalent real forms in real $N = 2$ superspace were analyzed in [8], where the relation to the complex $N = 4$ supersymmetric KdV hierarchy [9] was established.

Consecutively, the reduction of the complex $N = 4$ supersymmetric Toda chain hierarchy from complex $N = 2$ superspace to complex $N = 1$ superspace was analyzed in [10], where also its Lax-pair and Hamiltonian descriptions were developed in detail. Here, we call this reduction the *complex twisted N=2 supersymmetric Toda chain hierarchy*, due to the common symmetry properties of its three different real forms which will be discussed in what follows (see the paragraph after eq. (2.11)). The main goals of the present letter are firstly to analyze real forms of this hierarchy in real $N = 1$ superspace with one even and one odd real coordinate, secondly to derive a manifest twisted $N = 2$ supersymmetric representation of its simplest non-trivial even flows in twisted $N = 2$ superspace, and thirdly to clarify its relations (if any) with other known hierarchies (s-Toda [11], modified NLS and derivative NLS hierarchies).

Let us start with a short summary of the results that we shall need concerning the complex twisted $N = 2$ supersymmetric Toda chain hierarchy (see [10, 7, 1, 2] for more details).

The complex twisted $N = 2$ supersymmetric Toda chain hierarchy in complex $N = 1$ superspace comprises an infinite set of even and odd flows for two complex even $N = 1$ superfields $u(z, \theta)$ and $v(z, \theta)$, where z and θ are complex even and odd coordinates, respectively. The flows are generated by complex even and odd evolution derivatives $\{\frac{\partial}{\partial t_k},\ U_k\}$ and $\{D_k,\ Q_k\}$ ($k \in \mathbb{N}$), respectively, with the following length dimensions:

$$[\tfrac{\partial}{\partial t_k}] = [U_k] = -k, \quad [D_k] = [Q_k] = -k + \frac{1}{2}, \tag{1.1}$$

which are derived by the reduction of the supersymmetric KP hierarchy in $N = 1$ superspace [12], characterized by the Lax operator

$$L = Q + vD^{-1}u. \tag{1.2}$$

D and Q are the odd covariant derivative and the supersymmetry generator, respectively,

$$D \equiv \frac{\partial}{\partial\theta} + \theta\partial, \quad Q \equiv \frac{\partial}{\partial\theta} - \theta\partial. \tag{1.3}$$

They form the algebra[2]

$$\{D, D\} = +2\partial, \quad \{Q, Q\} = -2\partial. \tag{1.4}$$

The first few of these flows are:

$$\tfrac{\partial}{\partial t_0}\begin{pmatrix} v \\ u \end{pmatrix} = \begin{pmatrix} +v \\ -u \end{pmatrix}, \quad \tfrac{\partial}{\partial t_1}\begin{pmatrix} v \\ u \end{pmatrix} = \partial\begin{pmatrix} v \\ u \end{pmatrix}, \tag{1.5}$$

$$\begin{aligned} &\tfrac{\partial}{\partial t_2}v = +v\,'' - 2uv(DQv) + (DQv^2u) + v^2(DQu) - 2v(uv)^2, \\ &\tfrac{\partial}{\partial t_2}u = -u\,'' - 2uv(DQu) + (DQu^2v) + u^2(DQv) + 2u(uv)^2, \end{aligned} \tag{1.6}$$

[2]We explicitly present only non-zero brackets in this letter.

$$\begin{aligned}\frac{\partial}{\partial t_3}v &= v\,''' + 3(Dv)\,'(Quv) - 3(Qv)\,'(Duv) + 3v\,'(Du)(Qv) \\ &\quad - 3v\,'(Qu)(Dv) + 6vv\,'(DQu) - 6(uv)^2 v\,', \\ \frac{\partial}{\partial t_3}u &= u\,''' + 3(Qu)\,'(Duv) - 3(Du)\,'(Quv) + 3u\,'(Qv)(Du) \\ &\quad - 3u\,'(Dv)(Qu) + 6uu\,'(QDv) - 6(uv)^2 u\,',\end{aligned} \tag{1.7}$$

$$\begin{aligned}D_1 v &= -Dv + 2vQ^{-1}(uv), \quad D_1 u = -Du - 2uQ^{-1}(uv), \\ Q_1 v &= -Qv - 2vD^{-1}(uv), \quad Q_1 u = -Qu + 2uD^{-1}(uv),\end{aligned} \tag{1.8}$$

$$U_0 \begin{pmatrix} v \\ u \end{pmatrix} = \theta D \begin{pmatrix} v \\ u \end{pmatrix}. \tag{1.9}$$

Throughout this letter, we shall use the notation $u' = \partial u = \frac{\partial}{\partial z} u$. Using the explicit expressions of the flows (1.5)–(1.9), one can calculate their algebra which has the following nonzero brackets:

$$\Big\{D_k\,,\,D_l\Big\} = -2\,\frac{\partial}{\partial t_{k+l-1}}, \quad \Big\{Q_k\,,\,Q_l\Big\} = +2\,\frac{\partial}{\partial t_{k+l-1}}, \tag{1.10}$$

$$\Big[U_k\,,\,D_l\Big] = Q_{k+l}, \qquad \Big[U_k\,,\,Q_l\Big] = D_{k+l}. \tag{1.11}$$

This algebra produces an affinization of the algebra of global complex $N = 2$ supersymmetry, together with an affinization of its $gl(1,\mathbb{C})$ automorphisms. It is the algebra of symmetries of the nonlinear even flows (1.6)–(1.7). The generators may be realized in the superspace $\{t_k, \theta_k, \rho_k, h_k\}$,

$$\begin{aligned}D_k &= \frac{\partial}{\partial \theta_k} - \sum_{l=1}^{\infty} \theta_l \frac{\partial}{\partial t_{k+l-1}}, \quad Q_k = \frac{\partial}{\partial \rho_k} + \sum_{l=1}^{\infty} \rho_l \frac{\partial}{\partial t_{k+l-1}}, \\ U_k &= \frac{\partial}{\partial h_k} - \sum_{l=1}^{\infty} (\theta_l \frac{\partial}{\partial \rho_{k+l}} + \rho_l \frac{\partial}{\partial \theta_{k+l}}),\end{aligned} \tag{1.12}$$

where t_k, h_k (θ_k, ρ_k) are bosonic (fermionic) abelian evolution times with length dimensions

$$[t_k] = [h_k] = k, \quad [\theta_k] = [\rho_k] = k - \frac{1}{2} \tag{1.13}$$

which are in one-to-one correspondence with the length dimensions (1.1) of the corresponding evolution derivatives.

The flows $\{\frac{\partial}{\partial t_k},\ D_k,\ Q_k\}$ can be derived from the flows $\{\frac{\partial}{\partial t_k},\ D_k^+,\ D_k^-\}$ of the complex $N = 4$ Toda chain hierarchy [7] by the reduction constraint

$$\theta^+ = i\theta^- \equiv \theta \tag{1.14}$$

which leads to the correspondence $D_+ \equiv D$ and $D_- \equiv iQ$ with the fermionic derivatives of the present paper, where i is the imaginary unit and $\theta^\pm$ are the Grassmann coordinates of the $N = 2$ superspace in [7].

2 Real forms of the complex twisted N=2 Toda chain hierarchy

It is well known that different real forms derived from the same complex integrable hierarchy are nonequivalent in general. Keeping this in mind it seems important to find as many different real forms of the complex twisted $N=2$ Toda chain hierarchy as possible.

With this aim let us discuss various nonequivalent complex conjugations of the superfields $u(z,\theta)$ and $v(z,\theta)$, of the superspace coordinates $\{z,\ \theta\}$, and of the evolution derivatives $\{\frac{\partial}{\partial t_k},\ U_k,\ D_k,\ Q_k\}$ which should be consistent with the flows (1.5)–(1.9). We restrict our considerations to the case when iz and θ are coordinates of real $N=1$ superspace which satisfy the following standard complex conjugation properties:

$$(iz,\theta)^* = (iz,\theta). \tag{2.1}$$

We will also use the standard convention regarding complex conjugation of products involving odd operators and functions (see, e.g., the books [13]). In particular, if $\mathbb{D}$ is some even differential operator acting on a superfield F, we define the complex conjugate of $\mathbb{D}$ by $(\mathbb{D}F)^* = \mathbb{D}^* F^*$. Then, in the case under consideration one can derive, for example, the following relations

$$\begin{aligned} &\partial^* = -\partial, \quad \epsilon^* = \epsilon, \quad \varepsilon^* = \varepsilon, \quad (\epsilon\varepsilon)^* = -\epsilon\varepsilon, \\ &(\epsilon D)^* = \epsilon D, \quad (\varepsilon Q)^* = \varepsilon Q, \quad (DQ)^* = -DQ \end{aligned} \tag{2.2}$$

which we use in what follows. Here, ϵ and ε are constant odd real parameters.

Let us remark that, although most of the flows of the complex twisted $N=2$ supersymmetric Toda chain hierarchy can be derived by reduction (1.14), its real forms in $N=1$ superspace (2.1) cannot be derived in this way from the real forms of the complex $N=4$ Toda chain hierarchy in the real $N=2$ superspace

$$(iz,\theta^{\pm})^* = (iz,\theta^{\pm}) \tag{2.3}$$

found in [8]. This conflict arises because the constraint (1.14) is inconsistent with the reality properties (2.3) of the $N=2$ superspace.

We would like to underline that the flows (1.5)–(1.9) form a particular realization of the algebra (1.10)–(1.11) in terms of the $N=1$ superfields $u(z,\theta)$ and $v(z,\theta)$. Although the classification of real forms of affine and conformal superalgebras was given in a series of classical papers [14, 15] (see also interesting paper [16] for recent discussions and references therein) we cannot obtain the complex conjugations of the target space superfields $\{u(z,\theta), v(z,\theta)\}$ using only this base. It is a rather different, non-trivial task to construct the corresponding complex conjugations of various realizations of a superalgebra which are relevant in the context of integrable hierarchies. Moreover, different complex conjugations of a given (super)algebra realization may correspond to the same real form of the (super)algebra, while some of its other existing real forms may not be reproducible on the base of a given particular realization. In what follows we will demonstrate that this is exactly the case for the realization under consideration. We shall see that complex conjugations of the target space superfields $\{u(z,\theta), v(z,\theta)\}$ correspond to the twisted real $N=2$ supersymmetry.

Direct verification shows that the flows (1.5)–(1.9) admit the following three nonequivalent complex conjugations (meaning that it is not possible to relate them via obvious symmetries):

$$\begin{aligned}&(v,u)^{*}=(v,-u),\quad (iz,\theta)^{*}=(iz,\theta),\\&(t_p,U_p,\epsilon_pD_p,\varepsilon_pQ_p)^{*}=(-1)^p(t_p,U_p,-\epsilon_pD_p,-\varepsilon_pQ_p),\end{aligned}\tag{2.4}$$

$$\begin{aligned}&(v,u)^{\bullet}=(u,v),\quad (iz,\theta)^{\bullet}=(iz,\theta),\\&(t_p,U_p,\epsilon_pD_p,\varepsilon_pQ_p)^{\bullet}=(-t_p,U_p,\epsilon_pD_p,\varepsilon_pQ_p),\end{aligned}\tag{2.5}$$

$$\begin{aligned}&(v,\ u)^{\star}=(\ -u(QD\ln u+uv),\ \frac{1}{u}\ ),\quad (iz,\theta)^{\star}=(iz,\theta),\\&(t_p,U_p,\epsilon_pD_p,\varepsilon_pQ_p)^{\star}=(-t_p,U_p,-\epsilon_pD_p,-\varepsilon_pQ_p),\end{aligned}\tag{2.6}$$

where ϵ_p and ε_p are constant odd real parameters. We would like to underline that the complex conjugations of the evolution derivatives (the second lines of eqs. (2.4)–(2.6)) are defined and fixed completely by the explicit expressions (1.5)–(1.9) for the flows. These complex conjugations extract three different real forms of the complex integrable hierarchy we started with, while all the real forms of the flows algebra (1.10)–(1.11) correspond to the same algebra of a twisted global real $N=2$ supersymmetry. This last fact becomes obvious if one uses the $N=2$ basis of the algebra with the generators

$$\mathcal{D}_1\equiv\frac{1}{\sqrt{2}}(Q_1+D_1),\quad \overline{\mathcal{D}}_1\equiv\frac{1}{\sqrt{2}}(Q_1-D_1).\tag{2.7}$$

Then, the nonzero algebra brackets (1.10)–(1.11) and the complex conjugation rules (2.4)–(2.6) are the standard ones for the twisted $N=2$ supersymmetry algebra together with its non–compact $o(1,1)$ automorphism,

$$\left\{\mathcal{D}_1\,,\,\overline{\mathcal{D}}_1\right\}=2\,\frac{\partial}{\partial t_1},\quad \left[U_0\,,\,\mathcal{D}_1\right]=+\mathcal{D}_1,\qquad \left[U_0\,,\,\overline{\mathcal{D}}_1\right]=-\overline{\mathcal{D}}_1,\tag{2.8}$$

$$(\tfrac{\partial}{\partial t_1},U_0,\gamma_1\mathcal{D}_1,\overline{\gamma}_1\overline{\mathcal{D}}_1)^{*}=(-\tfrac{\partial}{\partial t_1},U_0,+\gamma_1\mathcal{D}_1,+\overline{\gamma}_1\overline{\mathcal{D}}_1),\quad (\gamma_1,\overline{\gamma}_1)^{*}=(\gamma_1,\overline{\gamma}_1),\tag{2.9}$$

$$(\tfrac{\partial}{\partial t_1},U_0,\gamma_1\mathcal{D}_1,\overline{\gamma}_1\overline{\mathcal{D}}_1)^{\bullet}=(-\tfrac{\partial}{\partial t_1},U_0,+\gamma_1\mathcal{D}_1,+\overline{\gamma}_1\overline{\mathcal{D}}_1),\quad (\gamma_1,\overline{\gamma}_1)^{\bullet}=(\gamma_1,\overline{\gamma}_1),\tag{2.10}$$

$$(\tfrac{\partial}{\partial t_1},U_0,\gamma_1\mathcal{D}_1,\overline{\gamma}_1\overline{\mathcal{D}}_1)^{\star}=(-\tfrac{\partial}{\partial t_1},U_0,-\gamma_1\mathcal{D}_1,-\overline{\gamma}_1\overline{\mathcal{D}}_1),\quad (\gamma_1,\overline{\gamma}_1)^{\star}=(\gamma_1,\overline{\gamma}_1),\tag{2.11}$$

where $\gamma_1,\overline{\gamma}_1$ are constant odd real parameters. Therefore, we conclude that the complex twisted $N=2$ supersymmetric Toda chain hierarchy with the complex conjugations (2.4)–(2.6) possesses twisted real $N=2$ supersymmetry. For this reason we like to call it the "twisted $N=2$ supersymmetric Toda chain hierarchy" (for the supersymmetric Toda chain hierarchy possessing untwisted $N=2$ supersymmetry see [17] and references therein).

Let us remark that a combination of the two involutions (2.6) and (2.5) generates the infinite-dimensional group of discrete Darboux transformations [10]

$$\begin{aligned}&(v,\ u)^{\star\bullet}=(\ v(QD\ln v-uv),\ \frac{1}{v}\ ),\quad (z,\theta)^{\star\bullet}=(z,\theta),\\&(t_p,U_p,D_p,Q_p)^{\star\bullet}=(t_p,U_p,-D_p,-Q_p).\end{aligned}\tag{2.12}$$

This way of deriving discrete symmetries was proposed in [18] and applied to the construction of discrete symmetry transformations of the $N=2$ supersymmetric GNLS hierarchies.

To close this section let us stress once more that we cannot claim to have exhausted *all* complex conjugations of the twisted $N=2$ Toda chain hierarchy by the three examples of complex conjugations (eqs. (2.4)–(2.6)) we have constructed. Finding complex conjugations for affine (super)algebras themselves is a problem solved by the classification of [14] but rather different from constructing complex conjugations for different *realizations* of affine (super)algebras. To our knowledge, no algorithm yet exists for solving this rather complicated second problem. Thus, classifying *all* complex conjugations is out of the scope of the present letter. Rather, we have constructed these examples in order to use them merely as tools to generate the important discrete symmetries (2.12) as well as to construct a convenient superfield basis and a manifest twisted $N=2$ superfield representation (see Sections 3 and 4), with the aim to clarify the relationships of the hierarchy under consideration to other physical hierarchies discussed in the literature (see Section 5).

3 A KdV-like basis with locally realized supersymmetries.

The third complex conjugation (2.6) looks rather complicated when compared to the first two ones (2.4)–(2.5). However, it drastically simplifies in another superfield basis defined as

$$J \equiv uv + QD\ln u, \quad \overline{J} \equiv -uv, \tag{3.1}$$

where $J \equiv J(z,\theta)$ and $\overline{J} \equiv \overline{J}(z,\theta)$ ($[J]=[\overline{J}]=-1$) are unconstrained even $N=1$ superfields. In this basis the complex conjugations (2.4)–(2.6) and the discrete Darboux transformations (2.12) are given by

$$(J,\ \overline{J})^{*} = -(J,\ \overline{J}), \tag{3.2}$$

$$(J,\ \overline{J})^{\bullet} = (\ J - QD\ln\overline{J},\ \overline{J}\ ), \tag{3.3}$$

$$(J,\ \overline{J})^{\star} = (\overline{J},\ J), \tag{3.4}$$

$$(J,\ \overline{J})^{\star\bullet} = (\ \overline{J},\ J - QD\ln\overline{J}\ ), \tag{3.5}$$

and the equations (1.6)–(1.9) become simpler as well,

$$\begin{aligned} \frac{\partial}{\partial t_2} J &= (-J\,' + 2JD^{-1}Q\overline{J} - J^2)\,', \\ \frac{\partial}{\partial t_2} \overline{J} &= (+\overline{J}\,' + 2\overline{J}D^{-1}QJ - \overline{J}^2)\,', \end{aligned} \tag{3.6}$$

$$\begin{aligned} \frac{\partial}{\partial t_3} J &= 3\left[\ \frac{1}{3}J\,'' + J\ J\,' - J\,'D^{-1}Q\overline{J} - 2J^2D^{-1}Q\overline{J} - JD^{-1}Q\overline{J}^2 + \frac{1}{3}J^3\ \right]', \\ \frac{\partial}{\partial t_3} \overline{J} &= 3\left[\ \frac{1}{3}\overline{J}\,'' - \overline{J}\ \overline{J}\,' + \overline{J}\,'D^{-1}QJ - 2\overline{J}^2D^{-1}QJ - \overline{J}D^{-1}QJ^2 + \frac{1}{3}\overline{J}^3\ \right]', \end{aligned} \tag{3.7}$$

and then

$$D_1\begin{pmatrix} J \\ \overline{J} \end{pmatrix} = D\begin{pmatrix} +J \\ -\overline{J} \end{pmatrix}, \quad Q_1\begin{pmatrix} J \\ \overline{J} \end{pmatrix} = Q\begin{pmatrix} +J \\ -\overline{J} \end{pmatrix}, \tag{3.8}$$

$$U_0\left(\frac{J}{\bar{J}}\right) = \theta D\left(\frac{J}{\bar{J}}\right). \tag{3.9}$$

Notice that the supersymmetry and $o(1,1)$ transformations (3.8)–(3.9) of the superfields J, $\bar{J}$ are local functions of the superfields. The evolution equations (3.6)–(3.7) are also local because the operator $D^{-1}Q$ is a purely differential one, $D^{-1}Q \equiv [\theta, D]$.

4 A manifest twisted N=2 supersymmetric representation

The existence of a basis with locally and linearly realized twisted $N=2$ supersymmetric flows (3.8) would give evidence in favour of a possible description of the hierarchy in terms of twisted $N=2$ superfields. It turns out that this is indeed the case. In order to show this, let us introduce a twisted $N=2$ superspace with even coordinate z and two odd real coordinates η and $\overline{\eta}$ ($\eta^* = \eta$, $\overline{\eta}^* = \overline{\eta}$), as well as odd covariant derivatives $\mathcal{D}$ and $\overline{\mathcal{D}}$ via

$$\mathcal{D} \equiv \frac{\partial}{\partial\eta} + \overline{\eta}\partial, \quad \overline{\mathcal{D}} \equiv \frac{\partial}{\partial\overline{\eta}} + \eta\partial, \quad \left\{\mathcal{D}\,, \overline{\mathcal{D}}\right\} = 2\partial, \quad \mathcal{D}^2 = \overline{\mathcal{D}}^2 = 0 \tag{4.1}$$

together with twisted $N=2$ supersymmetry generators $\mathcal{Q}$ and $\overline{\mathcal{Q}}$

$$\mathcal{Q} \equiv \frac{\partial}{\partial\eta} - \overline{\eta}\partial, \quad \overline{\mathcal{Q}} \equiv \frac{\partial}{\partial\overline{\eta}} - \eta\partial, \quad \left\{\mathcal{Q}\,, \overline{\mathcal{Q}}\right\} = -2\partial, \quad \mathcal{Q}^2 = \overline{\mathcal{Q}}^2 = 0. \tag{4.2}$$

In this space, we consider two chiral even twisted $N=2$ superfields $\{\mathcal{J}(z,\eta,\overline{\eta})\ \overline{\mathcal{J}}(z,\eta,\overline{\eta})\}$, which obey

$$\mathcal{D}\mathcal{J} = 0, \quad \mathcal{D}\overline{\mathcal{J}} = 0 \tag{4.3}$$

and are related to the $N=1$ superfields $\{J(z,\theta),\ \overline{J}(z,\theta)\}$ (3.1). More concretely, their independent components are related to those of J and $\overline{J}$ as follows,

$$\begin{aligned} &\mathcal{J}|_{\eta=\overline{\eta}=0} = J|_{\theta=0}, \quad \overline{\mathcal{D}}\ \mathcal{J}|_{\eta=\overline{\eta}=0} = +DJ|_{\theta=0}, \\ &\overline{\mathcal{J}}|_{\eta=\overline{\eta}=0} = \overline{J}|_{\theta=0}, \quad \overline{\mathcal{D}}\ \overline{\mathcal{J}}|_{\eta=\overline{\eta}=0} = -D\overline{J}|_{\theta=0}. \end{aligned} \tag{4.4}$$

Then, in terms of these superfields the equations (3.6)–(3.7) become

$$\begin{aligned} &\tfrac{\partial}{\partial t_2}\mathcal{J} = (-\mathcal{J}\,' - 2\mathcal{J}\overline{\mathcal{J}} - \mathcal{J}^2)\,', \\ &\tfrac{\partial}{\partial t_2}\overline{\mathcal{J}} = (+\overline{\mathcal{J}}\,' - 2\mathcal{J}\overline{\mathcal{J}} - \overline{\mathcal{J}}^2)\,', \end{aligned} \tag{4.5}$$

$$\begin{aligned} &\tfrac{\partial}{\partial t_3}\mathcal{J} = 3\left(\frac{1}{3}\mathcal{J}\,'' + \mathcal{J}\ \mathcal{J}\,' + \overline{\mathcal{J}}\mathcal{J}\,' + 2\mathcal{J}^2\overline{\mathcal{J}} + \mathcal{J}\overline{\mathcal{J}}^2 + \frac{1}{3}\mathcal{J}^3\right)', \\ &\tfrac{\partial}{\partial t_3}\overline{\mathcal{J}} = 3\left(\frac{1}{3}\overline{\mathcal{J}}\,'' - \overline{\mathcal{J}}\ \overline{\mathcal{J}}\,' - \mathcal{J}\overline{\mathcal{J}}\,' + 2\overline{\mathcal{J}}^2\mathcal{J} + \overline{\mathcal{J}}\ \mathcal{J}^2 + \frac{1}{3}\overline{\mathcal{J}}^3\right)', \end{aligned} \tag{4.6}$$

and it is obvious that they and the chirality constraints (4.3) are manifestly invariant with respect to the transformations generated by the twisted $N=2$ supersymmetry generators $\mathcal{Q}$ and $\overline{\mathcal{Q}}$ (4.2).

Let us also present a manifestly twisted $N=2$ supersymmetric form of the complex conjugations (3.2)–(3.4) and the discrete Darboux transformations (3.5) in terms of the superfields $\mathcal{J}(z,\eta,\overline{\eta})$ and $\overline{\mathcal{J}}(z,\eta,\overline{\eta})$ (4.4):

$$(\mathcal{J},\ \overline{\mathcal{J}})^{*} = -(\mathcal{J},\ \overline{\mathcal{J}}), \tag{4.7}$$

$$(\mathcal{J},\ \overline{\mathcal{J}})^{\bullet} = (\ \mathcal{J} - \partial \ln \overline{\mathcal{J}},\ \overline{\mathcal{J}}\ ), \tag{4.8}$$

$$(\mathcal{J},\ \overline{\mathcal{J}})^{\star} = (\overline{\mathcal{J}},\ \mathcal{J}), \tag{4.9}$$

$$(\mathcal{J},\ \overline{\mathcal{J}})^{\star\bullet} = (\ \overline{\mathcal{J}},\ \mathcal{J} - \partial \ln \overline{\mathcal{J}}\ ), \tag{4.10}$$

modulo the standard automorphism which changes the sign of all Grassmann odd objects.

5 Relation with the s-Toda, modified NLS and derivative NLS hierarchies

It is well known that there are often hidden relationships between a priori unrelated hierarchies. Some examples are the $N=2$ NLS and $N=2$ $\alpha=4$ KdV [19], the "quasi" $N=4$ KdV and $N=2$ $\alpha=-2$ Boussinesq [20], the $N=2$ (1,1)-GNLS and $N=4$ KdV [18, 21], the $N=4$ Toda and $N=4$ KdV [8]. These relationships may lead to a deeper understanding of the hierarchies. They may help to obtain a more complete description and to derive solutions.

The absence of odd derivatives in the equations (4.5)–(4.6), starting off the twisted $N=2$ supersymmetric Toda chain hierarchy, gives additional evidence in favour of a hidden relationship with some bosonic hierarchy. It turns out that such a relationship indeed exists. Let us search it first at the level of the Darboux transformations (4.10), then in the second flow equation (4.5).

For this purpose, we introduce new $N=1$ superfields $\{\Phi(z,\theta), \Psi(z,\theta)\}$ via

$$\begin{aligned} &\mathcal{J}|_{\eta=\overline{\eta}=0} \equiv (\Phi\Psi + \partial \ln \Psi)|_{\theta=0}, && \overline{\mathcal{D}}\ \mathcal{J}|_{\eta=\overline{\eta}=0} \equiv D(\Phi\Psi + \partial \ln \Psi)|_{\theta=0}, \\ &\overline{\mathcal{J}}|_{\eta=\overline{\eta}=0} \equiv -(\Phi\Psi)|_{\theta=0}, && \overline{\mathcal{D}}\ \overline{\mathcal{J}}|_{\eta=\overline{\eta}=0} \equiv -D(\Phi\Psi)|_{\theta=0}. \end{aligned} \tag{5.1}$$

The Darboux transformations (4.10), expressed in terms of those new superfields, exactly reproduce the Darboux-Backlund (s-Toda) transformations

$$(\Phi,\ \Psi)^{\star\bullet} = (\ \Phi(\partial \ln \Phi - \Phi\Psi),\ \frac{1}{\Phi}\ ) \tag{5.2}$$

proposed in [11] in the context of the reduction of the supersymmetric KP hierarchy in $N=1$ superspace characterized by the Lax operator

$$L = D - 2(D^{-1}\Phi\Psi) + \Phi D^{-1}\Psi. \tag{5.3}$$

For completeness, we also present the corresponding second flow equations,

$$\tfrac{\partial}{\partial t_2}\Phi = +\Phi\,'' - 2\Phi^2\Psi\,' - 2(\Phi\Psi)^2\Phi, \quad \tfrac{\partial}{\partial t_2}\Psi = -\Psi\,'' - 2\Psi^2\Phi\,' + 2(\Phi\Psi)^2\Psi, \tag{5.4}$$

which also follow from [11]. Therefore, we are led to the conclusion that the two integrable hierarchies related to the reductions (1.2) and (5.3) are *equivalent*. It would be interesting to establish a relationship (if any) between these two hierarchies in the more general case where v, u and Φ, Ψ entering the corresponding Lax operators (1.2) and (5.3) are rectangular (super)matrix-valued superfields [10], but this rather complicated question is outside the scope of the present letter. In general, these two families of $N = 2$ supersymmetric hierarchies correspond to a non-trivial supersymmetrization[3] of bosonic hierarchies, except for the simplest case we consider here. Indeed, a simple inspection shows that the equations (4.5)–(4.6) do not contain fermionic derivatives and belong to the hierarchy which is the trivial $N = 2$ supersymmetrization of the bosonic modified NLS or derivative NLS hierarchy. This last fact becomes obvious if one introduces yet a new superfield basis $\{b(z,\eta,\overline{\eta}), \overline{b}(z,\eta,\overline{\eta})\}$ through

$$\mathcal{J} \equiv (\ln \overline{b})\,', \quad \overline{\mathcal{J}} \equiv -b\overline{b}, \quad \mathcal{D}b = \mathcal{D}\overline{b} = 0, \tag{5.5}$$

in which the second flow (4.5) and the Darboux transformations (4.10) become

$$\tfrac{\partial}{\partial t_2} b = +b\,'' + 2b\overline{b}b\,', \quad \tfrac{\partial}{\partial t_2} \overline{b} = -\overline{b}\,'' + 2\overline{b}b\overline{b}\,', \tag{5.6}$$

$$b^{\star\bullet\star\bullet} = b\,(\ln b^{\star\bullet})\,', \quad \overline{b}^{\star\bullet\star\bullet} = \frac{1}{b}, \tag{5.7}$$

respectively, and the equation (5.6) reproduces the trivial $N = 2$ supersymmetrization of the modified NLS equation [22]. When passing to alternative superfields $g(z,\eta,\overline{\eta})$ and $\overline{g}(z,\eta,\overline{\eta})$ defined by the following invertible transformations

$$g = b\,\exp(-\partial^{-1}(b\overline{b})), \quad \overline{g} = \overline{b}\,\exp(+\partial^{-1}(b\overline{b})), \tag{5.8}$$

equation (5.6) becomes

$$\frac{\partial}{\partial t_2} g = (+g\,' + 2g\overline{g}g)\,', \quad \frac{\partial}{\partial t_2} \overline{g} = (-\overline{g}\,' + 2\overline{g}g\overline{g})\,' \tag{5.9}$$

and coincides with the derivative NLS equation [23].

Finally, we would like to remark that one can produce the *non-trivial* $N = 2$ supersymmetric modified KdV hierarchy by secondary reduction even though the twisted $N = 2$ Toda chain hierarchy is a *trivial* $N = 2$ supersymmetrization of the modified or derivative NLS hierarchy. One of such reductions was described in [10]. In terms of the superfields J and $\overline{J}$ (3.1), the reduction constraint is

$$J + \overline{J} = 0, \tag{5.10}$$

and only half of the flows from the set (1.1) are consistent with this reduction, namely

$$\{\ \tfrac{\partial}{\partial t_{2k-1}}\ U_{2k},\ D_{2k},\ Q_{2k}\ \} \tag{5.11}$$

(for details, see [10]). Substituting the constraint (5.10) into the third flow equation (3.7) of the reduced hierarchy, this flow becomes

$$\tfrac{\partial}{\partial t_3} u = (J\,'' + 3(QJ)(DJ) - 2J^3)\,'. \tag{5.12}$$

[3] By trivial supersymmetrization of bosonic equations we mean just replacing functions by superfunctions. In this case the resulting equations are supersymmetric, but they do not contain fermionic derivatives at all.

Now, one can easily recognize that the equation for the bosonic component reproduces the modified KdV equation and does not contain the fermionic component at all. Nevertheless, it seems that the supersymmetrization (5.12) is rather non-trivial, because it involves the odd operators D and Q but does not admit odd flows having length dimension $[D] = [Q] = -1/2$. Hence, it does not seem to be possible to avoid a dependence of D and Q in a cleverly chosen superfield basis. To close this discussion let us mention that the possible alternative constraint on the twisted $N = 2$ superfields $\mathcal{J}$ and $\overline{\mathcal{J}}$, namely

$$\mathcal{J} + \overline{\mathcal{J}} = 0, \tag{5.13}$$

leads again to the trivial $N = 2$ supersymmetrization of the modified KdV hierarchy.

6 Conclusion

In this letter we have described three distinct real forms of the twisted $N = 2$ Toda chain hierarchy introduced in [10]. It has been shown that the symmetry algebra of these real forms is the twisted $N = 2$ supersymmetry algebra. We have introduced a set of $N = 1$ superfields. They enjoy simple conjugation properties and allowed us to eliminate all nonlocalities in the flows. All flows and complex conjugation rules have been rewritten directly in twisted $N = 2$ superspace. As a byproduct, relationships between the twisted $N = 2$ Toda chain, s-Toda, modified NLS, and derivative NLS hierarchies have been established. These connections enable us to derive new real forms of the last three hierarchies, possessing a twisted $N = 2$ supersymmetry.

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