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Symmetry, Singularities and Integrability in Complex Dynamics I: The Reduction Problem

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Abstract

Quadratic systems generated using Yang-Baxter equations are integrable in a sense, but we display a deterioration in the possession of the Painlevé property as the number of equations in each 'integrable system' increases. Certain intermediate systems are constructed and also tested for the Painlevé property. The Lie symmetries are also computed for completeness.

1 Introduction

Golubchik and Sokolov [17] recently discussed systems of ordinary differential equations which are integrable by the standard factorisation method of Adler-Kostant-Symes [21] or the generalised factorisation method [16]. They established relationships between such reductions, operator Yang-Baxter equations and some kinds of nonassociative algebras. In their paper [17] a number of specific examples was given without much detailed comment on the solutions except for the possibility of the system passing or not passing the Painlevé test. The examples are as follows:

$$\begin{aligned} P_t &= P^2 - RP - QS \\ Q_t &= (\beta - 2)RQ + \beta PQ \\ R_t &= R^2 - RP - QS \\ S_t &= (3 - \beta)RS + (1 - \beta)PS \end{aligned} \tag{1.1}$$

$$\begin{aligned} P_t &= RQ \\ Q_t &= PQ \\ R_t &= \alpha RP \end{aligned} \tag{1.2}$$

$$\begin{aligned} P_t &= 2PR + \lambda QR \\ Q_t &= 2QR - \lambda PR \\ R_t &= P^2 + Q^2 + R^2 \end{aligned} \tag{1.3}$$

$$\begin{aligned} P_t &= (\nu - \mu)QR \\ Q_t &= 2\mu PQ + \mu Q^2 + \nu QR \\ R_t &= -2\nu PR - \nu R^2 - \mu QR \end{aligned} \tag{1.4}$$

and

$$\begin{aligned} \frac{\mathrm{d}X}{\mathrm{d}t} &= X^2 + 3XY = X(X + 3Y) \\ \frac{\mathrm{d}Y}{\mathrm{d}t} &= 3XY + Y^2 = Y(3X + Y) \end{aligned} \tag{1.5}$$

in which Greek letters are constants. We have given the sets of equations in the order and the form as presented by [17].

The integrability of these systems, apart from the two-dimensional system in (1.5), is not immediately obvious, although in the paper of Golubchik and Sokolov [17] we are informed that they are so constructed to be integrable in the sense of the generalised factorisation method. Unfortunately they do not connect this definition of integrability with the usual ones which are as follows:

(a) The existence of an explicit solution relating the variables, not necessarily the dependent variables as explicit functions of the independent variable, or, in the instance of systems, a number of explicit independent functions equal to the number of the dependent variables;

(b) The existence of a sufficient number of independent first integrals and invariants which could be used to give a local version via the implicit function theorem of (a) above;

(c) The existence of a sufficient number of symmetries either to reduce the differential equations of the system to algebraic equations or to obtain the independent first integrals and invariants of (b) above;

(d) The passing of the Painlevé test, either in the normal form or the weak form, which, in the form as given by Conte [8], will guarantee the existence of a solution as required by (a) above that is analytic in the complex t-plane or in a region of it apart from isolated moveable singularities which are poles or algebraic branch points.

The only hint which the two authors give is their comment associated with (1.1) to the effect that for general values of the parameter β, one would not expect the system to pass the Painlevé-Kovalevskaya test. One infers from this that they are not looking for a solution which is analytic apart from isolated singularities which are either poles or algebraic and so one can presume that they intend (a) above.

There are those who would insist that a solution be a (single-valued) function (cf Conte [8]) and, in the strict mathematical sense, they are quite correct. However, in practice the

domain of the variable, t, in which the solution is going to be used is much smaller than the complex t-plane. It is with this emphasis on the practicality of applications that (a) above is proposed.

It is the intention of this paper to take up the systems enumerated above where the authors of [17] left them. We shall see that the solutions of these systems of equations have certain interesting aspects which illustrates various properties of differential equations.

The most obvious one is that all of the systems quoted are quadratic systems which are of tremendous interest in many areas of mathematical modelling ranging from studies of population systems to the reactions in chemical systems. They, along with the closely related Lotka-Volterra systems, are usually studied from the point of view of dynamical systems since in general they are not integrable, but are well adapted for intensive and informative qualitative investigation. Nevertheless there has been a number of studies of these systems and the circumstances under which they possess first integrals and/or invariants [4, 5, 6]. The connection between the possession of the Painlevé property and the existence of first integrals/invariants has also been investigated [19].

A less obvious property of the systems presented by Golubchik and Sokolov is that all of them possess the two point symmetries of invariance under time translation and the selfsimilar transformation. These two symmetries immediately make them interesting from the point of view of the Painlevé property since there is a very close relationship between the nature of the singularity and the specific form of the selfsimilar symmetry [13]. The selfsimilar symmetry permits an analysis of the next to leading order behaviour and the Painlevé property follows from this when the next to leading order behaviour becomes a Laurent expansion about the singularity [14].

There is the question whether the possession of the two symmetries mentioned above is sufficient to guarantee integrability. As we have been informed in [17], all of these systems are integrable, we have an excellent opportunity to examine the symmetry structure of these systems and equations related to them in the knowledge that the results will characterise integrable systems. Of course it is an open question whether the examination will lead to a clear result. We defer our comments on the question until the conclusion. However, our path to the conclusion will reveal a number of interesting features of these systems and their connections to other well known facets of integrable systems.

On the presumption that complexity of appearance is a representative of complexity of fact we shall more or less work backwards through the list of systems above and attempt to examine them in increasing order of difficulty. In the case of some of the systems we make use of rescaling to make them look simpler. This may simply be a psychological trick, but it is one worth practising on all occasions. One of our principal activities will be the examination of these systems and any derivate equations for their Lie symmetries. As an aid in this examination we make use of Program Lie [18, 33] to attend to the dirty business of the calculation of the symmetries. This program, which is one of the more successful analysers for symmetry, is freely available on Simtel sites. Naturally we shall also be concerned with the possession of the Painlevé-Kovalevskaya property either by a direct computation or by a consideration of the structure of the solution of the system of equations.

Before we commence our analysis we mention some terminology introduced in reference [24]. In systems of differential equations invariant under time translation and rescaling all terms contribute to the dominant behaviour. However, it is possible that a subset of

terms has an additional or different rescaling property. This symmetry is termed 'subselfsimilarity' and the terms possessing this symmetry are called 'subdominant'. These terms also have to be analysed for possession of the Painlevé property, naturally with compability at the resonances no longer guaranteed. A simple example of the possession of subselfsimilarity is the generalised Chazy equation

$$\dddot{x} + ax\ddot{x} + b\dot{x}^2 = 0. \tag{1.6}$$

The equation as a whole has the two symmetries $G_1 = \partial/\partial t$ and $G_2 = -t\partial/\partial t + x\partial/\partial x$. The second and third terms have the two homogeneity symmetries $t\partial/\partial t$ and $x\partial/\partial x$ and so would also have to be analysed separately.

Finally we note that this is the first in a series of papers devoted to the intertwined subjects of symmetry, singularity and integrability. The second paper in the series [25] treats the singularity analysis and integrability of the class of second order equations invariant under time-translation and rescaling and the third [26] some aspects of complete symmetry groups [22].

2 The two-dimensional system

2.1 Painlevé analysis

We rewrite (1.5), *viz*

$$\begin{aligned} \frac{\mathrm{d}X}{\mathrm{d}t} &= X^2 + 3XY \\ \frac{\mathrm{d}Y}{\mathrm{d}t} &= 3XY + Y^2 \end{aligned} \tag{2.1}$$

as

$$\dot{x} = x^2 + 3xy \tag{2.2a}$$

$$\dot{y} = 3xy + y^2. \tag{2.2b}$$

When we perform the Painlevé analysis on the system (2.2), we find that, using the usual notations and methods, the singularity corresponding to the selfsimilar symmetry of the system gives the following behaviour

$$\begin{aligned} x &= -\tfrac{1}{4}\tau^{-1} + 3\mu\tau^{-\frac{1}{2}}, \quad p = -1, \quad r = -1, \tfrac{1}{2} \\ y &= -\tfrac{1}{4}\tau^{-1} - 2\mu\tau^{-\frac{1}{2}}, \quad q = -1, \quad r = -1, \tfrac{1}{2}, \end{aligned} \tag{2.3}$$

where $\tau = t - t_0$ and t_0 is the location of the movable singularity and μ is the second arbitrary constant of integration, which indicates the weak Painlevé property implied in the solution (see (2.31)).

There is also the possibility of a different type of singular behaviour for (2.2) which would be given by the two equivalent possibilities $p = -1, q > -1$ and $p > -1, q = -1$. This type of behaviour is found for example in the analysis of the Mixmaster universe [9, 10]. A simple-minded approach would be to write

$$x = \sum_{i=0}^{\infty} a_i \tau^{i-1}, \quad y = \sum_{i=0}^{\infty} \tau^i \tag{2.4}$$

for the first of these possibilities. This yields the particular solution

$$x = \frac{-1}{\tau}, \qquad y = 0. \tag{2.5}$$

However, bearing in mind the expansion in powers of $\tau^{\frac{1}{2}}$ found in (2.3) we do well to propose

$$x = \sum_{i=0}^{\infty} a_i \tau^{\frac{1}{2}i-1}, \quad y = \sum_{i=0}^{\infty} \tau^{\frac{1}{2}i-\frac{1}{2}}. \tag{2.6}$$

This gives the one-parameter solution

$$\begin{aligned} x &= -\tfrac{1}{6}\tau^{-1} + \tfrac{1}{6}\tau^{-\frac{1}{2}} - \tfrac{13}{54} + \dots \\ y &= -\tfrac{1}{3}\tau^{-\frac{1}{2}} - \tfrac{1}{9} + \tfrac{7}{27}\tau^{\frac{1}{2}} + \dots . \end{aligned} \tag{2.7}$$

Since (2.7) is a one-parameter system, this pattern of singular behaviour (and clearly the one corresponding to the interchange of the roles of x and y) does not pass the weak Painlevé test and so the system (2.2) does not possess the weak Painlevé Property. (In this we follow the definition of Tabor [34][p 330].)

2.2 First reduction

We now obtain the solution of (2.2) as follows. Firstly we are immediately struck by the resemblance to the coefficients in the expansion of a cubic binomial. This suggests the change of variables

$$u = x^2 + y^2 \qquad v = x + y \tag{2.8}$$

so that the system (2.2) takes the form

$$\dot{u} = 2v^3 \tag{2.9a}$$

$$\dot{v} = -2u + 3v^2 \tag{2.9b}$$

which by differentiation of the second of these and substitution for $\dot{u}$ gives the second order equation

$$\ddot{v} - 6v\dot{v} + 4v^3 = 0. \tag{2.10}$$

Equation (2.10) is of a type [20, 6.43, p 551] which has been studied extensively in the literature because of the frequency of its occurrence in different areas [7, 12, 15, 31, 32] and in particular is known, for certain values of the coefficients, to have eight symmetries instead of the obvious two of invariance under time translation and selfsimilarity, thereby making it linearisable [30].

For the system (2.9) we obtain two possible sets of leading order behaviour. They are $p = -2$ and $q = -1$ with the resonances $r = -1, 1$ and $p = q = -1$ with the resonances $r = -1, -2$. We give the critical parts of the expansions. They are

$$\begin{aligned} u &= \tfrac{1}{8}\tau^{-2} + 3\mu\tau^{-1} + \dots \\ v &= -\tfrac{1}{2}\tau^{-1} - 2\mu + \dots , \end{aligned} \tag{2.11}$$

which gives the expansion about the singularity at $t = t_0$, and

$$\begin{aligned} u &= t^{-1} + 2\mu t^{-2} + 3\nu t^{-3} \\ v &= t^{-1} + \mu t^{-2} + \nu t^{-3}, \end{aligned} \tag{2.12}$$

which represents the asymptotic expansion of the solution. The second constant of integration is ν and we have followed the practice introduced by Feix *et al* [13] of making the expansion in the variable t because of the asymptotic nature of the expansion.

The expansion in (2.11) has been termed [13] a right Painlevé series because it is a Laurent expansion about the singularity at t_0 in ascending powers of τ whereas that in (2.12) is called a left Painlevé series because it is a Laurent expansion in descending powers of the independent variable, t. The existence of what is now called the left Painlevé series, that is an asymptotic series as a representation of the solution, has only been recognised in recent years [27].

It is interesting that the subselfsimilarity symmetry property of the system (2.2) and its corresponding Painlevé analysis disappears under the transformation which leads to the system (2.9). The reason for this is obvious for the first of (2.9) contains only two terms and a minimum of three terms is required for the presence of a subselfsimilarity symmetry [13] (these can be additive or multiplicative). It is an interesting speculation as to whether there is always a transformation which will remove the possibility of the subselfsimilar behaviour.

2.3 Second reduction

An alternative route to the analysis of (2.2) initiated by the transformation (2.8) is given by the transformation

$$u = x + y, \quad v = x - y. \tag{2.13}$$

The transformed system (2.2) is now

$$\dot{u} = 2u^2 - v^2 \tag{2.14a}$$

$$\dot{v} = uv. \tag{2.14b}$$

We have two possible routes for the solution of (2.14), *viz* the differentiation of (2.14a) and elimination of v or the differentiation of (2.14b) and elimination of u. The first route produces the nonlinear second order ordinary differential equation

$$\ddot{u} - 2u\dot{u} - 4u^3 = 0 \tag{2.15}$$

which belongs to the same class of equations as (2.10). If we perform the leading order analysis on (2.15), we find that the only possible singularity is a simple pole since there is no possibility of subselfsimilarity. The coefficient of the leading power can be either $-\frac{1}{2}$ or 1. In the former case the resonances are at $r = -1, 3$ and in the latter case at $r = -1, 6$. There is no possibility of inconsistency at the second resonance because of the rescaling symmetry. Both expansions are right Painlevé series and the equation possesses the Painlevé property. If we look to the solution of (2.15) by means of reduction of order, we come to an Abel's equation of the second kind and firstly transformation is necessary

to obtain the solution. An alternative route is to use the Riccati transformation of (2.27) (with the same value of α) to obtain the third order equation

$$w\dddot{w} - 2\dot{w}\ddot{w} = 0 \tag{2.16}$$

which has a solution in terms of elliptic functions.

The second route produces the nonlinear second order ordinary differential equation

$$v\ddot{v} - 3\dot{v}^2 + v^4 = 0 \tag{2.17}$$

which belongs to a class of equations to be found in Kamke [20, 6,128, p 574]. Under the transformation $v = w^{-\frac{1}{2}}$ (2.17) takes the particularly simple form

$$\ddot{w} = 2 \tag{2.18}$$

which we shall again encounter in (3.5). The solution of (2.18) is trivial and so we have

$$v(t) = (A + 2Bt + t^2)^{-\frac{1}{2}}. \tag{2.19}$$

The other solution $u(t)$ follows easily from (2.14b). One easily checks that these two solutions lead to the same forms for the general solutions of the original system, (2.2).

Equation (2.15) has just the two Lie point symmetries of invariance under time translation and selfsimilarity. In marked contrast (2.17) possesses eight Lie point symmetries which, in a sense, makes it more like (2.10) than (2.15) is. We observe that the first two terms of (2.17) are invariant under the homogeneity symmetry $u\partial/\partial u$ and so expect a more complicated pattern of leading order behaviour. This is borne out by the analysis which gives in the case of all terms dominant an exponent of -1 with coefficients ± 1 and in the subdominant case exponents of 0 and $-\frac{1}{2}$. In the former case the resonances are given by $r = -1, -2$ and so for this type of leading order behaviour the equation passes the Painlevé test and has a left Painlevé series, that is, an asymptotic solution. This is easily seen from (2.19) if we expand it in the asymptotic form

$$v(t) = \frac{1}{t}\left(1 - \frac{B}{2t} + \frac{3B^2 - 4A}{8t^2} + \dots\right). \tag{2.20}$$

For the subdominant behaviour and $p = 0$ the first few terms of the Taylor expansion give

$$v(t) = a_0 + a_1 t + \frac{1}{2a_0}\left(3a_1^2 - a_0^4\right)t^2 + \left(1 - 3a_0^2\right)a_1 t^3 + \dots . \tag{2.21}$$

We observe that the third term in (2.17), which breaks the homogeneity symmetry of the first two terms, does not affect the existence of the Taylor series solution. The coefficients of the first two terms are arbitrary constants and so this is the series representation of a general solution. In the case that $p = -\frac{1}{2}$ we obtain

$$v(t) = a_0\tau^{-\frac{1}{2}} - \tfrac{1}{2}a_0^3\tau^{\frac{3}{2}} - \tfrac{5}{8}a_0^4\tau^{\frac{7}{2}}2 + \dots . \tag{2.22}$$

This series representation of the solution contains the two arbitrary constants a_0 and t_0. The equation satisfies the requirements of the weak Painlevé test. This is one occasion

when all possible leading order behaviours possess the Painlevé property and so there is no question that the equation (2.17) possesses the Painlevé property. For the sake of completeness, not to mention a little amusement, we perform the Painlevé analysis of the system (2.14). With the usual substitution for the leading order behaviour the exponents are

$$\begin{array}{lll} p-1 & 2p & 2q \\ q-1 & p+q & \end{array} \tag{2.23}$$

so that we have the possibility of all terms being dominant with $p = q = -1$ or of just one variable being dominant with $p = -1, q > -1$. In the former case the resonances occur at $r = -1, -2$ and so we have a left Painlevé series. Because of the symmetry of the system we shall not have inconsistencies at the resonances and we obtain the first few terms of the expansion as

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -1 \\ \beta \end{pmatrix} t^{-1} + \begin{pmatrix} 1 \\ -\beta \end{pmatrix} \mu t^{-2} + \begin{pmatrix} -2\beta \\ 1 \end{pmatrix} \nu t^{-3} + \dots , \tag{2.24}$$

where μ and ν are the two arbitrary constants required for a general solution. For the second case the substitution of the series expansion

$$u = \sum_{i=0}^{\infty} a_i \tau^{i-1} \quad v = \sum_{i=0}^{\infty} b_i \tau^i \tag{2.25}$$

results in the particular solution

$$u(t) = -\frac{1}{2\tau} \quad v(t) = 0. \tag{2.26}$$

We thus arrive at the interesting conclusion that, while we started with a system, (2.2), which does not possess the (weak) Painlevé property, we can map it through the nonlinear transformation (2.8) to the new system (2.9) that has the strong Painlevé property and through the linear nonsingular transformation (2.13) to the new system (2.14) which, although it has the (strong) Painlevé property, naturally splits into two single equations, *viz* (2.15) and (2.17), one possessing the (strong) Painlevé property and the other the weak Painlevé property.

2.4 General solution

It so happens that (2.10) has the combination of coefficients which gives eight Lie point symmetries. In principle one can use these symmetries to obtain the solution, but they are particularly complicated and there is an easier route to the solution. It is known [7] that the equations which belong to this class can be expressed as a simple third order equation by means of a Riccati transformation. We put

$$v = \alpha \frac{\dot{w}}{w}, \tag{2.27}$$

where α is a constant to be determined to give the third order equation an optimal simplicity [1]. Using (2.27) we find that (2.10) becomes

$$\begin{aligned} &\frac{\dddot{w}}{w} - (3+6\alpha)\frac{\dot{w}\ddot{w}}{w^2} + \left(2+6\alpha+4\alpha^2\right)\frac{\dot{w}^3}{w^3} = 0 \\ &\Leftrightarrow \dddot{w} = 0 \end{aligned} \tag{2.28}$$

when we put $\alpha = -\frac{1}{2}$ for both of the terms containing α then vanish. Equation (2.28) is trivial to solve and we obtain

$$v(t) = -\frac{B+Ct}{A+2Bt+Ct^2}, \tag{2.29}$$

where A, B and C are constants of integration. Equation (2.9a) is now a simple quadrature and we obtain

$$u(t) = \frac{2(B+Ct)^2 - (B^2-AC)}{(A+2Bt+Ct^2)^2}, \tag{2.30}$$

for which an additive constant of integration must be set at zero to maintain consistency with (2.9b). (Despite the presence of the three constants A, B and C they provide only two independent constants of integration.) Finally we obtain the solutions of the original system of equations. We have

$$x(t) = -\tfrac{1}{2}\frac{B+Ct \pm \sqrt{3(B+Ct)^2 - 2(B^2-AC)}}{A+2Bt+Ct^2} \tag{2.31}$$

and $y(t)$ has the same form with the opposite sign attached to the square root. We observe that in general the solutions $x(t)$ and $y(t)$ have branch point singularities whereas the only singularities of $u(t)$ and $v(t)$ are always poles.

2.5 Symmetry analysis

We complete our considerations of the system (2.2) by looking at its first integral and invariant and some of its Lie point symmetries. Since the systems (2.2) and (2.9) (or (2.2) and (2.14)) are related by a point transformation, the discussion of one system is equivalent to the discussion of the other system when it comes to Lie point symmetries and, as these underlie the existence of the first integral and invariant, of them as well. We have noted that all of the systems discussed in this paper have the two Lie point symmetries which leave them invariant under the transformations of time translation and rescaling, that is

$$G_1 = \frac{\partial}{\partial t} \quad \text{and} \quad G_2 = -t\frac{\partial}{\partial t} + x_i\frac{\partial}{\partial x_i}, \tag{2.32}$$

where the summation over i includes all of the dependent variables. The first integral is associated with G_1 and is simply obtained from the integration of the ratio of (2.2a) and (2.2b), *viz*

$$\frac{\mathrm{d}x}{\mathrm{d}y} = \frac{x^2+3xy}{3xy+y^2}. \tag{2.33}$$

It is

$$I = \frac{(x-y)^4}{xy}. \tag{2.34}$$

We use G_2 to obtain the invariant from the autonomous first order ordinary differential equation satisfied by the two characteristics, $x = ut$ and $y = vt$, which is

$$\frac{\mathrm{d}v}{\mathrm{d}u} = \frac{3uv + v^2 + v}{u^2 + 3uv + u}. \tag{2.35}$$

The integration of (2.35) is not as simple as that of (2.33) since for the latter the integration is simply a quadrature whereas for the former a first order ordinary differential equation has to be solved. (This is more or less the same as the situation as the integration of a second order ordinary differential equation with the two symmetries (2.32) [28].) In terms of the original variables the invariant is

$$J = \frac{t(x-y)^2 + (x+y)}{(xy)^{\frac{1}{2}}} \tag{2.36}$$

in which we see that the coefficient of the independent variable, t, is simply $I^{\frac{1}{2}}$. In principle we could solve (2.34) and (2.36) for x and y as functions of t.

The Lie point symmetries of the system (2.2) are the equivalent of generalised symmetries at the second order level and consequently there will be an infinity of them. We have no intention of presenting them all here! We do note, however, that, if G is a Lie point symmetry of (2.2), then so also is $f(I,J)G$. This is the equivalent result to that found in the case of second order ordinary differential equations [23]. We can limit the number of symmetries found by simply making an ansatz for the functional dependence of the coefficient functions. Thus, if we assume at most quadratic dependence on the variables, we obtain the three symmetries

$$\begin{aligned} G_1 &= \frac{\partial}{\partial t} \\ G_2 &= -t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} \\ G_3 &= \left(x^2 + 3xy\right)\frac{\partial}{\partial x} + \left(3xy + y^2\right)\frac{\partial}{\partial y} \end{aligned} \tag{2.37}$$

and, if we allow cubic dependence, the further four symmetries

$$\begin{aligned} G_4 &= t\left[\frac{\partial}{\partial t} + \left(x^2 + 3xy\right)\frac{\partial}{\partial x} + \left(3xy + y^2\right)\frac{\partial}{\partial y}\right] \\ G_5 &= x\left[\frac{\partial}{\partial t} + \left(x^2 + 3xy\right)\frac{\partial}{\partial x} + \left(3xy + y^2\right)\frac{\partial}{\partial y}\right] \\ G_6 &= y\left[\frac{\partial}{\partial t} + \left(x^2 + 3xy\right)\frac{\partial}{\partial x} + \left(3xy + y^2\right)\frac{\partial}{\partial y}\right] \\ G_7 &= (x-y)^2\left[(3x+y)\frac{\partial}{\partial x} + (x+3y)\frac{\partial}{\partial y}\right] \end{aligned} \tag{2.38}$$

are added. We note that the power of the independent variable, t, lags behind the powers of the dependent variables.

Table 1. The four sets of values for the coefficients of the leading terms of the Right Painlevé Series for (3.1).

α	β	γ
-1	1	1
1	1	-1
1	-1	1
-1	-1	-1

3 The simplest three-dimensional system

3.1 Painlevé analysis

From a casual observation of the three three-dimensional systems listed in the introduction one cannot, *a priori*, judge which of them is the simplest. We shall presume that it is (1.2) on the basis that the system contains the fewest number of terms of the three. Our first step is to write the system in the simplest form by rescaling the variables P and Q so that α becomes unity. Thus we examine the system

$$\dot{u} = vw \tag{3.1a}$$

$$\dot{v} = wu \tag{3.1b}$$

$$\dot{w} = uv \tag{3.1c}$$

which is a special case of the Rikitake System.

Because of the structure of the system (3.1) there can be no subselfsimilar symmetry and the leading order behaviour is that of a simple pole. To make up for this simplicity there are four possible sets of values for the coefficients of the leading terms. If we take the leading terms to be $\alpha\tau^{-1}$, $\beta\tau^{-1}$ and $\gamma\tau^{-1}$, the possible combinations are given in Table 1. The resonances are determined from the solution of the characteristic equation

$$\begin{vmatrix} r-1 & -\gamma & -\beta \\ -\gamma & r-1 & -\alpha \\ -\beta & -\alpha & r-1 \end{vmatrix} = 0. \tag{3.2}$$

They are $r = -1, 2(2)$. The repetition of the resonance at $r = 2$ is not a cause for concern since the matrix for the determination of the resonances is a real symmetric matrix and so the geometric multiplicity of the eigen vectors equals the algebraic multiplicity of the eigenvalues. The expansion up to the resonance is

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} \tau^{-1} + \left\{ \begin{bmatrix} \gamma \\ 1 \\ 0 \end{bmatrix} \mu + \begin{bmatrix} \beta \\ 0 \\ 1 \end{bmatrix} \nu \right\} \tau \tag{3.3}$$

and so the system (3.1) possesses the (strong) Painlevé property.

3.2 First reduction and symmetry analysis

Differentiation of (3.1a) and the substitution of (3.1b) and (3.1c), division by u and the same process of differentiation and substitution gives the third order equation

$$u\dddot{u} - \dot{u}\ddot{u} - 4\dot{u}u^3 = 0 \tag{3.4}$$

which, at the level of contact symmetries, only has the two symmetries relating to time translation and rescaling. The Painlevé analysis of (3.4) reveals three patterns of leading order behaviour. The all terms dominant case has a simple pole and the resonances are $r = -1, 2$ and 4. There is no inconsistency at the second resonance and this case passes the Painlevé test. Subselfsimilar behaviour can be found with the first two terms which share the two homogeneity symmetries, $t\partial/\partial t$ and $u\partial/\partial u$. The leading order terms are in τ^0 and τ^1. For both we obtain a Taylor series containing three arbitrary constants. Consequently (3.4) possesses the (strong) Painlevé property.

We reduce the order of (3.4) using u^2 and $\dot{u}^2$, the two invariants of $G_1 = \partial/\partial t$, as the new independent (x) and dependent (y) variables to obtain the second order ordinary differential equation

$$y'' = 2. \tag{3.5}$$

Equation (3.5) is a linear second order equation of very simple appearance. This indicates that (3.4) has nonlocal symmetries in sufficient supply to give the necessary right point symmetries for the second order equation. Equation (3.5) has the solution

$$y(x) = A + Bx + x^2 \tag{3.6}$$

and so the solution of (3.4) is

$$\pm(t - t_0) = \int \frac{\mathrm{d}u}{\sqrt{A + Bu^2 + u^4}} \tag{3.7}$$

in which the integral can be evaluated in terms of elliptic integrals, the expression of which is not informative and so we omit it. Because of the cyclic symmetry of (3.1) v and w are given by expressions of the same form as (3.7).

It is a trivial matter to obtain the two autonomous integrals of the system (3.1). They are

$$I = u^2 - v^2 \quad \text{and} \quad J = u^2 - w^2. \tag{3.8}$$

The third cyclic expression is not independent of I and J. The determination of the necessary invariant to complete the set (with I and J) using the selfsimilar symmetry, G_2, seems to be impossible. The one invariant which comes moderately easy is simply the ratio of I and J which is of no use.

As in the case of the two-dimensional system the system (3.1) possesses an infinite number of Lie point symmetries and we confine our attention to those symmetries which are up to cubic in the variables for the coefficient functions. Thus we have

$$\begin{aligned}
G_1 &= \frac{\partial}{\partial t} \\
G_2 &= -t\frac{\partial}{\partial t} + u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v} + w\frac{\partial}{\partial w} \\
G_3 &= vw\frac{\partial}{\partial u} + wu\frac{\partial}{\partial v} + uv\frac{\partial}{\partial w} \\
G_4 &= t\left[\frac{\partial}{\partial t} + vw\frac{\partial}{\partial u} + wu\frac{\partial}{\partial v} + uv\frac{\partial}{\partial w}\right]
\end{aligned}$$

$$
\begin{aligned}
G_5 &= u\left[\frac{\partial}{\partial t} + vw\frac{\partial}{\partial u} + wu\frac{\partial}{\partial v} + uv\frac{\partial}{\partial w}\right] \\
G_6 &= v\left[\frac{\partial}{\partial t} + vw\frac{\partial}{\partial u} + wu\frac{\partial}{\partial v} + uv\frac{\partial}{\partial w}\right] \\
G_7 &= w\left[\frac{\partial}{\partial t} + vw\frac{\partial}{\partial u} + wu\frac{\partial}{\partial v} + uv\frac{\partial}{\partial w}\right] \\
G_8 &= \left(u^2 - v^2\right)\frac{\partial}{\partial t} \\
G_9 &= \left(u^2 - w^2\right)\frac{\partial}{\partial t} \\
G_{10} &= \left(u^2 - v^2\right)\left[-t\frac{\partial}{\partial t} + u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v} + w\frac{\partial}{\partial w}\right] \\
G_{11} &= \left(u^2 - w^2\right)\left[-t\frac{\partial}{\partial t} + u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v} + w\frac{\partial}{\partial w}\right].
\end{aligned}
\tag{3.9}
$$

We observe that the two first integrals, I and J, occur as coefficients in G_8 to G_{11}. Although we noted the possibility of this occurrence for the system (2.2), the form of the first integral and invariant for that system was such that symmetries of that type would not be found using the ansatz we have adopted. In the case of the system (3.1) the two first integrals fitted in with the ansatz and so we see the appearance of symmetries containing them as common multipliers. Given the fact that we cannot determine the invariant for this system, the likelihood of us making an ansatz for the structure of the symmetry which would include symmetries containing the invariant as a multiplier is most unlikely. Indeed, *a priori* the likelihood of an Ansatz producing such a multiplier would be so low that its occurrence would verge on the miraculous! A point which may be worth future consideration is whether the symmetries associated with a particular first integral have the integrals/invariants as multipliers or is there simply no ordained structure as we proposed for the system (2.2)? please consider!) We recall that for the system (2.2) we did not have I but $I^{\frac{1}{2}}$ as the coefficient of t, whereas here we have I and J themselves.

4 The second three-dimensional system

4.1 Painlevé analysis

In (1.4) the variables can be rescaled to give the system

$$\dot{u} = vw \tag{4.1a}$$

$$\dot{v} = v(au + v + w) \tag{4.1b}$$

$$\dot{w} = -w(bu + v + w), \tag{4.1c}$$

where $a = 1 - \mu/\nu$ and $b = 1 - \nu/\mu$. We assume that $a \neq b$.

In (4.1) we set the variables at

$$u = \alpha\tau^p, \quad v = \beta\tau^q, \quad w = \gamma\tau^r \tag{4.2}$$

and obtain the following sets of powers of τ

$$\begin{array}{llll} \text{p- 1} & \text{q+r} & & \\ \text{q- 1} & \text{q+p} & \text{2q} & \text{q+r} \\ \text{r- 1} & \text{r+p} & \text{q+r} & \text{2r} \end{array} . \tag{4.3}$$

Evidently one set of dominant behaviours given by $p = q = r = -1$. The coefficients are determined from the set of equations

$$\begin{aligned} -\alpha &= \beta\gamma \\ -\beta &= \beta(a\alpha + \beta + \gamma) \\ -\gamma &= -\gamma(b\alpha + \beta + \gamma) \end{aligned} \tag{4.4}$$

and we have

$$\alpha = -\frac{2}{a-b}, \quad \beta + \gamma = \frac{a+b}{a-b}, \quad \beta\gamma = \frac{2}{a-b} \tag{4.5}$$

with particularly unpleasant expressions for β and γ which we do not bother to write.

There are two other sets of possible dominant behaviours given by

$$q = -1, \quad r > -1, \quad p = r \quad \text{or} \quad r = -1, \quad q > -1, \quad p = q. \tag{4.6}$$

There is a symmetry between these two possibilities and so we just consider one, the second. Since the powers are required to be integral, q and p cannot be negative integers as the analysis requires and so we must make a full substitution. We take

$$\begin{aligned} u &= \sum_{i=0} a_i \tau^i \\ v &= \sum_{i=0} b_i \tau^i \\ w &= \sum_{i=0} c_i \tau^{i-1}. \end{aligned} \tag{4.7}$$

On equating coefficients of like powers of τ to zero we obtain the following set of information

i	0	1	2
a	a_0	a_1	$(a-b)a_0a_1$
b	0	a_1	$\frac{1}{2}(2a-b)a_0a_1$
c	1	$-\frac{1}{2}ba_0$	$\frac{1}{12}b^2a_0^2 - \frac{1}{3}(b+1)a_1$

(4.8)

with the rest of the coefficients following naturally. We observe that a_0 and a_1 are arbitrary and these, in combination with t_0, provide the three arbitrary constants of integration required to give the general solution of (4.1). Thus the subselfsimilarity dominant behaviour leads to a general solution and not a particular (or singular, an unfortunate word in this context) solution.

It remains to look at the resonances of the all terms dominant singular behaviour. We set

$$\begin{aligned} u &= \alpha\tau^{-1} + \mu\tau^{r-1} \\ v &= \beta\tau^{-1} + \nu\tau^{r-1} \\ w &= \gamma\tau^{-1} + \sigma\tau^{r-1}, \end{aligned} \tag{4.9}$$

observe the following simplifications

$$a\alpha + \beta + \gamma = -1 \qquad b\alpha + \beta + \gamma = 1 \tag{4.10}$$

and obtain the characteristic equation for the resonances

$$\begin{vmatrix} r-1 & -\gamma & -\beta \\ -a\beta & r-\beta & -\beta \\ b\gamma & \gamma & r+\gamma \end{vmatrix} = 0. \tag{4.11}$$

We obtain

$$r = -1, 2, \beta - \gamma. \tag{4.12}$$

The first two values for the resonances are fine, but the third is almost certainly going to be a disaster except for some very specific values of the parameters a and b. Since the third contains the difference of the two roots of the quadratic, even for "nice" values of a and b the value of r could be terrible.

We conclude that generically (4.1) does not possess the Painlevé property and is not integrable in the sense of Painlevé. There is an amusing contrast with the results of the singularity analysis of the equations for the Mixmaster universe. In that analysis the all terms dominant behaviour was better than the not all terms dominant behaviour [9, 10].

4.2 First reduction and symmetry analysis

If we define $r = vw$, in (4.1a) and (4.17), we reduce (4.1) to the following two-dimensional system

$$\begin{aligned} \dot{u} &= r \\ \dot{r} &= (a-b)ru. \end{aligned} \tag{4.13}$$

When we make the Painlevé analysis of (4.13), we find that the leading order behaviour is

$$u = -\frac{2}{(a-b)}\tau^{-1} \qquad r = \frac{2}{(a-b)}\tau^{-2}. \tag{4.14}$$

The resonances occur at $r = -1, 2$ and, as one would expect from a system of such symmetry, there is no incompatibility at the second resonance. We obtain

$$\begin{aligned} u &= -\frac{2}{(a-b)}\tau^{-1} + \alpha\tau \\ r &= \frac{2}{(a-b)}\tau^{-2} + \alpha. \end{aligned} \tag{4.15}$$

The Lie point symmetries of (4.13) up to the third order in the variables are

$$
\begin{aligned}
G_1 &= \frac{\partial}{\partial t}\\
G_2 &= -t\frac{\partial}{\partial t} + u\frac{\partial}{\partial u} + r\frac{\partial}{\partial r}\\
G_3 &= r\frac{\partial}{\partial u} + (a-b)ur\frac{\partial}{\partial r}\\
G_4 &= t\left[\frac{\partial}{\partial t} + r\frac{\partial}{\partial u} + (a-b)ur\frac{\partial}{\partial r}\right]\\
G_5 &= u\left[\frac{\partial}{\partial t} + r\frac{\partial}{\partial u} + (a-b)ur\frac{\partial}{\partial r}\right]\\
G_6 &= r\left[\frac{\partial}{\partial t} + r\frac{\partial}{\partial u} + (a-b)ur\frac{\partial}{\partial r}\right]\\
G_7 &= 2I\frac{\partial}{\partial t}\\
G_8 &= \left(2I - (a-b)u^2\right)t\frac{\partial}{\partial t} + u\left((a-b)u^2 - 4r\right)\frac{\partial}{\partial u} - 4r^2\frac{\partial}{\partial r}.
\end{aligned}
\tag{4.16}
$$

with the simpler system (4.13) it is possible to make the computations at least to order four in the variables, but there is little point in writing them down as most of them are of the type of G_4 except with more elaborate coefficients.

4.3 Symmetries and solution

The combination (4.1b) $w + v$(4.1c) gives

$$(vw)^{\cdot} = (a-b)u(vw) \tag{4.17}$$

which, when combined with (4.1a), becomes

$$\ddot{u} = (a-b)u\dot{u}. \tag{4.18}$$

We note that (4.18) has only the two symmetries

$$
\begin{aligned}
G_1 &= \frac{\partial}{\partial t}\\
G_2 &= -t\frac{\partial}{\partial t} + u\frac{\partial}{\partial u}.
\end{aligned}
\tag{4.19}
$$

We observe that to integrate the system (4.1) it is only necessary to go to a second order ordinary differential equation. Equation (4.18) is integrated once to give

$$\dot{u} = I + \tfrac{1}{2}(a-b)u^2 \tag{4.20}$$

which immediately leads to the quadrature

$$
\begin{aligned}
t - t_0 &= \int \frac{\mathrm{d}u}{I + \frac{1}{2}(a-b)u^2}\\
\tfrac{1}{2}(a-b)(t-t_0) &= \left(\frac{a-b}{2I}\right)^{\frac{1}{2}} \arctan\left(\frac{a-b}{2I}\right)^{\frac{1}{2}} u.
\end{aligned}
\tag{4.21}
$$

inversion of (4.21) and the trivial quadrature of (4.17) immediately give

$$\begin{aligned} u &= A\tan\Omega(t-t_0) \\ vw &= B\sec^2\Omega(t-t_0) \end{aligned} \tag{4.22}$$

in which $A = [2I/(a-b)]^{\frac{1}{2}}$, $\Omega = [\frac{1}{2}I(a-b)]^{\frac{1}{2}}$, B is another constant of integration and we have assumed that the parameters in (4.20) are positive. We consider this case only as the other possibilities require a similar discussion and this would be repetitive.

We are now left with the integration of (4.1c) which now has the simple form

$$\dot{w} + bA\tan\Omega(t-t_0)w + w^2 + B\sec^2\Omega(t-t_0) = 0. \tag{4.23}$$

Equation (4.23) is a Riccati equation and we transform it to a linear second order equation by means of the transformation $w = \dot{\eta}/\eta$ to obtain

$$\ddot{\eta} + bA\tan\Omega(t-t_0)\dot{\eta} + B\sec^2\Omega(t-t_0)\eta = 0. \tag{4.24}$$

Equation (4.24) can be improved in appearance by the change of variables

$$\eta(t) = y(x) \qquad x(t) = \tan\Omega(t-t_0). \tag{4.25}$$

We now have

$$\left(1+x^2\right)y'' + \left(2 + \frac{2b}{a-b}\right)xy' + \frac{2B}{I(a-b)}y = 0. \tag{4.26}$$

Equation (4.26), being a linear second order differential equation, has eight symmetries. This knowledge is not of much use in the solution of the equation since seven of these symmetries require a knowledge of the solution. The eighth is the homogeneity symmetry, $y\partial/\partial y$, which was used to obtain this equation from the Riccati equation. Equation (4.26) is a little disturbing because it contains two constants of integration and one would prefer to discuss the solution with only the parameters, a and b, present. The constants of integration can be removed by dividing (4.26) by y and differentiating with respect to x. Unfortunately the resulting third order equation looks too awful to contemplate and that line of action is not pursued here. In general the solution of (4.26) is in terms of the hypergeometric function which is a bit too diffuse for our purposes. There are certain values of the constants for which solutions are known [29] for these have been used in the solution of the problem of the Tikekar superdense stars [35, 36, 37]. Kamke [20, 2.261, p 470] provides a method for obtaining polynomial solutions, again for certain values of the constants.

From the solutions of (4.26) we can find w and hence v from (4.22). Consequently the passing of the Painlevé property depends critically on the solution of (4.26). On the other hand there is no constraint of this type on u.

The presence of the two parameters a and b in the system (4.1) makes the analysis of the Lie point symmetries of the system problematic. The symmetries up to the quadratic

symmetries are

$$\begin{aligned}
G_1 &= \frac{\partial}{\partial t}\\
G_2 &= -t\frac{\partial}{\partial t} + u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v} + w\frac{\partial}{\partial w}\\
G_3 &= vw\frac{\partial}{\partial u} + v(au+v+w)\frac{\partial}{\partial v} - w(bu+v+w)\frac{\partial}{\partial w}\\
G_4 &= \left[(a-b)u^2 - 2vw\right]\frac{\partial}{\partial t}.
\end{aligned} \tag{4.27}$$

In G_4 we recognise that the coefficient of $\partial/\partial t$ is simply the first integral in (4.20).

5 The third three-dimensional system

5.1 Painlevé analysis

We write (1.3) in the equivalent form

$$\dot{u} = (2u + \lambda v)w \tag{5.1}$$
$$\dot{v} = (-\lambda u + 2v)w \tag{5.2}$$
$$\dot{w} = u^2 + v^2 + w^2. \tag{5.3}$$

We consider the Painlevé analysis of the system (5.1) – (5.3). We find that the powers in the different terms are

$$\begin{array}{llll}
-1 & r & -p+q+r & \\
-1 & -q+p+r & r & \\
-1 & 2p-r & 2q-r & r,
\end{array} \tag{5.4}$$

where we have assumed the leading terms to be

$$\begin{aligned}
u &= \alpha\tau^p\\
v &= \beta\tau^q\\
w &= \gamma\tau^r.
\end{aligned} \tag{5.5}$$

The only possible singular behaviour is $p = q = r = -1$. The coefficients α, β and γ are determined from the system

$$-\alpha = (2\alpha + \lambda\beta)\gamma \tag{5.6}$$
$$-\beta = (-\lambda\alpha + 2\beta)\gamma \tag{5.7}$$
$$-\gamma = \alpha^2 + \beta^2 + \gamma^2. \tag{5.8}$$

The combination of $\alpha(5.6) + \beta(5.7)$ gives

$$-\left(\alpha^2 + \beta^2\right) = 2\left(\alpha^2 + \beta^2\right)\gamma \tag{5.9}$$

from which it follows that either $\gamma = -\frac{1}{2}$ or $\alpha^2 + \beta^2 = 0$. We consider each in turn.

From (5.8) it follows that

$$\alpha^2 + \beta^2 = \tfrac{1}{4}. \tag{5.10}$$

With this value of γ (5.6) and (5.7) become

$$\lambda\beta = 0 \quad \lambda\alpha = 0 \quad \Rightarrow \quad \lambda = 0 \tag{5.11}$$

since the assumption of leading order behaviour requires that α and β be nonzero. A similar analysis for the second possibility leads to the result that $\lambda = \pm i$.

Case: $\lambda = 0$ The system is now

$$\begin{aligned} \dot{u} &= 2uw \\ \dot{v} &= 2vw \\ \dot{w} &= u^2 + v^2 + w^2 \end{aligned} \tag{5.12}$$

and to determine the resonances we make the substitution

$$\begin{aligned} u &= \alpha\tau^{-1} + \mu\tau^{r-1} \\ v &= \beta\tau^{-1} + \nu\tau^{r-1} \\ w &= \gamma\tau^{-1} + \sigma\tau^{r-1}. \end{aligned} \tag{5.13}$$

The characteristic equation for r is

$$\begin{vmatrix} r-1-2\gamma & 0 & -2\alpha \\ 0 & r-1-2\gamma & -2\beta \\ -2\alpha & -2\beta & r-1-2\gamma \end{vmatrix} = 0 \tag{5.14}$$

which gives the values

$$r = -1, 0, 1 \tag{5.15}$$

when the values of the constants are substituted. There is only one eigenvector which is

$$\mathbf{e} = \begin{pmatrix} 2\beta \\ -2\alpha \\ 0 \end{pmatrix}. \tag{5.16}$$

That there is only one eigenvector indicates that it is necessary to introduce a logarithmic term and the 0 for the entry of w indicates that the logarithmic term must come in with the other two variables, u and v.

Case: $\lambda = \pm i$ With the same substitution as that in (5.13) the characteristic equation for the resonances is given by

$$\begin{vmatrix} r-1-2\gamma & -\lambda\gamma & 2\alpha+\lambda\beta \\ \lambda\gamma & r-1-2\gamma & -\lambda\alpha+2\beta \\ -2\alpha & -2\beta & r-1-2\gamma \end{vmatrix} = 0 \tag{5.17}$$

and with the constraints on the constants gives the values

$$r = -2, -1, 0. \tag{5.18}$$

The value $r = 0$ is the critical one. Two possible results occur. In the case that $\alpha = \lambda\beta$ μ and ν are arbitrary because the rank of the matrix becomes one. There is no need to introduce a logarithmic term. In the case that $\alpha \neq \lambda\beta$ this is not the case because μ and ν are related by $2\alpha\mu + 2\beta\nu = 0$. A logarithmic term must be introduced.

We conclude that only for the specific values of λ shown above does the system (5.1) – (5.3) pass the Painlevé test.

5.2 The two reductions

We now consider the integration of the system (5.1) – (5.3). We observe that the combination $u(5.1) + v(5.2)$ gives

$$u\dot{u} + v\dot{v} = 2(u^2 + v^2)w \tag{5.19}$$

so that differentiation of (5.3) with respect to time gives the second order ordinary differential equation

$$\ddot{w} - 6w\dot{w} + 4w^3 = 0 \tag{5.20}$$

which we recognise as the equation we already met in (2.10). We know that it possesses eight Lie point symmetries, that under a Riccati transformation it becomes the simplest third order equation and that the solution is

$$w = -\frac{B + Ct}{A + 2Bt + Ct^2}. \tag{5.21}$$

Now that we know the functional form of $w(t)$ we see that (5.1) and (5.2) now constitute a linear system. This system is rendered autonomous by the introduction of a new time variable [2, 3] defined by

$$\frac{\mathrm{d}\tau}{\mathrm{d}t} = w \Leftrightarrow \tau = \log\left(A + 2Bt + Ct^2\right)^{-\frac{1}{2}}. \tag{5.22}$$

If we denote differentiation with respect to τ by a $'$, the linear system is

$$\begin{pmatrix} u \\ v \end{pmatrix}' = \begin{pmatrix} 2 & \lambda \\ -\lambda & 2 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}. \tag{5.23}$$

The solution of (5.22) is

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -i \\ 1 \end{pmatrix} \mu \mathrm{e}^{(2+i\lambda)\tau} + \begin{pmatrix} i \\ 1 \end{pmatrix} \nu \mathrm{e}^{(2-i\lambda)\tau}. \tag{5.24}$$

There appear to be too many constants of integration, but the three in the expression for w are in reality two and a small calculation shows that the constants of integration in (5.24), *viz* μ and ν, are related to the other constants via

$$2\mu\nu = B^2 - AC \tag{5.25}$$

so that the number of arbitrary constants is in fact three. In terms of the variable τ the solutions for u and v are analytic. Given the definition of τ in (5.22), it is evident that

the solution in terms of the original variable, t, is not analytic in general. In terms of the variable t the solution for w has simple poles as the only singularities. If we express this solution in terms of the new time, we find that

$$w = \pm e^{2\tau}\sqrt{B^2 - AC + Ce^{-2\tau}} \tag{5.26}$$

which has a branch point singularity. Of course, to define a new time in terms of the solution of one of the variables means that the problem has been reduced from a three-dimensional system to a two-dimensional system and a linear one at that.

In fact we can interpose an intermediate system between (1.3) and (5.20) by introducing a new variable, r, defined by

$$r^2 = u^2 + v^2. \tag{5.27}$$

The system is

$$\begin{aligned} \dot{r} &= 2rw \\ \dot{w} &= r^2 + w^2. \end{aligned} \tag{5.28}$$

The solution to the system (5.28) is (5.21) and

$$r = \frac{B^2 - AC}{(A + 2Bt + Ct^2)^2} \tag{5.29}$$

in which the additional constant of integration has already been identified in terms of the three constants found in the solution for the function w.

We now consider the Painlevé analysis of (5.20), (5.28) and (1.3) (in the present variables) in turn. In the case of (5.20) the only possible singular behaviour is given by $p = -1$, but there are two possible families of solutions since the coefficient of the leading term can be either -1 or $-\frac{1}{2}$. To determine the resonances we write

$$w = \alpha\tau^{-1} + \beta\tau^{r-1} \tag{5.30}$$

so that we can treat both cases simultaneously. We find that

$$r = -1, 6\alpha + 4 \tag{5.31}$$

so that we have

$$\alpha = -\tfrac{1}{2} \quad r = -1, 1 \qquad \alpha = -1 \quad r = -1, -2. \tag{5.32}$$

Evidently the first gives the right Painlevé series, that is the Laurent expansion in the neighbourhood of the singularity, and the second the left Painlevé series, the asymptotic expansion, that is the Laurent expansion away from the singularity.

For the system (5.28) the solution for w is given in (5.21) and it is a simple quadrature to find that

$$r = \frac{B^2 - AC}{A + 2Bt + Ct^2}. \tag{5.33}$$

We determine the leading order behaviour by writing

$$r = \alpha\tau^p \qquad w = \beta\tau^q \tag{5.34}$$

and substituting this into (5.28). We have

$$\begin{aligned} \alpha p\tau^{p-1} &= 2\alpha\beta\tau p + q \\ \beta q\tau^{q-1} &= \alpha^2\tau 2^p + \beta^2\tau^{2q}. \end{aligned} \tag{5.35}$$

Evidently there are two possible behaviours. We can have $p = q = -1$ or $p > -1, q = -1$. The roles of p and q cannot be reversed in the second possibility. We find that $\alpha = \pm\frac{1}{2}$ and $\beta = -\frac{1}{2}$. The resonances are given by $r = \pm 1$. The first two terms of the right Painlevé series are

$$\begin{pmatrix} r \\ w \end{pmatrix} = \begin{pmatrix} \pm\frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \tau^{-1} + \begin{pmatrix} 1 \\ \pm 1 \end{pmatrix} \mu, \tag{5.36}$$

where μ is the second arbitrary constant required for the general solution. For the second possibility we set

$$r = \sum_{i=0} a_i\tau^i \qquad w = \sum_{i=0} b_i\tau^{i-1} \tag{5.37}$$

since the standard Painlevé analysis breaks down in the case that the leading order behaviour is not singular in all variables [9, 10]. We substitute (5.37) into (5.28) to obtain

$$\begin{aligned} \sum_{i=0} i a_i\tau^{i-1} &= \sum_{i=0}\sum_{j=0} a_i b_j \tau^{i+j-1} \\ \sum_{i=0}(i-1) b_i\tau^{i-2} &= \sum_{i=0}\sum_{j=0} a_i a_j \tau^{i+j} + \sum_{i=0}\sum_{j=0} b_i b_j \tau^{i+j-2}. \end{aligned} \tag{5.38}$$

When we compare coefficients of like powers of τ, we find that all coefficients except b_0 vanish so that the solution is a singular (particular) solution and is unrelated to the general solution. The same result is found if we commence the series for r at a higher power of τ. Thus we see that the Painlevé property holds equally well for (5.28) as it did for (5.20). Bearing in mind the results of the Painlevé analysis of the system (1.3) we see that the decomposition of the system (5.28) into the system (1.3) and even the possibility of the preservation of the Painlevé property is restricted to those special cases for the value of λ obtained above.

In the case of this system going from a system of two equations to a system of three equations has had a drastic effect upon the possibility of the possession of the Painlevé property. Generically, *ie* for arbitrary values of λ, this system of three equations does not have the Painlevé property.

5.3 Symmetries

The Lie point symmetries for (5.20) are

$$G_1 = \frac{\partial}{\partial t}$$

$$
\begin{aligned}
G_2 &= -t\frac{\partial}{\partial t} + w\frac{\partial}{\partial w} \\
G_3 &= w\left(\frac{\partial}{\partial t} + w^2\frac{\partial}{\partial w}\right) \\
G_4 &= tw\frac{\partial}{\partial t} + w^2(1+tw)\frac{\partial}{\partial w} \\
G_5 &= t^2w\frac{\partial}{\partial t} + w\left(1+2tw+2t^2w^2\right)\frac{\partial}{\partial w} \\
G_6 &= t^2\left(3+2tw\right)\frac{\partial}{\partial t} + \left(6t^2w^2+4t^3w^3\right)\frac{\partial}{\partial w} \\
G_7 &= t^2\left(1+tw\right)\frac{\partial}{\partial t} + tw\left(1+3tw+2t^2w^2\right)\frac{\partial}{\partial w} \\
G_8 &= t^3\left(1+tw\right)\frac{\partial}{\partial t} + t\left(1+3tw+4t^2w^2+2t^3w^3\right)\frac{\partial}{\partial w},
\end{aligned}
\tag{5.39}
$$

those of (5.28) up to cubic in the variables are

$$
\begin{aligned}
G_1 &= \frac{\partial}{\partial t} \\
G_2 &= -t\frac{\partial}{\partial t} + r\frac{\partial}{\partial r} + w\frac{\partial}{\partial w} \\
G_3 &= 4rw\frac{\partial}{\partial r} + \left(r^2+w^2\right)\frac{\partial}{\partial w} \\
G_4 &= t\left[\frac{\partial}{\partial t} + 4rw\frac{\partial}{\partial r} + \left(r^2+w^2\right)\frac{\partial}{\partial w}\right] \\
G_5 &= r\left[\frac{\partial}{\partial t} + 4rw\frac{\partial}{\partial r} + \left(r^2+w^2\right)\frac{\partial}{\partial w}\right] \\
G_6 &= w\left[\frac{\partial}{\partial t} + 4rw\frac{\partial}{\partial r} + \left(r^2+w^2\right)\frac{\partial}{\partial w}\right]
\end{aligned}
\tag{5.40}
$$

and of the original system, (1.3), (in the present variables) are

$$
\begin{aligned}
G_1 &= \frac{\partial}{\partial t} \\
G_2 &= -t\frac{\partial}{\partial t} + u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v} + w\frac{\partial}{\partial w} \\
G_3 &= (2u+\lambda v)w\frac{\partial}{\partial u} + (-\lambda u+2v)w\frac{\partial}{\partial v} + (u^2+v^2+w^2)\frac{\partial}{\partial w} \\
G_4 &= t\left[\frac{\partial}{\partial t} + (2u+\lambda v)w\frac{\partial}{\partial u} + (-\lambda u+2v)w\frac{\partial}{\partial v} + (u^2+v^2+w^2)\frac{\partial}{\partial w}\right] \\
G_5 &= u\left[\frac{\partial}{\partial t} + (2u+\lambda v)w\frac{\partial}{\partial u} + (-\lambda u+2v)w\frac{\partial}{\partial v} + (u^2+v^2+w^2)\frac{\partial}{\partial w}\right] \\
G_6 &= v\left[\frac{\partial}{\partial t} + (2u+\lambda v)w\frac{\partial}{\partial u} + (-\lambda u+2v)w\frac{\partial}{\partial v} + (u^2+v^2+w^2)\frac{\partial}{\partial w}\right] \\
G_7 &= w\left[\frac{\partial}{\partial t} + (2u+\lambda v)w\frac{\partial}{\partial u} + (-\lambda u+2v)w\frac{\partial}{\partial v} + (u^2+v^2+w^2)\frac{\partial}{\partial w}\right].
\end{aligned}
\tag{5.41}
$$

The symmetries in (5.40) and in (5.41) are essentially the same with the latter having an additional symmetry due to the presence of the additional variable.

6 The four-dimensional system

6.1 Painlevé analysis

The parameter β in (1.1) is an essential parameter which cannot be removed by rescaling. In fact any attempt to remove it will simply lead to the appearance of another parameter in the other terms. We rewrite the system as

$$\dot{u} = u(u-w) - vx \tag{6.1a}$$
$$\dot{v} = v[bu + (b-2)w] \tag{6.1b}$$
$$\dot{w} = -w(u-w) - vx \tag{6.1c}$$
$$\dot{x} = -x[(b-1)u + (b-3)w]. \tag{6.1d}$$

We perform the Painlevé analysis of (6.1). In the case of all terms being dominant the singularity is a simple pole and the coefficients of the leading order terms must satisfy the following set of equations

$$\begin{aligned} -\alpha &= \alpha^2 - \alpha\gamma - \beta\gamma \\ -\beta &= b\alpha\beta + (b-2)\beta\gamma \\ -\gamma &= -\alpha\gamma + \gamma^2 - \beta\delta \\ -\delta &= -(b-1)\alpha\delta - (b-3)\gamma\delta. \end{aligned} \tag{6.2}$$

Since the leading order behaviour assumes that the coefficients are nonzero, the second and fourth of (6.2) give

$$\begin{aligned} b\alpha + (b-2)\gamma &= -1 \\ (b-1)\alpha + (b-3)\gamma &= 1 \end{aligned} \tag{6.3}$$

from which it follows that either $\alpha = \gamma = -1$ or

$$\alpha + \gamma = -2 \quad \text{and} \quad \alpha + \gamma = -1 \tag{6.4}$$

which is a contradiction. Substitution for the value of α and γ into (6.2) requires that the product $\beta\gamma = -1$ and, more importantly, $b = -\frac{3}{2}$. Thus the Painlevé analysis of (6.1) cannot even begin without a restriction of the value of b. We observe in (6.25) and (6.26) that this value of b removes the possible singularity in the trigonometric term, but we are still left with the square root to give the solution a branch point singularity.

Nevertheless for the sake of completeness we analyse the situation for this particular value of b. On making the usual substitution for the leading order behaviour we obtain the following set of exponents

$$\begin{array}{llll} p-1 & 2p & p+r & q+s \\ q-1 & p+q & q+r & \\ r-1 & p+r & 2r & q+s \\ s-1 & p+s & r+s & \\ & & & \\ -1 & p & r & \end{array} \tag{6.5}$$

in which the separated bottom line gives the essence of the content of the second and fourth lines. When all terms are dominant, the singularity is of order -1. However, there are various options for subdominant behaviour. These are listed below as

p	q	r	s	Comment
-1	-1	-1	-1	generic
-1	≥ 0	≥ 0	≥ 0	plus
-1	-1	≥ 0	≥ 0	variants
-1	≥ 0	≥ 0	-1	$p \leftrightarrow r$.

(6.6)

We note that the leading order behaviour in the subdominant cases must contain either p or r.

Only in the case of all terms dominant can we apply the standard algorithm. The coefficients of the leading order terms are found from the solution of the system

$$\begin{aligned} -\alpha &= \alpha^2 - \alpha\gamma - \beta\delta \\ -2\beta &= 3\alpha\beta - \beta\delta \\ -\gamma &= \alpha\gamma + \gamma^2 - \beta\delta \\ -2\delta &= -\alpha\delta + 3\gamma\delta \end{aligned} \tag{6.7}$$

and are

$$\alpha = -1 \quad \gamma = -1 \quad \beta\delta = -1, \tag{6.8}$$

ie one of the coefficients is arbitrary. The resonance behaviour is determined by the solution of the system of equations

$$\begin{pmatrix} r & \delta & -1 & \beta \\ -3\beta & 2r & \beta & 0 \\ -1 & \gamma & r & \beta \\ \delta & 0 & -3\delta & 2r \end{pmatrix} \begin{pmatrix} \mu \\ \nu \\ \sigma \\ \rho \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{6.9}$$

The resonances are given by $r = -1(2), 0, 2$. The double resonance at -1 means that we cannot have the general solution since there will be at most only three arbitrary constants. For the resonances at $r = 0$ and $r = 2$ we find that the corresponding vectors are respectively

$$\mathbf{a}_0 = \begin{pmatrix} -1 \\ \pm 1 \\ 1 \\ \begin{matrix} - \\ + \end{matrix} \; 1 \end{pmatrix} \quad \text{and} \quad \mathbf{a}_2 = \begin{pmatrix} 1 \\ \pm\frac{1}{2} \\ 1 \\ \begin{matrix} - \\ + \end{matrix} \; \frac{1}{2} \end{pmatrix} \mu. \tag{6.10}$$

Thus we see that the resonance $r = 0$ indicates the existence of two possible solutions. Each of these solutions will contain an arbitrary constant introduced at the $r = 2$ resonance.

In the cases of not all terms being dominant the standard algorithm cannot be applied and so we must substitute Anzätse appropriate to the particular pattern desired. For the

second set of indices in (6.6) we write

$$\begin{aligned} u &= \sum_{i=0} a_i \tau^{i-1} \\ v &= \sum_{i=0} b_i \tau^{i} \\ w &= \sum_{i=o} c_i \tau^{i} \\ x &= \sum_{i=0} d_i \tau^{i} \end{aligned} \tag{6.11}$$

and after equating the coefficients of like powers of τ to zero we obtain the following set of coefficients

$$\begin{array}{llll} a_0 = -1 & b_0 = 0 & c_0 = 0 & d_0 = 0 \\ a_1 = 0 & b_1 = 0 & c_1 = c_1 & d_1 = 0 \\ a_2 = \frac{1}{3}c_0 & b_2 = 0 & c_2 = 0 & d_2 = 0 \\ a_3 = 0 & b_3 = 0 & c_3 = \frac{1}{3}c_1 & d_3 = 0. \end{array} \tag{6.12}$$

Evidently the functions v and x are identically zero for this pattern of singularity.

For the third and fourth sets of indices in (6.6) we obtain the same expansion as in (6.12). All the solutions for the different possible singularity structures are particular solutions and, as we expected above, the system (6.1) does not satisfy the requirements of the Painlevé test for any of the values of the parameter b.

6.2 The general solutions

From the combinations of (6.1a) − (6.1c), (6.1b)$x + v$(6.1d) and (6.1a)$w + u$(6.1c) we obtain respectively

$$\dot{u} - \dot{w} = u^2 - w^2 \tag{6.13}$$

$$(vx)^{\cdot} = (u + w)vx \tag{6.14}$$

$$(uw)^{\cdot} = -(u + w)vx. \tag{6.15}$$

From the combination of (6.14) and (6.15) we obtain the first integral

$$-I^2 = uw + vx \tag{6.16}$$

and of (6.13) and (6.15) the first integral

$$J = (u - w)vx, \tag{6.17}$$

where I and J are constants of integration. In (6.16) we have chosen the label for the first integral as $-I^2$ for later convenience. The other cases which we do not treat here can be treated in a fashion similar to what we are about to do.

When we substitute (6.16) into (6.1a), we obtain

$$\dot{u} = u^2 + I^2 \tag{6.18}$$

which can be integrated by an elementary quadrature to give

$$u(t) = I\tan(t - t_0), \tag{6.19}$$

where t_0 is the constant of integration. By the elimination of vx between (6.16) and (6.17) we obtain a quadratic equation for $w(t)$ which, when we substitute for $u(t)$ using (6.19), gives

$$w(t) = \frac{1}{\sin 2(t - t_0)}\left\{-I\cos 2(t - t_0) \pm \sqrt{[I^2 + J\sin 2(t - t_0)(1 + \cos(t - t_0))/I]}\right\} \tag{6.20}$$

so that (6.17) gives

$$vx = -\tfrac{1}{2}I\sec^2(t - t_0)\left\{I \pm \sqrt{[I^2 + J\sin 2(t - t_0)(I + \cos(t - t_0))/I]}\right\}. \tag{6.21}$$

We observe that the parameter, b, of the system (6.1) does not appear in the solutions obtained above. The presence of the square root suggests that contrary to the comment of Golubchik and Sokolov [17] that the system (6.1) probably does not pass the Painlevé-Kowalevskaya test for generic values of the parameter is something of an understatement. To determine the functions $v(t)$ and $x(t)$ we can now treat (6.1b) and (6.1d) as linear equations for the two variables. We obtain

$$v(t) = K\exp\left[\int (bu + (b - 2)w))\mathrm{d}t\right] \tag{6.22}$$

and

$$x(t) = \bar{K}\exp\left[-\int ((b - 1)u + (b - 3)w))\mathrm{d}t\right] \tag{6.23}$$

from which it is evident that

$$vx(t) = K\bar{K}\exp\left[\int (u + w)\mathrm{d}t\right], \tag{6.24}$$

where the notation $vx(t)$ is meant to indicate vx as a function of time, which gives us a neat way to write down the solution without going to the effort of evaluating the integral in either (6.22) or (6.23). Given the appearance of the square root in w one would not expect to be able to evaluate either integral with ease. Using this device we find that

$$v(t) = K\left(\frac{vx(t)}{K\bar{K}}\right)^{b-2}\sec^{2I}(t - t_0) \tag{6.25}$$

and

$$x(t) = \bar{K}\left(\frac{vx(t)}{K\bar{K}}\right)^{3-b}\cos^{2I}(t - t_0). \tag{6.26}$$

The exponents in the expressions in (6.25) and (6.26) definitely indicate that one should not expect the system, (6.1), to satisfy the Painlevé test for generic values of the parameter, b.

6.3 Second reduction

Another way to reduce the order of the system (6.1) is by the following procedure. We define the variables

$$\begin{aligned} s &= u - w \\ y &= u + w \\ z &= vx \end{aligned} \tag{6.27}$$

which, with the use of the system (6.1), satisfy the three-dimensional system of first order equations

$$\begin{aligned} \dot{s} &= sy \\ \dot{y} &= s^2 - 2z \\ \dot{z} &= yz. \end{aligned} \tag{6.28}$$

Consider the system (6.28). When we make the usual substitution for the leading order behaviour, we require the following terms to balance:

$$\begin{aligned} \alpha p \tau^{p-1} &= \alpha\beta\tau^{p+q} \\ \beta q \tau^{q-1} &= \alpha^2\tau^{2p} - 2\gamma\tau^r \\ \gamma r \tau r - 1 &= \beta\gamma\tau^{q+r}. \end{aligned} \tag{6.29}$$

From the first and last of (6.29) it is obvious that $q = -1$. If all terms are to be dominant we also have $p = -1$ and $r = -2$. However, when we solve for the coefficients, we find that the first of (6.29) gives $\beta = -1$ and the third $\beta = -2$ unless both α and γ are zero. This immediately implies that $\beta = 0$ which contradicts the idea of leading order behaviour. Consequently we must conclude that the case of all terms dominant is impossible.

There are two possible sets of subdominant behaviour. These are $p = -1, r > -2$ and $p > -1, r = -2$. Effectively these two cases are $p = -1, r = -1$ and $p = 0, r = -2$.

We commence with the second and, since one of the putative solutions is not singular, we make the ansatz

$$\begin{aligned} s &= \sum_{i=0} a_i \tau^{i-1} \\ y &= \sum_{i=0} b_i \tau^{i-1} \\ z &= \sum_{i=0} c_i \tau^{i-1} \end{aligned} \tag{6.30}$$

and substitute these into (6.28). After a modicum of calculation we find that

$$\begin{array}{lll} a_0 = \pm 1 & b_0 = -1 & c_0 = b_1 \\ a_1 = \pm b_1 & b_1 = b_1 & c_1 = \frac{4}{5} b_1^2 \\ a_2 = \pm \frac{2}{5} b_1^2 & b_2 = -\frac{1}{5} b_1^2 & c_2 = -\frac{3}{5} b_1^3 \\ a_3 = \pm \frac{3}{5} b_1^3 & b_3 = \frac{8}{5} b_1^3 & c_3 = \ldots \end{array} \tag{6.31}$$

in which it is quite obvious that there are only two arbitrary constants, the location of the singularity and the coefficient b_1. Consequently the Laurent series does not represent the full solution in the neighbourhood of the singularity, but a particular solution of the system (6.28).

For the first possibility of subdominant behaviour we substitute

$$\begin{aligned} s &= \sum_{i=0} a_i \tau^i \\ y &= \sum_{i=0} b_i \tau^{i-1} \\ z &= \sum_{i=0} c_i \tau^{i-2} \end{aligned} \tag{6.32}$$

and find

$$\begin{array}{lll} a_0 = 0 & b_0 = -2 & c_0 = -1 \\ a_1 = 0 & b_1 = 0 & c_1 = 0 \\ a_2 = 0 & b_2 = -2c_2 & c_2 = c_2 \\ a_3 = 0 & b_3 = 0 & c_3 = 0 \\ a_4 = 0 & b_4 = \frac{2}{5}c_2^2 & c_4 = -\frac{3}{5} \end{array} \tag{6.33}$$

with only the even coefficients b_{2i} and c_{2i} being nonzero. The solution depends upon two arbitrary constants, but is definitely very particular for the system since one of the functions is identically zero. It is probably incorrect to regard this solution as the general solution for the two functions $y(t)$ and $z(t)$.

Somewhat dismayed by the result displayed in (6.32) we are smitten by a stroke of genius and think to apply a left Painlevé series to see if we can obtain a solution which does contain the function $s(t)$. Thus we write

$$\begin{aligned} s &= \sum_{i=0} a_i \tau^{-i-2} \\ y &= \sum_{i=0} b_i \tau^{-i-1} \\ z &= \sum_{i=0} c_i \tau^{-i-2} \end{aligned} \tag{6.34}$$

and find

$$\begin{array}{lll} a_0 = \pm(2b_2 - b_1^2)^{\frac{1}{2}} & b_0 = -2 & c_0 = -1 \\ a_1 = \pm(2b_2 - b_1^2)^{\frac{1}{2}} b_1 & b_1 = b_1 & c_1 = b_1 \\ a_2 = \pm\frac{1}{2}(2b_2 - b_1^2)^{\frac{1}{2}}(b_1^2 - b_2^2) & b_2 = b_2 & c_2 = \frac{1}{2}\left(b_2 - b_1^2\right) \\ a_3 = \pm\frac{1}{6}(2b_2 - b_1^2)^{\frac{1}{2}}(\frac{24}{5}b_1b_2 - 2b_1^3) & b_3 = \frac{9}{10}b_1b_2 - \frac{1}{2}b_1^3 & c_3 = -\frac{1}{5}b_1b_2 \end{array} \tag{6.35}$$

in which both b_1 and b_2 are arbitrary. However, this is a Laurent expansion at infinity and so the expansion is in negative powers of the variable t and so we have a particular

asymptotic solution and not the general solution of (6.28). We recall that the single third-order equation (6.37) had partial expansions for both left and right series for one of the values of the coefficient of the leading order power. Nevertheless we are inclined to think that this is possibly the first time that a left Painlevé series indicating a particular solution has been demonstrated in the case of subdominant behaviour.

By differentiating the second of (6.28) and utilising the other two equations we obtain

$$\ddot{y} - y\dot{y} = s^2 y. \tag{6.36}$$

With a repetition of the same procedure on (6.36) we obtain a third order equation for the variable y, *viz*

$$y\dddot{y} - \dot{y}\ddot{y} - 3y^2\ddot{y} + 2y^3\dot{y} = 0. \tag{6.37}$$

Equation (6.37) possesses just the two symmetries due to invariance under time translation and rescaling, even if the computation is carried out for contact symmetries. If we perform the Painlevé analysis on (6.37), we find that the only admissible singular behaviour is a simple pole. There are two values for the coefficient of this term, *viz* $\alpha = -1, -2$. For the first of these the resonances are given by $r = -1, 1, 2$ and so, in conjunction with the two symmetries, (6.37) passes the Painlevé test. In the case of the second value of α the resonances are $r = -1, \pm 2$. Consequently there is no possibility of the Painlevé test being satisfied for $\alpha = -2$. If we take the value $+2$, we obtain a right Painlevé series containing only two arbitrary constants. If we take the value -2, we obtain a left Painlevé series also containing only two arbitrary constants. One of these would be a particular solution in the vicinity of the singularity and the other a particular asymptotic solution, both in the sense defined by Cotsakis and Leach [11]).

This immediately raises the question of the integrability of (6.37) since it does not pass the Painlevé test for all possible singularity patterns. If we divide (6.37) by y^2, the equation becomes exact and we obtain the first integral

$$-L^2 = \frac{\ddot{y}}{y} - 3\dot{y} + y^2, \tag{6.38}$$

where we have taken the expression for the left-hand side to be more convenient for later usage. We can rewrite (6.38) as the differential equation

$$\ddot{y} - 3y\dot{y} + y^3 + L^2 y = 0. \tag{6.39}$$

We observe that the constant of integration, L, can be scaled out of (6.39) without affecting the other terms. We also observe that (6.39) is simply a variation on the equation we met in §§2 and 3 ((2.10) and (5.20)). The presence of the linear term does not affect the number of symmetries, which remains at eight and so implies that there exists a point transformation which linearises the equation, although it does make the transformation look more complex.

To solve (6.39) we introduce the Riccati transformation

$$y = -\frac{\dot{\eta}}{\eta} \tag{6.40}$$

to obtain the third order linear equation of maximal symmetry,

$$\dddot{\eta} + L^2\dot{\eta} = 0 \tag{6.41}$$

with solution

$$\eta(t) = A + B\sin L(t - to) \tag{6.42}$$

so that

$$y(t) = \frac{BL\cos L(t - t_0)}{A + B\sin L(t - t_0)} \tag{6.43}$$

which, provided $|A/B| \leq 1$, obviously has only poles as singularities.

Consequently we have a counterexample to the claim by Tabor [34, p 330] that the possession of the Painlevé property requires that all possible patterns of singular behaviour pass the Painlevé test.

We can make a few other observations. We have noted that (6.39) possesses eight Lie point symmetries with or without the presence of the linear term. As in the case of (2.10) and (5.20), (6.39) without the linear term possesses both a left Painlevé series and a right Painlevé series. This is not the case when the linear term is present. Then the only possibility is a right Painlevé series. This does not affect the integrability of (6.39) in terms of functions with only poles as moveable singularities. The existence of the same algebra of Lie point symmetries means that there exists a point transformation of (6.39) which removes the linear term. That there is a change in the nature of the singularity behaviour means that this transformation cannot be homeographic. Naturally it must have only polelike singularities since both solutions possess the Painlevé property.

This concludes the discussion of the fourth order system.

7 Discussion

From the analyses of the systems of equations (1.1), (1.2), (1.3), (1.4) and (1.5) several conclusions can be drawn.

The development of the systems of equations, which in the original paper [17] were obtained as realisations of some kinds of nonassociative algebras, considered in this paper may be viewed as a process of division of a single higher order equation. In the case of the two-dimensional quadratic system, (1.5), one can easily imagine it as being derived from a second order equation by a standard process of reduction to a system of two first order equations. It is not surprising that there was no loss of the Painlevé property in this splitting. In the case of (1.2) the three-dimensional first order quadratic system was derived naturally from a third order nonlinear equation. Although this equation (3.4) had only two contact symmetries, reduction of order brought it to a linear second order equation with eight symmetries and consequently (3.4) must have seven nonlocal symmetries with nice reduction properties. It is not surprising that (3.1) possesses the Painlevé property. In the case of the equations, (1.3) and (1.4), there is a distinct change in behaviour since the parameters which are present in the equations are essential parameters and cannot be removed by rescaling. The basic equation is really a nonlinear second order equation and

one could imagine this being split into two first order equations followed by a further split of an arbitrary nature of one of the equations into two first order equations containing the arbitrary parameter. This situation is repeated in the case of (1.1) at a higher order in that one starts from a third order nonlinear equation and obtains three first order equations which have the Painlevé property and which do not contain a parameter. The fourth first order equation is obtained by a parameter-dependent split of one of these three first order equations and the Painlevé property is lost.

One can envisage a situation in which a prediction of the preservation of the Painlevé property can be made. Given an nth order scalar ordinary differential equation possessing the Painlevé property, one could expect to be able to make a system of n first order ordinary differential equations by some sort of natural reduction and the system would still possess the Painlevé property. Any attempt to have more than n first order equations by some sort of further splitting may inevitably lead to a loss of the Painlevé property. It is interesting that the second order equations possessing the Painlevé property, *viz* (2.10), (5.20) and (4.6), which can be viewed as the origins of the systems (1.2), (1.3) and (1.5), respectively are essentially the same equation and that the third order equation from which the system (1.1) was derived is closely related to that equation. This does pose some interesting questions about the basis of the work of Golubchik and Sokolov [17] from the point of view of the algebraic structure of scalar ordinary differential equations of the second and high orders.

A third question which has only been suggested here is the question of the algebraic structure of the symmetries of the first order systems. There seems to be some structure in the symmetries listed. We note that this structure appears to be independent of the possession of the Painlevé property which is not surprising. Algebraic properties do not require an analytic basis. Thus a system can be integrable in the algebraic sense and not in the sense of Painlevé. However, because symmetry is at the basis of the process of reduction, a system which is integrable in the sense of Painlevé must also be integrable in the algebraic sense.

Finally we mention the interesting observation that a left Painlevé series can be found in subdominant behaviour. Related to that is the whole question of the Painlevé integrability of (6.37). The solution (6.43) is manifestly possessed of only poles as singularities. The equation has the Painlevé property for one possible variety of leading order behaviour, but not for the other. This makes one wonder just how much is required for one to conclude that there exists a solution expressible in terms of a Laurent expansion about a polelike singularity.

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