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Some Fourth-Order Ordinary Differential Equations which Pass the Painlevé Test

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Abstract

An approach to the Painlevé analysis of fourth-order ordinary differential equations is presented. Some fourth-order ordinary differential equations which pass the Painlevé test are found.

As is well known, at the turn of the century Painlevé and his school discovered six ordinary differential equations (ODEs) that define new functions. This was achieved by classifying second order ODEs of a certain form having what is today referred to as the Painlevé property (the general solution should be free of movable critical points).

The aim of this paper is to find some fourth-order differential equations having the Painlevé property. To do this we consider fourth order differential equations of two forms

$$y_{zzzz}+b\,y^5+c\,y^3y_z+d\,yy_z^2+f\,y^2y_{zz}+g\,y_zy_{zz}+h\,yy_{zzz}+F_1\,(y,z)=0, \tag{1}$$

$$y_{zzzz}+b\,y^3+c\,y_z^2+d\,yy_{zz}+F_2\,(y,z)=0, \tag{2}$$

where b, c, d, f, g and h are constant coefficients, $F_1\,(y,z)$ and $F_2\,(y,z)$ are exspressions that take the form

$$F_1\,(y,z)=\sum_{k=0}^{4} q_k\,(z)\;y^k, \tag{3}$$

$$F_2\,(y,z)=\sum_{k=0}^{2} q_k\,(z)\,y^k. \tag{4}$$

Here $q_k\,(z)$ are smooth functions of z.

We are looking for such values of constants b, c, d, f, g, h and the expressions $F_1\,(y,z)$ and $F_2\,(y,z)$ so that eqs. (1) and (2) pass the Painlevé test.

Let us consider the approach to the following problem: to find values of constant coefficients in eqs. (1) and (2) so that these equations pass the Painlevé test. These equations contain all leading terms if we do not take into account $F_1\,(y,z)$ and $F_2\,(y,z)$. Substituting [1, 2]

$$y \cong a_0x^p, \qquad x=z-z_0 \tag{5}$$

into eq. (1) shows that all terms without F_1 in the equation may balance for value $p=-1$. Requiring that the leading terms do balance we obtain the following equation that determines a_0:

$$ba_0^4-ca_0^3+(d+2f)\,a_0^2-(2g+6h)\,a_0+24=0. \tag{6}$$

We do not have any possibility to find solutions of this equation because we do not know the constants b, c, d, f, g and h. Let us note, that using variables

$$y = Ay', \qquad z = Bz', \tag{7}$$

one can see that one of the roots of (6) may be taken equal 1.

In this case we have the equation

$$b-c+d+2f-2g+6h+24 = 0 \tag{8}$$

from eq. (6). Substituting

$$y \cong x^{-1} + \beta x^{j-1} \tag{9}$$

into the leading terms of eq. (1) we obtain the equation for the Fuchs indices in the form

$$\begin{aligned} &j^4 + (h-10)\,j^3 + (35+f-g-6h)\,j^2 + (c+5g+11h-50-2d-3f)\,j \\ &\qquad +3d-4c-4g+6f+5b-12h+24 = 0. \end{aligned} \tag{10}$$

At this step of the investigation we want to determine the value of the coefficient h. We are interested in integer Fuchs indices that are determined by eq. (10) because we can hope that eqs. (1) have the Painlevé property in this case. One can see that

$$\sum_{k=1}^{4} j_k = 10 - h \tag{11}$$

from eq. (10), where j_k are roots of the above. We have to choose h as integer. Later in this work we take h equal 0; 1; 2; 3; 4 and 5. We know that one of the roots of eq. (11) is -1. Let $j_1 = -1$, then we have

$$j_2 + j_3 + j_4 = 11 - h. \tag{12}$$

We assume that only this family of solution has the positive Fuchs indices. Without loss of generality we consider the solution of eq. (12) $j_2 = 2$, $j_3 = 3$ and $j_4 = 6$ in the case $h = 0$, since we studied other cases in the same way. From eq. (10) follows:

$$24f-10g+2c-9d+5b+120 = 0, \tag{13}$$

$$4f+2g-2c-d+5b = 0, \tag{14}$$

$$6f+2g-c-3d+5b = 0. \tag{15}$$

The solution of the set of equations (8), (13)–(15) takes the form

$$f = g-10, \tag{16}$$

$$d = b+2g-16, \tag{17}$$

$$c = 2b+2g-12, \tag{18}$$

Taking into account relations (16)–(18) one can check arbitrary coefficients in the Laurent series for solutions of eq. (1). Assuming that [3]

$$y = \frac{1}{x} + a_1 + a_2 x + a_3 x^2 + a_4 x^3 + a_5 x^4 + a_6 x^6 + \cdots \tag{19}$$

and substituting (19) into eq. (1) we find that a_2 and a_3 are arbitrary coefficients but a_6 is arbitrary only in the case

$$g = 6 - b. \tag{20}$$

Using (16)–(18) and (20) we obtain

$$ba_0^4 - (2b + 12)\, a_0^2 - 2\,(6 - b)\, a_0 + 24 = 0 \tag{21}$$

from eq. (6). The solution of eq. (21) takes the form

$$a_0^{(1)} = 1, \quad a_0^{(2)} = -2, \qquad a_0^{(3,4)} = \frac{b \pm \sqrt{b^2 + 48b}}{2b}. \tag{22}$$

We can now investigate the second solution family of eq. (1) in this case.

Assuming

$$y \cong -\frac{2}{x} + \beta_2 x^{j-1} \tag{23}$$

and substituting (23) into eq. (1) at conditions (16)–(18), (20) we get the equation for the Fuchs indices that has solution

$$j_1 = -1, \qquad j_2 = 6, \qquad j_{3,4} = \frac{5}{2} \pm \frac{1}{2}\sqrt{24b - 23}. \tag{24}$$

Let $24b - 23 = m^2$ then

$$b = \frac{23 + m^2}{24}. \tag{25}$$

Assuming $m = 2n + 1$ $(n = 0, 1, 2, \ldots, 25)$ we find the integer Fuchs indices from eq. (24) and values of coefficient b from eq. (25). We therefore have roots a_0 from eq. (21). We consider the cases when roots of a_0 are rational numbers and found all cases which lead to integer Fuchs indices for all values a_0.

In a similar manner we check eqs. (1) at $h = 1, 2, 3, 4$ and 5 for all cases. We find values of coefficients b and consequently values c, d, f and g when the necessary conditions in the Painlevé test for eqs. (1) are satisfied.

Taking into account these values we look for those of eqs. (1) that pass the Painlevé test at $F_1 \neq 0$, when $F_1\,(y, z)$ is determined by eq. (3).

We obtained a list of equations that pass the Painlevé test

$$y_{zzzz} + 6y^5 - 10yy_z^2 - 10y^2 y_{zz} = \varepsilon_1 zy + \varepsilon_0; \tag{26}$$

$$y_{zzzz} + y^5 - 5yy_z^2 - 5y^2 y_{zz} + 5y_z y_{zz} = \varepsilon_1 zy + \varepsilon_0, \tag{27}$$

where ε_0 and ε_1 are arbitrary constants;

$$y_{zzzz} + (g - 6)\left(2y^3 y_z + 2yy_z^2 + y^2 y_{zz}\right) + gy_z y_{zz} + 2yy_{zzz} = F\,(z)\,, \tag{28}$$

where

$$F\,(z) = \begin{cases} q_0\,(z)\,, & g = 6; \\ \varepsilon_0, & g \neq 6; \end{cases}$$

$$y_{zzzz}+6yy_z^2+3y^2y_{zz}+9y_zy_{zz}+3yy_{zzz}=q_0(z), \tag{29}$$

where $q_0(z)$ is the smooth function of z;

$$y_{zzzz}+4y^3y_z+12yy_z^2+6y^2y_{zz}+10y_zy_{zz}+4yy_{zzz}=q_0(z); \tag{30}$$

$$y_{zzzz}+y^5+10y^3y_z+15yy_z^2+10y^2y_{zz}+10y_zy_{zz}+5yy_{zzz}=q_0(z)+q_1(z)\,y. \tag{31}$$

It turnes out that all these equations possess the Painlevé property. Eqs. (26) and (28) are reductions of the modified Korteweg-de Vries and Caudrey–Dodd–Gibbon equations of fifth order. Eq. (28) at $g=6$ reduces to the Riccati equation. At $g\neq 6$ eq. (28) can be presented in the form

$$\omega_{zz}+\frac{1}{2}(g-6)\,\omega^2=\varepsilon_0 z+c_1, \tag{32}$$

where

$$\omega=y_z+y^2. \tag{33}$$

Eq. (32) is the first Painlevé equation and its solutions are the Painlevé transcendents. Solution of eq. (28) at $g=6$ can be found from equation (33) that can be linearized by transformation

$$y=\frac{\varphi_z}{\varphi}. \tag{34}$$

One can see that eqs. (29) and (30) can be presented in the form

$$y_{zz}+y^3+3yy_z=Q_2(z), \tag{35}$$

$$y_{zzz}+y^4+6y^2y_z+3y_z^2+4yy_{zz}=Q_1(z) \tag{36}$$

after integration over z. These equations are not conceptually fourth order differential equations. These ones and eq. (31) can be linearized by transformation (34) as well. They take the form

$$\varphi_{zzz}=Q_2(z)\,\varphi,\qquad \varphi_{zzzz}=Q_1(z)\,\varphi,\qquad \varphi_{zzzzz}=q_0(z)\cdot\varphi+q_1(z)\cdot\varphi. \tag{37}$$

Analogously, all eqs. (26)–(31) possess the Painlevé property.

Let us study eqs. (2) to find the values of coefficients b, c, d and $F_2(y,z)$ when these equations possess the Painlevé property. With this in mind we substitute

$$y\cong a_0x^{-2},\qquad x=z-z_0 \tag{38}$$

into eqs. (2) at $F_2=0$ and equate terms to zero at x^{-6}. The equation then takes the form

$$ba_0^2+(6c+4d)\,a_0+120=0. \tag{39}$$

Assuming $a_0^{(1)}=-1$ in this equation we have

$$b+120-6c-4d=0. \tag{40}$$

After that, assuming

$$y\cong -x^{-2}+\beta_1x^{j-2},\qquad x=z-z_0 \tag{41}$$

and substituting into eqs. (2) we obtain the equation for the Fuchs indices in the form

$$j^4 - 14j^3 + (71 - c)\, j^3 + (5c + 4d - 154)\, j + 120 - 8d - 12c + 3b = 0. \tag{42}$$

One can see that

$$\sum_{k=1}^{4} j_k = 14 \tag{43}$$

from eq. (42). Here j_k are roots of eq. (42).

We assume that one of the solution family of eqs. (2) has the positive indices except -1. Let $j_1 = -1$. In this case we have

$$j_2 + j_3 + j_4 = 15 \tag{44}$$

from eq. (43). We are interested in integers and the different Fuchs indices of eq. (44) and one can find 12 solutions of this equations at $j_k > 0$ ($k = 2, 3, 4$). Substituting these solutions into eq. (42) give set of equations for b, c and d. Taking into account these equations and eq. (40) we obtain different variants of values b, c and d. We can find then another root $a_0^{(2)}$ for every variant of values b, c and d.

Using these values and substituting

$$y \cong a_0^{(2)} x^{-2} + \beta_2 x^{j-2} \tag{45}$$

one can get the equation for the Fuchs indices of the second solution family.

We found 5 variants of values b, c and d when eqs. (2) pass the Painlevé test.

These values correspond to the equations of form (2) at $F_2 = 0$. Assume that $F_2\,(y, z)$ is determined by eq. (4) for these equations we obtain the list of the equations (2) that pass the Painlevé test:

$$y_{zzzz} + 60y^3 + 30yy_{zz} + \varepsilon_0 + \varepsilon_1 y = 0, \tag{46}$$

$$y_{zzzz} + 40y^3 + 20yy_{zz} + 10y_z^2 + n_1 z + n_0 = 0, \tag{47}$$

$$y_{zzzz} + 24y^3 + 18yy_{zz} + 9y_z^2 + n_0 + n_1 z = 0, \tag{48}$$

$$y_{zzzz} + 15y^3 + 15yy_{zz} + \frac{45}{4} y_z^2 + \varepsilon_0 + \varepsilon_1 y = 0, \tag{49}$$

$$y_{zzzz} + 12yy_{zz} + 12y_z^2 = 0, \tag{50}$$

where ε_0, ε_1, n_0 and n_1 are arbitrary constants.

Eq. (50) can be presented in the form of the first Painlevé equation

$$y_{zz} + 6y^2 + c_1 z + c_2 \tag{51}$$

after integration over z. Eqs. (47) and (48) are reductions of the Schwarzian KdV and Schwarzian Caudrey Dodd–Gibbons equations. Eqs. (46) and (49) correspond to the ordinary differential equations that can be found from the Kaup–Kuperschmidt and the Caudrey–Dodd–Gibbon equations [4] if we look for solutions in the form of travelling wave. All these equations possess the Painlevé property.

In conclusion we have to note that we did not obtain new equations in the mentioned list of fourth order differential equations. We found that these equations are reductions of known ordinary differential equations and similarly reductions of known partial differential equations. The main result of this paper is a list of equations all of which pass the Painlevé test and we have no new equations of the Painlevé type of such forms. There are four equations in the above mentioned list that have solutions in the form of transcendental functions with respect to constants of integration. It was shown [5, 6] that eqs. (26), (27), (47) and (48) have such type of solutions. We applied our approach to find other forms of equations different from (1) and (2). In particular we found an equation of the form

$$yy_{zzzz}+\frac{7}{2}y^6-\frac{15}{2}y^2y_z^2-5y^3y_{zz}-\frac{5}{2}y_{zz}^2+ay^4+\beta y^2=0, \tag{52}$$

where a and β are arbitrary constants.

This is a new equation and we found this equation to passed the Painlevé test. Eq. (52) has four solutions families: $\left(a_0^{(1,2)},1\right)=(\pm 1,1)$ and $\left(a_0^{(3,4)},1\right)=(\pm 2,1)$. These families have the following Fuchs indices: $j_k^{(1,2)}=(-1,1,3,7)$ and $j_k^{(3,4)}=(-1,6,-2,7)$. Unfortunately we do not know much about this equation and we need to investigate it in the future.

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