

KAM Theorem for the Nonlinear Schrödinger Equation

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Abstract

We prove the persistence of finite dimensional invariant tori associated with the defocusing nonlinear Schrödinger equation under small Hamiltonian perturbations. The invariant tori are not necessarily small.

1 Introduction

Consider the defocusing nonlinear Schrödinger equation on a circle of unit length,

$$i\partial_t\varphi = -\partial_{xx}\varphi + 2|\varphi|^2\varphi \quad (t \in \mathbb{R}, x \in S^1). \tag{1}$$

It is a completely integrable Hamiltonian system of infinite dimension with phase space $H^N \equiv H^N(S^1, \mathbb{C})$ ($N \in \mathbb{R}_{\geq 1}$) and Hamiltonian $H_0 \equiv H_0(\varphi, \bar{\varphi})$. Here

$$H^N(S^1, \mathbb{C}) = \left\{ f(x) = \sum_{k \in \mathbb{Z}} \hat{f}(k) e^{2i\pi kx} \mid \|f\|_N < \infty \right\},$$

where

$$\|f\|_N^2 := \sum_{k \in \mathbb{Z}} (1 + |k|)^{2N} |\hat{f}(k)|^2$$

and

$$H_0(\varphi, \bar{\varphi}) := \int_0^1 (|\varphi_x|^2 + |\varphi|^4) dx.$$

The Poisson structure is given by the regular Poisson bracket

$$\{F, G\} := i \int_{S^1} \left(\frac{\partial F}{\partial \varphi(x)} \frac{\partial G}{\partial \bar{\varphi}(x)} - \frac{\partial F}{\partial \bar{\varphi}(x)} \frac{\partial G}{\partial \varphi(x)} \right) dx,$$

where F, G are functionals on $L^2 \equiv L^2(S^1, \mathbb{C})$ of class C^1 . When written as a Hamiltonian system, (1) takes the form

$$\partial_t \varphi = \{H_0, \varphi\} = -i \frac{\partial H_0}{\partial \bar{\varphi}}. \quad (2)$$

Our aim is to prove the existence of quasiperiodic solutions, not necessarily small, of small Hamiltonian perturbations of (2), i.e. of the equation

$$\partial_t \varphi = -i \frac{\partial H}{\partial \bar{\varphi}}, \quad (3)$$

where

$$H(\varphi, \bar{\varphi}) = H_0(\varphi, \bar{\varphi}) + \varepsilon K(\varphi, \bar{\varphi}), \quad \varepsilon \text{ small}. \quad (4)$$

To obtain solutions which are not necessarily close to zero we use a method developed in [11] for the KdV equation: First we prove the existence of global Birkhoff variables $(x_j, y_j)_{j \in \mathbb{Z}}$ (see Section 2). In these new variables, the NLS equation takes the canonical form

$$\begin{cases} \dot{x}_k = -w_k y_k, \\ \dot{y}_k = w_k x_k, \end{cases} \quad (5)$$

where $(\dot{\cdot})$ denotes the time derivative and $w(I) = (w_k(I))_{k \in \mathbb{Z}}$ is the sequence of frequencies which depend only on the actions $I_j = (x_j^2 + y_j^2)/2$, $j \in \mathbb{Z}$.

We then verify non-resonance conditions for the frequencies of the unperturbed system (2) reduced by certain symmetries which allow us to apply a refined version [15] of a KAM-theorem of Kuksin [12].

The results of this work have been announced in [6] and proved in a series of articles [7, 8, 9] and [10].

2 Existence of global Birkhoff variables

It is well known that NLS admits a Lax pair representation

$$\frac{\partial L}{\partial t} = [L, M] := LM - ML, \quad (6)$$

where

$$L = L(\varphi) := i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{d}{dx} + \begin{pmatrix} 0 & \varphi \\ \bar{\varphi} & 0 \end{pmatrix} \quad (7)$$

is the Zakharov–Shabat operator and M is a rather complicated operator given in [4]. As a consequence, the periodic spectrum of $L(\varphi)$, $\text{spec}(\varphi) := \{\lambda \in \mathbb{C}, \exists F \in H_{\text{loc}}^1(\mathbb{R}, \mathbb{C}^2)$ with $F \neq 0$, $L(\varphi)F = \lambda F$ and $F(x+2) = F(x)$, $x \in \mathbb{R}\}$, remains invariant under the NLS flow. The periodic spectrum consists of two interlacing sequences $(\lambda_k^+(\varphi))_{k \in \mathbb{Z}}$, $(\lambda_k^-(\varphi))_{k \in \mathbb{Z}}$ of real numbers satisfying $\lambda_k^\pm \sim k\pi$ ($|k|$ large) and $\lambda_k^- \leq \lambda_k^+ < \lambda_{k+1}^-$, $k \in \mathbb{Z}$ (cf [14], [5]). Furthermore, $\text{spec}(\varphi)$ is a complete set of integrals for the NLS equation (cf. [4]). This fact is used to prove the following Theorem (cf. [1] and [7]),

Theorem 1. *For any $N \geq 1$, there exists a bianalytic, bijective symplectomorphism*

$$\Phi : l_N^2(\mathbb{Z}, \mathbb{R}^2) \rightarrow H^N(S^1, \mathbb{R}^2)$$

such that $(x_j, y_j)_{j \in \mathbb{Z}} = \Phi^{-1}(\varphi)$ are Birkhoff coordinates for NLS, i.e. $I_j = \frac{1}{2}(x_j^2 + y_j^2)$ are action and $\theta_j = \arctg\left(\frac{y_j}{x_j}\right)$ are angle variables.

Here

$$l_N^2(\mathbb{Z}, \mathbb{R}^2) := \{(a_j, b_j)_{j \in \mathbb{Z}}, \|(a_j)_{j \in \mathbb{Z}}\|_N + \|(b_j)_{j \in \mathbb{Z}}\|_N < +\infty\},$$

where

$$\|(a_j)_{j \in \mathbb{Z}}\|_N^2 = \sum_{j \in \mathbb{Z}} (1 + |j|)^{2N} |a_j|^2,$$

and $l_N^2(\mathbb{Z}, \mathbb{R}^2)$ is endowed with the canonical symplectic structure

$$\Omega((a_j, b_j)_{j \in \mathbb{Z}}, (x_j, y_j)_{j \in \mathbb{Z}}) = 2 \sum_{j \in \mathbb{Z}} a_j y_j - b_j x_j.$$

In action angle variables, the Hamiltonian H_0 depends only on the actions, $H_0(\varphi, \bar{\varphi}) = \mathcal{H}(I)$, and NLS is equivalent to the system (5), where $w_k(I) = \frac{\partial \mathcal{H}}{\partial I_k}(I)$.

In particular, given $I \in l_{2N}^1(\mathbb{Z}, \mathbb{R}_+)$ with finite support,

$$\mathcal{I}_I = \Phi\left(\{(x_j, y_j)_{j \in \mathbb{Z}}, x_j^2 + y_j^2 = 2I_j, j \in \mathbb{Z}\}\right) \quad (8)$$

is an invariant set diffeomorphic to a torus whose dimension is $\#\{k \in \mathbb{Z}, I_k \neq 0\}$.

The solution $\phi(x, t) \equiv \varphi_t(x)$ of the initial value problem for NLS, with initial profile $\varphi_0 = \Phi\left(\left(\sqrt{2I_j} e^{i\theta_j}\right)_{j \in \mathbb{Z}}\right)$ in $H^N(S^1, \mathbb{C})$, is given by

$$\varphi_t = \Phi\left(\left(\left(\sqrt{2I_j} e^{i(\theta_j + tw_j(I))}\right)_{j \in \mathbb{Z}}\right)\right).$$

3 KAM Theorem for NLS

An asymptotic expansion shows that the frequencies have asymptotic resonances,

$$w_{\pm k}(I) \sim 4\pi^2 k^2 \quad \text{for } k \text{ large.}$$

In order to control their effect on perturbed equations we impose symmetry conditions on the perturbation. These conditions (see [8]) allow to consider as phase spaces the subspaces $H_\alpha^N(S^1, \mathbb{C})$, $\alpha \in \mathbb{R}$, defined by,

$$H_\alpha^N(S^1, \mathbb{C}) := \Phi(l_{N;\alpha}^2(\mathbb{Z}, \mathbb{R}^2)), \quad (9)$$

where $(\sqrt{2I_j} e^{i\theta_j})_{j \in \mathbb{Z}} \in l_{N;\alpha}^2$ iff $(\sqrt{2I_j} e^{i\theta_j})_{j \in \mathbb{Z}} \in l_N^2$ and satisfies

$$I_{-j} = I_j \quad \forall j \geq 1, \quad (10)$$

and

$$\theta_{-j} \equiv \theta_j + \alpha \pmod{2\pi} \quad \forall j \geq 0 \quad \text{with} \quad I_j \neq 0. \quad (11)$$

(Notice that for $\alpha \not\equiv 0 \pmod{2\pi}$, (11) implies that $I_0(\varphi) = 0$ for all $\varphi \in H_\alpha^N$.)

One verifies that the subspaces $H_\alpha^N(S^1, \mathbb{C})$ are invariant under the NLS-flow by showing (cf. [9]) that the symmetries of the NLS Hamiltonian $\mathcal{H}(I)$ imply that $\mathcal{H}(\mathcal{J}(I)) = \mathcal{H}(I)$, where $\mathcal{J}(I)_k = I_{-k} \forall k \in \mathbb{Z}$. As a consequence, the frequencies $w_j = \frac{\partial \mathcal{H}}{\partial I_j}$ are symmetric at points where $\mathcal{J}(I) = I$.

Moreover, in [8] we provide the following characterization of $H_\alpha^N(S^1, \mathbb{C})$,

$$H_\alpha^N(S^1, \mathbb{C}) = \{ \varphi \in H^N(S^1, \mathbb{C}) \mid e^{i\alpha} \check{\varphi} \equiv \varphi \},$$

where $\check{\varphi}(x) = \varphi(-x)$. In particular, $H_\pi^N \cap C^\infty$ (resp. $H_0^N \cap C^\infty$) is the phase-space of $\varphi \in H^N \cap C^\infty$ satisfying generalized Dirichlet (resp. Neumann) conditions, i.e. $\partial_x^{2k} \varphi(0) = \partial_x^{2k} \varphi(1) = 0$ (resp. $\partial_x^{2k+1} \varphi(0) = \partial_x^{2k+1} \varphi(1) = 0 \forall k \in \mathbb{Z}$). By a slight abuse of notation, the restriction of Φ to $l_{N,\alpha}^2(\mathbb{Z}, \mathbb{R}^2)$ is again denoted by Φ .

For $\alpha \in \mathbb{R}/2\pi\mathbb{Z}$, a finite subset $A \subseteq \mathbb{Z}_{\geq 0}$ (with $0 \notin A$ if $\alpha \neq 0$) and $I_A \in (\mathbb{R}_{>0})^{|A|}$ we denote by $T_{I_A}^\alpha$ the $|A|$ dimensional torus of the model space $l^2(\mathbb{Z}; \mathbb{R}^2)$, defined by

$$\begin{aligned} T_{I_A}^\alpha := \{ & (\sqrt{2J_j} e^{i\theta_j})_{j \in \mathbb{Z}} \mid J_j = J_{-j} = I_j, \forall j \in A; \\ & J_j = J_{-j} = 0, \forall j \notin A; \theta_j = \theta_{-j} + \alpha, \forall j \in A \} \end{aligned} \quad (12)$$

and by $\mathcal{T}_{I_A}^\alpha$ the $|A|$ dimensional torus in H_α^N , invariant under NLS,

$$\mathcal{T}_{I_A}^\alpha := \Phi(T_{I_A}^\alpha). \quad (13)$$

For $\Gamma \subseteq (\mathbb{R}_{>0})^{|A|}$ compact and of positive Lebesgue measure, introduce

$$\mathcal{T}_\Gamma^\alpha := \cup_{I_A \in \Gamma} \mathcal{T}_{I_A}^\alpha. \quad (14)$$

The set $\mathcal{T}_\Gamma^\alpha$ consists of symmetric $2|A|$ -gap potentials (if $0 \notin A$) or $(2|A|-1)$ -gap potentials (if $0 \in A$), i.e potentials $\varphi \in H_\alpha^0$ with $\lambda_j^+(\varphi) \neq \lambda_j^-(\varphi)$ iff $|j| \in A$ and $\lambda_{-j}^+(\varphi) - \lambda_{-j}^-(\varphi) = \lambda_j^+(\varphi) - \lambda_j^-(\varphi) \forall j \geq 1$ (cf. [8]).

We consider Hamiltonian perturbations, $H_\varepsilon = H_0 + \varepsilon K$ on $H_\alpha^N(S^1, \mathbb{C})$ with the following properties:

- (P1) K is real analytic on some symmetric neighborhood U_Γ of $\{(\varphi, \bar{\varphi}), \varphi \in \mathcal{T}_\Gamma^\alpha\}$ in $(H^N(S^1, \mathbb{C}))^{2 \cdot 1}$.
- (P2) $\frac{\partial K}{\partial \varphi}, \frac{\partial K}{\partial \psi}$ are bounded as functions from U_Γ into $H^N(S^1, \mathbb{C})$ and verify the normalization condition

$$\sup \left\{ \left\| \frac{\partial K}{\partial \varphi}(\varphi, \psi) \right\|_N + \left\| \frac{\partial K}{\partial \psi}(\varphi, \psi) \right\|_N \mid (\varphi, \psi) \in U_\Gamma \right\} \leq 1.$$

- (P3) K satisfies the symmetry condition, $((\varphi, \psi) \in U_\Gamma)$

$$K(\varphi, \psi) = K(e^{i\alpha} \check{\varphi}, e^{-i\alpha} \check{\psi}).$$

¹ U_Γ is said to be symmetric iff $(e^{i\alpha} \check{\varphi}, e^{-i\alpha} \check{\psi}) \in U_\Gamma$ for any $(\varphi, \psi) \in U_\Gamma$.

Notice that condition (P3) insures that solutions of $\frac{\partial \varphi}{\partial t} = i \frac{\partial H_\varepsilon}{\partial \varphi}$ for initial data in $H_\alpha^N(S^1, \mathbb{C})$ evolve in the same space $H_\alpha^N(S^1, \mathbb{C})$.

Our KAM Theorem states that, for ε small enough, many of the NLS-invariant tori $\mathcal{T}_{I_A}^\alpha$ persist under perturbation of the NLS Hamiltonian by εK with K satisfying (P1), (P2), and (P3). Moreover these tori and their linear flows are only slightly perturbed.

Denote by T^n the n -dimensional torus $(\mathbb{R}/\mathbb{Z})^n$.

Theorem 2. *Let $N \geq 1$, $A, \Gamma, \alpha, U_\Gamma$ be given as above. Then, for K satisfying (P1), (P2) and (P3), there exists ε_0 so that for any ε with $|\varepsilon| \leq \varepsilon_0$ the following statements hold:*

- (i) *there exists a Cantor set $\Gamma_\varepsilon \subset \Gamma$ with $\text{meas}(\Gamma \setminus \Gamma_\varepsilon) \xrightarrow{\varepsilon \rightarrow 0} 0$,*
- (ii) *there exists a Lipschitz family of real analytic torus embeddings*

$$\Psi : T^{|A|} \times \Gamma_\varepsilon \rightarrow U_\Gamma \cap \{(\varphi, \bar{\varphi}) \mid \varphi \in H_\alpha^N\}$$

and

- (iii) *there exists a Lipschitz map $f : \Gamma_\varepsilon \rightarrow \mathbb{R}^{|A|}$*

such that for $I_A \in \Gamma_\varepsilon$ and $\theta_A \in T^{|A|}$, $\Psi(\theta_A + tf(I_A), I_A)$ is a quasiperiodic solution of $\partial_t \varphi = i \frac{\partial H_0}{\partial \varphi} + i\varepsilon \frac{\partial K}{\partial \varphi}$. Moreover, the deformed invariant tori, $\Psi(T^{|A|} \times \{I_A\})$, are linearly stable.

Remarks:

1. Theorem 2 generalizes results due to Kuksin–Pöschel [13] which concern the special case where $\Gamma \subseteq \mathbb{R}_+^{|A|}$ is contained in a sufficiently small neighbourhood of $0 \in \mathbb{R}^{|A|}$ and the phase space consists of elements satisfying generalized Dirichlet boundary conditions. In this situation, action-angle variables are not needed as the Fourier coefficients $(\hat{\varphi}(k))_{k \in \mathbb{Z}}$ are a sufficiently good approximation of the Birkhoff coordinates close to the origin.
2. Similarly, the results of [3] and their generalization in [2], while not directly comparable with our Theorem 2, concern only small perturbations of NLS around $\varphi = 0$ as well.
3. Our results and methods continue the investigation in [11] on the Korteweg-de Vries equation. The purpose of this paper is to document similar features of the NLS equation.

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