# A Note on Finite Groups all of Whose Subgroups are C-Normal

**Z. Mostaghim**

School of Mathematics, Iran University of Science and Technology, Tehran, Iran

Abstract - A subgroup $H$ of a group $G$ is said to be $c$-normal in $G$ if there exists a normal subgroup $N$ of $G$ such that $HN=G$ and $H\cap N \leq Core(H)$ where $Core(H)$ is the largest normal subgroup of $G$ contained in $H$. In this paper we consider finite $p$-groups of order at most $p^4$ where $p$ is a prime and show that all of their subgroups are $c$-normal. Also we study some classes of finite groups whose all of subgroups are $c$-normal.

Index Terms - $c$-normal subgroups, $p$-groups, maximal class, supersolvable groups.

## I. Introduction

The notion of $c$-normal subgroup was introduced for the first time by *Wang*[1]. He used the $c$-normality of maximal subgroups to give some conditions for the solvability and supersolvability of a finite group. For example, he showed that $G$ is solvable if and only if $M$ is $c$-normal in $G$ for every maximal subgroup $M$ of $G$. In this paper we consider finite $p$-groups of order at most $p^4$ where $p$ is a prime and show that all of their subgroups are $c$-normal. Also we study some classes of finite groups whose all of subgroups are $c$-normal.

Throughout, all groups are assumed to be finite groups. Our terminology and notation is standard, see[2].

## II. Preliminaries

In this section, we give some definitions and basic results which are essential in the sequel.

**Definition 2.1.**[1]

Let $G$ be a group. We call a subgroup $H$ c-normal in $G$ if there exists a normal subgroup $N$ of $G$ such that $HN=G$ and $H\cap N \leq Core(H)$

It is clear that a normal subgroup of $G$ is a $c$-normal subgroup of $G$ but the converse is not true.

**Definition 2.2.**[1]

We call a group $G$ is c-simple if $G$ has no c-normal subgroup except the identity group 1 and $G$.

We can easily show that $G$ is $c$-simple if and only if $G$ is simple.

**Lemma 2.3.**[1]

Let $G$ be a group. Then

(1) If $H$ is normal in $G$, then $H$ is $c$-normal in $G$;
$G$ is $c$-simple if and only if $G$ is simple;

(2) If $H$ is $c$-normal in $G$, $H \leq K \leq G$, then $H$ is $c$-normal in $K$;

(3) Let $K$ be normal in $G$ and $K \leq H$. Then $H$ is $c$-normal in $G$ if and only if $H/K$ is $c$-normal in $G/K$.

Let $p$ be a prime. Now we will give some properties of non-abelian groups of order $p^4$.

**Lemma 2.4.**[3]

Let $G$ be a finite non-abelian $p$-group of order $p^4$. Then

(1) $|Z(G)| = p \ or\ p^2$;

(2) $|G'| \leq p^2$.

**Lemma 2.5.**[3]

Let $G$ be a finite non-abelian $p$-group of order $p^4$. If $Z(G)$ is cyclic of order $p$, then $G'$ has order $p^2$. Moreover $Z(G) < G'$ and $G/Z(G)$ is not abelian.

**Definition 2.6.**[4]

A $p$-group $G$ is said to be a special $p$-group if either $G$ is an elementary abelian $p$-group or we have $\Phi(G) = Z(G) = G'$ and $G'$ is elementary abelian. If the center of a non-abelian special $p$-group $G$ is cyclic, then $G$ is called extraspecial.

**Definition 2.7.**[4]

A group of order $p^n$ is said to be a group of maximal class if the class of $G$ is $n-1$.

**Theorem 2.8.**[4]

The groups $G = D_{2m}$ , $Q_{2m}$, $SD_{2m}$ have the following properties.

(1) The center $Z(G)$ has order 2 and $G/Z(G) \cong D_m$.

(2) The derived group coincides with $\Phi(G)$ and the class of $G$ is $n-1$ where $|G| = 2^n$.

(3) The group $Q_{2m}$ contains exactly one element of order 2.

## III. Main Results

**Lemma 3.1.**

Let $G$ be an extraspecial $p$-group. Then every subgroup of order $p$ is $c$-normal.

**Proof.** It is easy to see that $|G'| = p$. Let $H$ be a subgroup of order. If $H \cap G' = G'$, then $H \lhd G$ . By lemma 2.3 $H$ is $c$-normal in $G$. If $H \cap G' = 1$, then there exists a maximal subgroup $M$ such that $H \nsubseteq M$ .So $G = HM$ and $H \cap M \leq Core(H)$. Therefore $H$ is $c$-normal in $G$.

**Theorem 3.2.**

Let $G$ be a $p$-group of order at most $p^4$. Then all of subgroups of $G$ are -normal.

**Proof.**

If $G$ be an abelian group, then all of subgroups of $G$ are normal and by lemma 2.3, they are $c$-normal. If $G$ be a non-abelian group of order $p^3$, then by lemma 3.1 all of subgroups of order $p$ are $c$-normal. It is easy to see that other subgroups of $G$ are normal. If $G$ be a non-abelian group of order $p^4$, then by lemmas 2.4 and 2.5 we have two cases:

**Case1.**

Let $|G'| = p$, therefore $|Z(G)| = p^2$.

Let $H$ be a subgroup of order $p$. If $H \cap G' \neq 1$, then $H = G'$ and $H$ is a normal subgroup.

Let $H \cap G' = 1$. If there exists a maximal subgroup $M$ such that $H \nsubseteq M$, then $G = HM$ and $H \cap M \leq Core(H)$. Therefore $H$ is a $c$-normal subgroup.

Let $H \leq \Phi(G)$, so $\Phi(G) = HG'$ and $|\Phi(G)| = p^2$. Since $G/Z(G)$ is an elementary abelian group, so $\Phi(G) \leq Z(G)$ and therefore $H$ is a normal subgroup.

Let $H$ be a subgroup of order $p^2$. We have $|H \cap Z(G)| = 1, p$ or $p^2$. If $|H \cap Z(G)| = p^2$, then $H = Z(G)$ and $H$ is a normal subgroup. If $|H \cap Z(G)| = 1$, then $G = HZ(G)$ and $H \cap Z(G) \leq Core(H)$. Hence $H$ is $c$-normal in $G$.

Let $|H \cap Z(G)| = p$. It is easy to see that $Z(G) = \Phi(G)$. If $H = \Phi(G)$, then $H$ is a normal subgroup. Otherwise there exists a maximal subgroup $M$ such that $HM = G$ and $|H \cap M| = p$. It is easy to see that $H \cap M \leq Core(H)$ and therefore $H$ is a $c$-normal subgroup.

**Case2.**

Let $|G'| = p^2$, therefore $|Z(G)| = p$.

Let $H$ be a subgroup of order $p$. If $H \cap Z(G) = H$, then $H$ is a normal subgroup. Let $H \cap Z(G) = 1$. If there exists a maximal subgroup $M$ such that $H \cap M = 1$, then $H$ is a $c$-normal subgroup. Otherwise $H \leq \Phi(G) = G' = HZ(G)$. It is easy to see that $H ch G'$ and then $H$ is a normal subgroup.

Let $H$ be a subgroup of order $p^2$. Hence $|H \cap G'| = p$ or $p^2$.If $|H \cap G'| = p^2$, then $H$ is a normal subgroup.

Let $|H \cap G'| = p$. If $H = \Phi(G)$, then $H$ is a normal subgroup. Otherwise, there exists a maximal subgroup $M$ such that $G = HM$ and $|H \cap M| = p$. Since $H \cap G' ch\, G'$, therefore $Core(H) = H \cap G' = H \cap M$. Hence $H$ is a $c$-normal subgroup.

**Theorem 3.3.**

Let $G = D_{2n} = \langle a, b | a^n = 1, b^2 = 1, bab = a^{-1} \rangle$. Then all of subgroups of $G$ are $c$-normal.

**Proof.**

When considered geometrically, $D_{2n}$ consist of $n$ rotations and $n$ reflections of the regular $n$-gon. The subgroups of $D_{2n}$ are two types:

(1) Those containing rotations only.
(2) Those containing rotations and reflections.

Let $H$ be a subgroup of $G$. We consider two cases.

**Case1.**

Let $H$ has no reflection. Then $H = \langle a^j \rangle$ for $0 \leq j \leq n-1$. Thus by lemma 2.3 $H$ is $c$-normal in $G$.

**Case2.**

Let $H$ be of type 2.

(1) Let $a^j \notin H$ for $0 < j \leq n-1$, so we have $|H| = 2$. Now let $N = \langle a \rangle$. Then $N$ is a normal subgroup, $G = HN, H \cap N = 1$. Hence $H$ is a $c$-normal subgroup.

(2) Let there exists $i > 0$ such that $a^i \in H$. Now let $m = \min\{i | i > 0, a^i \in H\}$ and $N = \langle a \rangle$, then $|H| = 2l$, $(1 < l \leq n)$ and $HN = G$. Also we have $H \cap N = \langle a^m \rangle$, then $H \cap N \leq Core(H)$. Hence $H$ is $c$-normal in $G$.

**Theorem 3.4.**

Let $G$ be a 2-group of maximal class. Then all of subgroups of $G$ are $c$-normal.

**Proof.**

Since $G$ is a 2-group of maximal class, then $G$ is $D_{2^m} (m \geq 3), Q_{2^m}, SD_{2^m}$. We consider three cases.

**Case1.**

Let $G = D_{2^m} (m \geq 3)$. Then by theorem 3.3 all of subgroups of $G$ are $c$-normal.

**Case2.**

Let $G = Q_m$, then by theorem 2.8 $G/Z(G) \cong D_{2^{m-1}}$. Let $H$ be a subgroup of $G$ such that $Z(G) \subseteq H$, then by lemma 2.3 $H/Z(G)$ is a $c$-normal subgroup of $G/Z(G)$. By using lemma 2.3 we have $H$ is a $c$-normal subgroup of $G$. If $H$ be a subgroup of $G$ and $Z(G) \nleq H$,then $x^j \notin H$ for $0 < j \leq 2^{m-1}$ so we have $|H| = 2$. Now let $N = \langle x \rangle$. Thus $N$ is a normal subgroup of $G$ and $G = HN, H \cap N = 1$. Hence $H$ is $c$-normal in $G$.

**Corollary 3.5.**

Let $G$ be one of the following groups.

(1) A non-nilpotent finite group that all of proper subgroups are Nilpotent.
(2) A non-abelian finite group that all of proper subgroups are abelian. Then every $p$-Sylow subgroup of $G$ is $c$-normal.

**Proof.**

For case $(i)$ we can see $|G| = p^\alpha q^\beta$, where $p$ and $q$ are distinct primes. Also one of Sylow subgroups of $G$ is cyclic and another is normal. Then every $p$-Sylow subgroup of $G$ is $c$-normal. Case $(ii)$ is similar.

**Corollary 3.6.**

Let $G$ be a finite supersolvable group and $p||G|$ where $p$ is the smallest prime divisor of $|G|$. Then $p$-Sylow subgroup of $G$ is $c$-normal.

**Proof.**

Let $|G| = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_n^{\alpha_n}$, where $p_i$ are primes such that $p_1 > p_2 > \cdots > p_n .(p = p_n)$ Let $P_i$ be a $p_i$-Sylow subgroup of $G$ for $1 \leq i \leq n$,then $P_1 P_2 \cdots P_k$ is a normal subgroup for all $1 \leq k \leq n$. It is easy to see that $P_n$ is $c$-normal in $G$.

## IV. GAP Program

In this section we use GAP[5] and give a program for finding c-normal subgroups. By using this program we can find all c-normal subgroups of a finite group with two generations. With a few changes in this program we can find a program for finding c-normal subgroups in a finite group with any number of generations and relations.

F:=FreeGroup("a","b");

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a:=GeneratorsOfGroup(F)[1];
b:=GeneratorsOfGroup(F)[2];
Read("r");
G:=F/r;
n:=Order(G);
z:=LowIndexSubgroupsFpGroup(G,TrivialSubgroup(G),n);
s:=[];
        for i in [1..Size(z)] do
        t:=ConjugacyClassSubgroups(G,z[i]);
             for j in [1..Size(t)] do
             Add(s,t[j]);
             od;
         od;
cnorm:=[];
N:=[];
H:=[];
         for i in [1..Size(s)] do
         vi:=IsNormal(G,s[i]);
          if vi=true then Add(N,s[i]);fi;
          if vi=false then Add(H,s[i]);fi;
          od;
for y in [1..Size(H)] do
l:=0;
m:=0;
h:=false;
  while (m=0 or h=false) and l<=Size(N) do
   l:=l+1;
   eH:=Elements(H[y]);
   eN:=Elements(N[l]);
   HN:=[];
     for i in [1..Order(H[y])] do
        for j in [1..Order(N[l])] do
        u:=eH[i]*eN[j];
        AddSet(HN,u);
        od;
     od;
 h:=IsSubgroup(Core(G,H[y]),Intersection(H[y],N[l]));
 if HN=G then m:=1;fi;
    od;
if HN=G and h=true then Add(cnorm,H[y]);fi;
od;
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## References And Notes


[1] Y. Wang , J.Algebra 180, (1996).
[2] D. Gorenstein, Finite groups,Chelsea Publishing Company (1980).
[3] H.Behravesh and H.Mousavi ,Proc.Indian Acad.Sci(Math. Sci) 119, (2009).
[4] M. Suzuki, Group Theory II, Springer-Verlag,New York, (1986).
[5] The GAPGroup,GAP-Groups,Algorithms and Programming, (2005).