A Note on Finite Groups all of Whose Subgroups are C-Normal

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Abstract - A subgroup H of a group G is said to be c-normal in G if there exists a normal subgroup N of G such that $HN=G$ and $H \cap N \leq Core(H)$ where $Core(H)$ is the largest normal subgroup of G contained in H. In this paper we consider finite p-groups of order at most p^4 where p is a prime and show that all of their subgroups are c-normal. Also we study some classes of finite groups whose all of subgroups are c-normal.

Index Terms - c-normal subgroups, p-groups, maximal class, supersolvable groups.

I. Introduction

The notion of c-normal subgroup was introduced for the first time by *Wang*[1]. He used the c-normality of maximal subgroups to give some conditions for the solvability and supersolvability of a finite group. For example, he showed that G is solvable if and only if M is c-normal in G for every maximal subgroup M of G. In this paper we consider finite p-groups of order at most p^4 where p is a prime and show that all of their subgroups are c-normal. Also we study some classes of finite groups whose all of subgroups are c-normal.

Throughout, all groups are assumed to be finite groups. Our terminology and notation is standard, see[2].

II. Preliminaries

In this section, we give some definitions and basic results which are essential in the sequel.

Definition 2.1.[1]

Let G be a group. We call a subgroup H c-normal in G if there exists a normal subgroup N of G such that $HN=G$ and $H \cap N \leq Core(H)$

It is clear that a normal subgroup of G is a c-normal subgroup of G but the converse is not true.

Definition 2.2.[1]

We call a group G is c-simple if G has no c-normal subgroup except the identity group 1 and G.

We can easily show that G is c-simple if and only if G is simple.

Lemma 2.3.[1]

Let G be a group. Then

(1) If H is normal in G, then H is c-normal in G; G is c-simple if and only if G is simple;

(2) If H is c-normal in G, $H \leq K \leq G$, then H is c-normal in K;

(3) Let K be normal in G and $K \leq H$. Then H is c-normal in G if and only if H/K is c-normal in G/K.

Let p be a prime. Now we will give some properties of non-abelian groups of order p^4.

Lemma 2.4.[3]

Let G be a finite non-abelian p-group of order p^4. Then

(1) $|Z(G)| = p \ or \ p^2$;

(2) $|G'| \leq p^2$.

Lemma 2.5.[3]

Let G be a finite non-abelian p-group of order p^4. If $Z(G)$ is cyclic of order p, then G' has order p^2. Moreover $Z(G) < G'$ and $G/Z(G)$ is not abelian.

Definition 2.6.[4]

A p-group G is said to be a special p-group if either G is an elementary abelian p-group or we have $\Phi(G) = Z(G) = G'$ and G' is elementary abelian. If the center of a non-abelian special p-group G is cyclic, then G is called extraspecial.

Definition 2.7.[4]

A group of order p^n is said to be a group of maximal class if the class of G is $n-1$.

Theorem 2.8.[4]

The groups $G = D_{2m}$, Q_{2m}, SD_{2m} have the following properties.

(1) The center $Z(G)$ has order 2 and $G/Z(G) \cong D_m$.

(2) The derived group coincides with $\Phi(G)$ and the class of G is $n-1$ where $|G| = 2^n$.

(3) The group Q_{2m} contains exactly one element of order 2.

III. Main Results

Lemma 3.1.

Let G be an extraspecial p-group. Then every subgroup of order p is c-normal.

Proof. It is easy to see that $|G'| = p$. Let H be a subgroup of order. If $H \cap G' = G'$, then $H \lhd G$. By lemma 2.3 H is c-normal in G. If $H \cap G' = 1$, then there exists a maximal subgroup M such that $H \nsubseteq M$ .So $G = HM$ and $H \cap M \leq Core(H)$. Therefore H is c-normal in G.

Theorem 3.2.

Let G be a p-group of order at most p^4. Then all of subgroups of G are -normal.

Proof.

If G be an abelian group, then all of subgroups of G are normal and by lemma 2.3, they are c-normal. If G be a non-abelian group of order p^3, then by lemma 3.1 all of subgroups of order p are c-normal. It is easy to see that other subgroups of G are normal. If G be a non-abelian group of order p^4, then by lemmas 2.4 and 2.5 we have two cases:

Case1.

Let $|G'| = p$, therefore $|Z(G)| = p^2$.

Let H be a subgroup of order p. If $H \cap G' \neq 1$, then $H = G'$ and H is a normal subgroup.

Let $H \cap G' = 1$. If there exists a maximal subgroup M such that $H \nsubseteq M$, then $G = HM$ and $H \cap M \leq Core(H)$. Therefore H is a c-normal subgroup.

Let $H \leq \Phi(G)$, so $\Phi(G) = HG'$ and $|\Phi(G)| = p^2$. Since $G/Z(G)$ is an elementary abelian group, so $\Phi(G) \leq Z(G)$ and therefore H is a normal subgroup.

Let H be a subgroup of order p^2. We have $|H \cap Z(G)| = 1, p$ or p^2. If $|H \cap Z(G)| = p^2$, then $H = Z(G)$ and H is a normal subgroup. If $|H \cap Z(G)| = 1$, then $G = HZ(G)$ and $H \cap Z(G) \leq Core(H)$. Hence H is c-normal in G.

Let $|H \cap Z(G)| = p$. It is easy to see that $Z(G) = \Phi(G)$. If $H = \Phi(G)$, then H is a normal subgroup. Otherwise there exists a maximal subgroup M such that $HM = G$ and $|H \cap M| = p$. It is easy to see that $H \cap M \leq Core(H)$ and therefore H is a c-normal subgroup.

Case2.

Let $|G'| = p^2$, therefore $|Z(G)| = p$.

Let H be a subgroup of order p. If $H \cap Z(G) = H$, then H is a normal subgroup. Let $H \cap Z(G) = 1$. If there exists a maximal subgroup M such that $H \cap M = 1$, then H is a c-normal subgroup. Otherwise $H \leq \Phi(G) = G' = HZ(G)$. It is easy to see that $H ch G'$ and then H is a normal subgroup.

Let H be a subgroup of order p^2. Hence $|H \cap G'| = p$ or p^2.If $|H \cap G'| = p^2$, then H is a normal subgroup.

Let $|H \cap G'| = p$. If $H = \Phi(G)$, then H is a normal subgroup. Otherwise, there exists a maximal subgroup M such that $G = HM$ and $|H \cap M| = p$. Since $H \cap G' ch\ G'$, therefore $Core(H) = H \cap G' = H \cap M$. Hence H is a c-normal subgroup.

Theorem 3.3.

Let $G = D_{2n} =< a, b | a^n = 1, b^2 = 1, bab = a^{-1} >$. Then all of subgroups of G are c-normal.

Proof.

When considered geometrically, D_{2n} consist of n rotations and n reflections of the regular n-gon. The subgroups of D_{2n} are two types:

(1) Those containing rotations only.

(2) Those containing rotations and reflections.

Let H be a subgroup of G. We consider two cases.

Case1.

Let H has no reflection. Then $H =< a^j >$ for $0 \leq j \leq n-1$. Thus by lemma 2.3 H is c-normal in G.

Case2.

Let H be of type 2.

(1) Let $a^j \notin H$ for $0 < j \leq n-1$, so we have $|H| = 2$. Now let $N =< a >$. Then N is a normal subgroup, $G = HN, H \cap N = 1$. Hence H is a c-normal subgroup.

(2) Let there exists $i > 0$ such that $a^i \in H$. Now let $m = \min\{i | i > 0, a^i \in H\}$ and $N =< a >$, then $|H| = 2l$, $(1 < l \leq n)$ and $HN = G$. Also we have $H \cap N =< a^m >$, then $H \cap N \leq Core(H)$. Hence H is c-normal in G.

Theorem 3.4.

Let G be a 2-group of maximal class. Then all of subgroups of G are c-normal.

Proof.

Since G is a 2-group of maximal class, then G is $D_{2^m} (m \geq 3), Q_{2^m}, SD_{2^m}$. We consider three cases.

Case1.

Let $G = D_{2^m} (m \geq 3)$. Then by theorem 3.3 all of subgroups of G are c-normal.

Case2.

Let $G = Q_m$, then by theorem 2.8 $G/Z(G) \cong D_{2^{m-1}}$. Let H be a subgroup of G such that $Z(G) \subseteq H$, then by lemma 2.3 $H/Z(G)$ is a c-normal subgroup of $G/Z(G)$. By using lemma 2.3 we have H is a c-normal subgroup of G. If H be a subgroup of G and $Z(G) \nleq H$,then $x^j \notin H$ for $0 < j \leq 2^{m-1}$ so we have $|H| = 2$. Now let $N =< x >$. Thus N is a normal subgroup of G and $G = HN, H \cap N = 1$. Hence H is c-normal in G.

Corollary 3.5.

Let G be one of the following groups.

(1) A non-nilpotent finite group that all of proper subgroups are Nilpotent.

(2) A non-abelian finite group that all of proper subgroups are abelian. Then every p-Sylow subgroup of G is c-normal.

Proof.

For case (i) we can see $|G| = p^\alpha q^\beta$, where p and q are distinct primes. Also one of Sylow subgroups of G is cyclic and another is normal. Then every p-Sylow subgroup of G is c-normal. Case (ii) is similar.

Corollary 3.6.

Let G be a finite supersolvable group and $p||G|$ where p is the smallest prime divisor of $|G|$. Then p-Sylow subgroup of G is c-normal.

Proof.

Let $|G| = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_n^{\alpha_n}$, where p_i are primes such that $p_1 > p_2 > \cdots > p_n$.$(p = p_n)$ Let P_i be a p_i-Sylow subgroup of G for $1 \leq i \leq n$,then $P_1 P_2 \cdots P_k$ is a normal subgroup for all $1 \leq k \leq n$. It is easy to see that P_n is c-normal in G.

IV. GAP Program

In this section we use GAP[5] and give a program for finding c-normal subgroups. By using this program we can find all c-normal subgroups of a finite group with two generations. With a few changes in this program we can find a program for finding c-normal subgroups in a finite group with any number of generations and relations.

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F:=FreeGroup("a","b");
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a:=GeneratorsOfGroup(F)[1];
b:=GeneratorsOfGroup(F)[2];
Read("r");
G:=F/r;
n:=Order(G);
z:=LowIndexSubgroupsFpGroup(G,TrivialSubgroup(G),n);
s:=[];
                for i in [1..Size(z)] do
                t:=ConjugacyClassSubgroups(G,z[i]);
                          for j in [1..Size(t)] do
                          Add(s,t[j]);
                          od;
                  od;
cnorm:=[];
N:=[];
H:=[];
                   for i in [1..Size(s)] do
                   vi:=IsNormal(G,s[i]);
                    if vi=true then Add(N,s[i]);fi;
                    if vi=false then Add(H,s[i]);fi;
                     od;
for y in [1..Size(H)] do
l:=0;
m:=0;
h:=false;
   while (m=0 or h=false) and l<=Size(N) do
    l:=l+1;
    eH:=Elements(H[y]);
    eN:=Elements(N[l]);
    HN:=[];
        for i in [1..Order(H[y])] do
              for j in [1..Order(N[l])] do
              u:=eH[i]*eN[j];
              AddSet(HN,u);
              od;
         od;
  h:=IsSubgroup(Core(G,H[y]),Intersection(H[y],N[l]));
  if HN=G then m:=1;fi;
      od;
if HN=G and h=true then Add(cnorm,H[y]);fi;
od;
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References And Notes

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