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A Tree of Linearisable Second-Order Evolution Equations by Generalised Hodograph Transformations

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To the memory of
Wilhelm Fushchych

Abstract

We present a list of $(1+1)$-dimensional second-order evolution equations all connected via a proposed generalised hodograph transformation, resulting in a tree of equations transformable to the linear second-order autonomous evolution equation. The list includes autonomous and nonautonomous equations.

1 Introduction

In [1] we report on the linearisation of the hierarchy of evolution equations

$$u_t = R^m[u](u^{-2}u_x)_x, \qquad R[u] = D_x^2 u^{-1} D_x^{-1}, \qquad m = 0, 1, 2, \ldots \tag{1.1}$$

by an extended hodograph transformation and define an autohodograph transformation for this hierarchy. The autohodograph transformation is revealed by the composition of the extended hodograph transformation and the linearising contact transformation. The extended hodograph transformation for the case $m = 0$, first introduced in [2], is of the form

$$\begin{aligned} dX(x,t) &= u\,dx + \left(\int^x u_t(\xi,t)\,d\xi\right) dt \\ dT(x,t) &= dt \\ U(X,T) &= x. \end{aligned} \tag{1.2}$$

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In the present paper we generalise the extended hodograph transformation and name it an *x-generalised hodograph transformation*. We are interested to derive a class (or *tree*, as we prefer to call it) of $(1+1)$-dimensional second-order evolution equations which are linearisable. This tree of equations, containing arbitrary nonconstant functions in $C^2(\Re)$, is constructed by nonlinearising the general second-order linear autonomous equation using the x-generalised hodograph transformation. In this way both nonlinear autonomous and nonlinear nonautonomous equations are revealed. The linearising transformations are obtained by composing and inverting the appropriate x-generalised hodograph transformations. Besides the obvious examples, such as Burgers' equation and (1.1) (with $m=0$), our tree of equations consists of new linearisable equations as well as several cases found in the literature (for example [2, 3, 4, 5, 6, 7, 8]). In particular, the results obtained by Sokolov, Svinolupov and Wolf [4] are special cases of our tree of equations.

The paper is organized as follows: In Section 2 we define the x-generalised hodograph transformation and introduce the notation. The most general form of the second-order equation linearisable by the proposed method is established in this section. In Section 3 we consider autonomous second-order evolution equations and derive a tree of linearisable equations. The linearising transformations are listed explicitly. The x-generalised hodograph transformations generating the equations are given in the Appendix. Only one equation from the tree of equations admits an autohodograph transformation in the sense of [1]. It should be pointed out that under the proposed x-generalised hodograph transformation the tree of autonomous linearisable equations (see Diagram 1) is complete. Some examples are given. In Section 4 we list nonautonomous linearisable second-order evolution equations which are generated from the tree of autonomous linearisable equations (Diagram 1) by x-generalised hodograph transformations. This case is not complete as we consider only the case where the coefficient of the highest derivative is autonomous. Once again we give the linearising transformations explicitly as well as some examples. The corresponding x-generalised hodograph transformations are listed in the Appendix. In the nonautonomous case each linearisable equation contains two arbitrary functions in $C^2(\Re)$; one function depending on the dependent variable and one depending on the independent "space"-variable x.

2 The x-Generalised hodograph transformation

Definition. The transformation

$$
{}_n\mathbf{H}^i_j : \begin{cases} dx_i(x_j,t_j) = f_1(x_j,u_j)dx_j + f_2(x_j,u_j,u_{jx_j},u_{jx_jx_j},\dots,u_{jx_j^{n-1}})dt_j \\ dt_i(x_j,t_j) = dt_j \\ u_i(x_i,t_i) = g(x_j), \end{cases} \tag{2.1}
$$

with $i \neq j$, $n = 2,3,\dots$ and

$$
u_{jt_j}\frac{\partial f_1}{\partial u_j} = \frac{\partial f_2}{\partial x_j} + u_{jx_j}\frac{\partial f_2}{\partial u_j} + u_{jx_jx_j}\frac{\partial f_2}{\partial u_{jx_j}} + \cdots + u_{jx_j^n}\frac{\partial f_2}{\partial u_{jx_j^{n-1}}}, \tag{2.2}
$$

is called an x-generalised hodograph transformation.

Remarks: We named the above transformation x-generalised in order to have the possibility in future to introduce other generalisations of the extended hodograph transformation. Condition (2.2) follows from the Lemma of Poincaré, i.e., $d(dx_i) \equiv 0$.

Here and below the subscripts denote partial derivatives, e.g.

$$u_{jx_jx_j} = \frac{\partial^2 u_j}{\partial {x_j}^2}.$$

Consider a general $(1+1)$-dimensional second-order autonomous evolution equation with dependent variable u_i and independent variables $x_i,\ t_i$, viz.

$$u_{it_i} = F(u_i, u_{ix_i}, u_{ix_ix_i}). \tag{2.3}$$

Applying the x-generalised hodograph transformation (2.1) leads to the following particular form for f_2:

$$f_2(x_j, u_j, u_{jx_j}) = -\frac{f_1(x_j,u_j)}{\dot g(x_j)}\left[F(u_i, u_{ix_i}, u_{ix_ix_i})\right]\Big|_{\Omega}\ , \tag{2.4}$$

where

$$\Omega = \Bigg\{ u_i = g(x_j),\ u_{ix_i} = \frac{\dot g(x_j)}{f_1(x_j,u_j)}\ ,$$
$$u_{ix_ix_i} = \frac{\ddot g(x_j)}{f_1^2(x_j,u_j)} - \frac{\dot g(x_j)}{f_1^3(x_j,u_j)}\left(\frac{\partial f_1}{\partial x_j} + \frac{\partial f_1}{\partial u_j}u_{jx_j}\right)\Bigg\}\ . \tag{2.5}$$

The most general equation which results when transforming (2.3) by the x-generalised hodograph transformation (2.1) with (2.4) is

$$\begin{aligned}\frac{\partial f_1}{\partial u_j}u_{jt_j} = &\Bigg[\frac{1}{f_1^2}\left(\frac{\partial f_1}{\partial u_j}u_{jx_jx_j} + \frac{\partial^2 f_1}{\partial u_j^2}u_{jx_j}^2 + 2\frac{\partial^2 f_1}{\partial x_j \partial u_j}u_{jx_j} + \frac{\partial^2 f_1}{\partial {x_j}^2}\right)\\ &-\frac{3}{f_1^3}\left(\frac{\partial f_1}{\partial x_j} + \frac{\partial f_1}{\partial u_j}u_{jx_j}\right)^2 + \frac{3\ddot g}{\dot g f_1^2}\left(\frac{\partial f_1}{\partial x_j} + \frac{\partial f_1}{\partial u_j}u_{jx_j}\right) - \frac{\dddot g}{\dot g f_1}\Bigg]\left[\frac{\partial F}{\partial u_{ix_ix_i}}\right]\Big|_{\Omega}\\ &+\left[\frac{1}{f_1}\left(\frac{\partial f_1}{\partial x_j} + \frac{\partial f_1}{\partial u_j}u_{jx_j}\right) - \frac{\ddot g}{\dot g}\right]\left[\frac{\partial F}{\partial u_{ix_i}}\right]\Big|_{\Omega} - f_1\left[\frac{\partial F}{\partial u_i}\right]\Big|_{\Omega}\\ &-\left[\frac{1}{\dot g}\left(\frac{\partial f_1}{\partial x_j} + \frac{\partial f_1}{\partial u_j}u_{jx_j}\right) - \frac{\ddot g f_1}{\dot g^2}\right]\left[F\right]\Big|_{\Omega}\ .\end{aligned} \tag{2.6}$$

Here and below $\dot g$ denotes the derivative w.r.t. x_j, $\ddot g$ the second derivative w.r.t. x_j, etc.

Using (2.5) it can easily be shown that the most general equation which may be constructed by applying (2.1) to the linear equation

$$u_{it_i} = u_{ix_ix_i} + \lambda_1 u_{ix_i} + \lambda_2 u_i, \qquad \lambda_1, \lambda_2 \in \Re \tag{2.7}$$

is of the form

$$u_{jt_j} = F_1(x_j, u_j)u_{x_jx_j} + F_2(x_j, u_j)u_{x_j} + F_3(x_j, u_j)u_{x_j}^2 + F_4(x_j, u_j) \tag{2.8}$$

for all iterations of the x-generalised hodograph transformation. The following statment is therefore true:

Proposition: *The most general* $(1+1)$*-dimensional second-order evolution equation which may be constructed to be linearisable in (2.7) by the* x*-generalised hodograph transformation (2.1) is necessarily of the form (2.8).*

Remark: In the sense of [1] an x-generalised hodograph transformation which keeps an equation invariant is known as an autohodograph transformation.

Finally we introduce an important notation which we use throughout this paper in order to abbreviate the derivatives of some arbitrary functions that appear in our tree of equations: Let $f = f(\xi) \in C^2(\Re)$ with $df/d\xi \neq 0$.

Then we define the following bracket:

$$\{f\}_\xi := -\frac{1}{2}\frac{df}{d\xi} + f\left(\frac{df}{d\xi}\right)^{-1}\frac{d^2 f}{d\xi^2}. \tag{2.9}$$

3 Linearisable autonomous second-order equations

Here we give the second-order linearisable autonomous evolution equations constructed by (2.1). We found eight cases, listed below, resulting in a tree of equations shown in Diagram 1. By nonlinearising (2.7) with (2.1) and restricting ourselves to autonomous equations, we obtain Cases I, II, III, V and VII. These equations follow when (2.1) is applied to each resulting autonomous equation until the iteration stops. That is, until no new autonomous equation appears. This happens at eq. (3.3), i.e., Case III. Applying the same procedure but starting from the second-order semilinear equation

$$u_{it_i} = u_{ix_ix_i} + G(u_i, u_{ix_i})$$

results in Cases IV, VI, and VIII. The corresponding linearising transformations, given below for each case, are obtained by composing and inverting the appropriate x-generalised hodograph transformations, given in the Appendix.

Case I: Let $\lambda_1 \in \Re$ and $h_1 \in C^2(\Re)$ with $dh_1/du_1 \neq 0$. Then

$$u_{1t_1} = h_1(u_1)u_{1x_1x_1} + \{h_1\}_{u_1} u_{1x_1}^2 \tag{3.1}$$

is linearised to $u_{0t_0} = u_{0x_0x_0} + \lambda_1 u_{0x_0}$ by the transformation

$${}_2\mathbf{L}_0^1 : \begin{cases} x_1(x_0, t_0) = u_0 \\ dt_1(x_0, t_0) = dt_0 \\ h_1(u_1(x_1, t_1)) = u_{0x_0}^2. \end{cases}$$

Case II: Let $\lambda \in \Re\backslash\{0\}$, $\lambda_1 \in \Re$ and $h_2 \in C^2(\Re)$ with $dh_2/du_2 \neq 0$. Then

$$u_{2t_2} = h_2(u_2)u_{2x_2x_2} + \lambda h_2(u_2)u_{2x_2} + \{h_2\}_{u_2} u_{2x_2}^2 \tag{3.2}$$

is linearised to $u_{0t_0} = u_{0x_0x_0} + \lambda_1 u_{0x_0}$ by the transformation

$${}_2\mathbf{L}_0^2 : \begin{cases} x_2(x_0, t_0) = \dfrac{1}{\lambda} \ln |u_{0x_0}| \\ dt_2(x_0, t_0) = dt_0 \\ h_2(u_2(x_2, t_2)) = \dfrac{1}{\lambda^2} \left(\dfrac{u_{0x_0x_0}}{u_{0x_0}}\right)^2. \end{cases}$$

Case III: Let $\lambda_1 \in \Re$, $\lambda_2 \in \Re\backslash\{0\}$ and $h_3 \in C^2(\Re)$ with $dh_3/du_3 \neq 0$. Then

$$u_{3t_3} = h_3(u_3)u_{3x_3x_3} + \{h_3\}_{u_3} u_{3x_3}^2 + 2\lambda_2 h_3^{3/2}(u_3) \left(\frac{dh_3}{du_3}\right)^{-1} \tag{3.3}$$

is linearised to $u_{0t_0} = u_{0x_0x_0} + \lambda_1 u_{0x_0}$ by the transformation

$${}_2\mathbf{L}_0^3 : \begin{cases} x_3(x_0, t_0) = \dfrac{2}{\lambda_2} \left(\dfrac{u_{0x_0x_0}}{u_{0x_0}}\right) \\ dt_3(x_0, t_0) = dt_0 \\ h_3(u_3(x_3, t_3)) = \dfrac{4}{\lambda_2^2} \left[\dfrac{\partial}{\partial x_0} \left(\dfrac{u_{0x_0x_0}}{u_{0x_0}}\right)\right]^2. \end{cases}$$

Case IV.1: Let $\{\lambda_1, \lambda_4\} \in \Re$, $\{\lambda, \lambda_2\} \in \Re\backslash\{0\}$ and $h_4 \in C^1(\Re)\backslash\{0\}$. Then

$$u_{4t_4} = u_{4x_4x_4} + \lambda_4 u_{4x_4} + \frac{1}{h_4(u_4)} \left(\lambda_2 - \frac{dh_4}{du_4}\right) u_{4x_4}^2 + h_4(u_4) \tag{3.4}$$

is linearised to $u_{0t_0} = u_{0x_0x_0} + \lambda_1 u_{0x_0} + \lambda_2 u_0$ by the transformation

$${}_2\mathbf{L}_0^{4.1} : \begin{cases} dx_4(x_0, t_0) = dx_0 + (\lambda_1 - \lambda_4)dt_0 \\ dt_4(x_0, t_0) = dt_0 \\ \displaystyle\int^{u_4(x_4,t_4)} \frac{d\xi}{h_4(\xi)} = \frac{1}{\lambda_2} \ln |\lambda u_0(x_0, t_0)|. \end{cases}$$

Case IV.2: Let $\lambda_1 \in \Re$, $\lambda_3 \in \Re\backslash\{0\}$, $\lambda_4 \in \Re$ and $h_4 \in C^1(\Re)\backslash\{0\}$. Then

$$u_{4t_4} = u_{4x_4x_4} + \lambda_4 u_{4x_4} - \frac{1}{h_4(u_4)} \frac{dh_4}{du_4} u_{4x_4}^2 + h_4(u_4) \tag{3.5}$$

is linearised to $u_{0t_0} = u_{0x_0x_0} + \lambda_1 u_{0x_0}$ by the transformation

$${}_2\mathbf{L}_0^{4.2} : \begin{cases} dx_4(x_0, t_0) = dx_0 + (\lambda_1 - \lambda_4)dt_0 \\ dt_4(x_0, t_0) = dt_0 \\ \dfrac{1}{h_4(u_4(x_4, t_4))} \dfrac{\partial u_4}{\partial x_4} = -\dfrac{u_0(x_0, t_0)}{\lambda_3}. \end{cases}$$

Case V: Let $\{\lambda, \lambda_2\} \in \Re\backslash\{0\}$, $\lambda_1 \in \Re$ and $h_5 \in C^2(\Re)$ with $dh_5/du_5 \neq 0$. Then

$$u_{5t_5} = h_5(u_5)u_{5x_5x_5} + \left(\lambda h_5(u_5) - \frac{\lambda_2}{\lambda}\right) u_{5x_5} + \{h_5\}_{u_5} u_{5x_5}^2 \tag{3.6}$$

is linearised to $u_{0t_0} = u_{0x_0x_0} + \lambda_1 u_{0x_0} + \lambda_2 u_0$ by the transformation

$$_2\mathbf{L}_0^5 : \begin{cases} x_5(x_0,t_0) = \dfrac{1}{\lambda}\ln|\lambda u_0| \\ dt_5(x_0,t_0) = dt_0 \\ h_5(u_5(x_5,t_5)) = \dfrac{1}{\lambda^2}\left(\dfrac{u_{0x_0}}{u_0}\right)^2 \end{cases}$$

Case VI: Let $\lambda_1 \in \Re$ and $h_6 \in C^2(\Re)$ with $dh_6/du_6 \neq 0$. Then

$$u_{6t_6} = u_{6x_6x_6} + h_6(u_6)u_{6x_6} + \frac{d^2h_6}{du_6^2}\left(\frac{dh_6}{du_6}\right)^{-1} u_{6x_6}^2 \tag{3.7}$$

is linearised to $u_{0t_0} = u_{0x_0x_0} + \lambda_1 u_{0x_0}$ by the transformation

$$_2\mathbf{L}_0^6 : \begin{cases} dx_6(x_0,t_0) = dx_0 + \lambda_1 dt_0 \\ dt_6(x_0,t_0) = dt_0 \\ h_6(u_6(x_6,t_6)) = 2\left(\dfrac{u_{0x_0x_0}}{u_{0x_0}}\right). \end{cases}$$

Case VII: Let $\lambda_1 \in \Re$, $\lambda_3 \in \Re\backslash\{0\}$ and $h_7 \in C^2(\Re)$ with $dh_7/du_7 \neq 0$. Then

$$u_{7t_7} = h_7(u_7)u_{7x_7x_7} + \lambda_3 u_{7x_7} + \{h_7\}_{u_7} u_{7x_7}^2 \tag{3.8}$$

is linearised to $u_{0t_0} = u_{0x_0x_0} + \lambda_1 u_{0x_0}$ by the transformation

$$_2\mathbf{L}_0^7 : \begin{cases} dx_7(x_0,t_0) = u_0 dx_0 + (u_{0x_0} + \lambda_1 u_0 - \lambda_3)\, dt_0 \\ dt_7(x_0,t_0) = dt_0 \\ h_7(u_7(x_7,t_7)) = u_0^2. \end{cases}$$

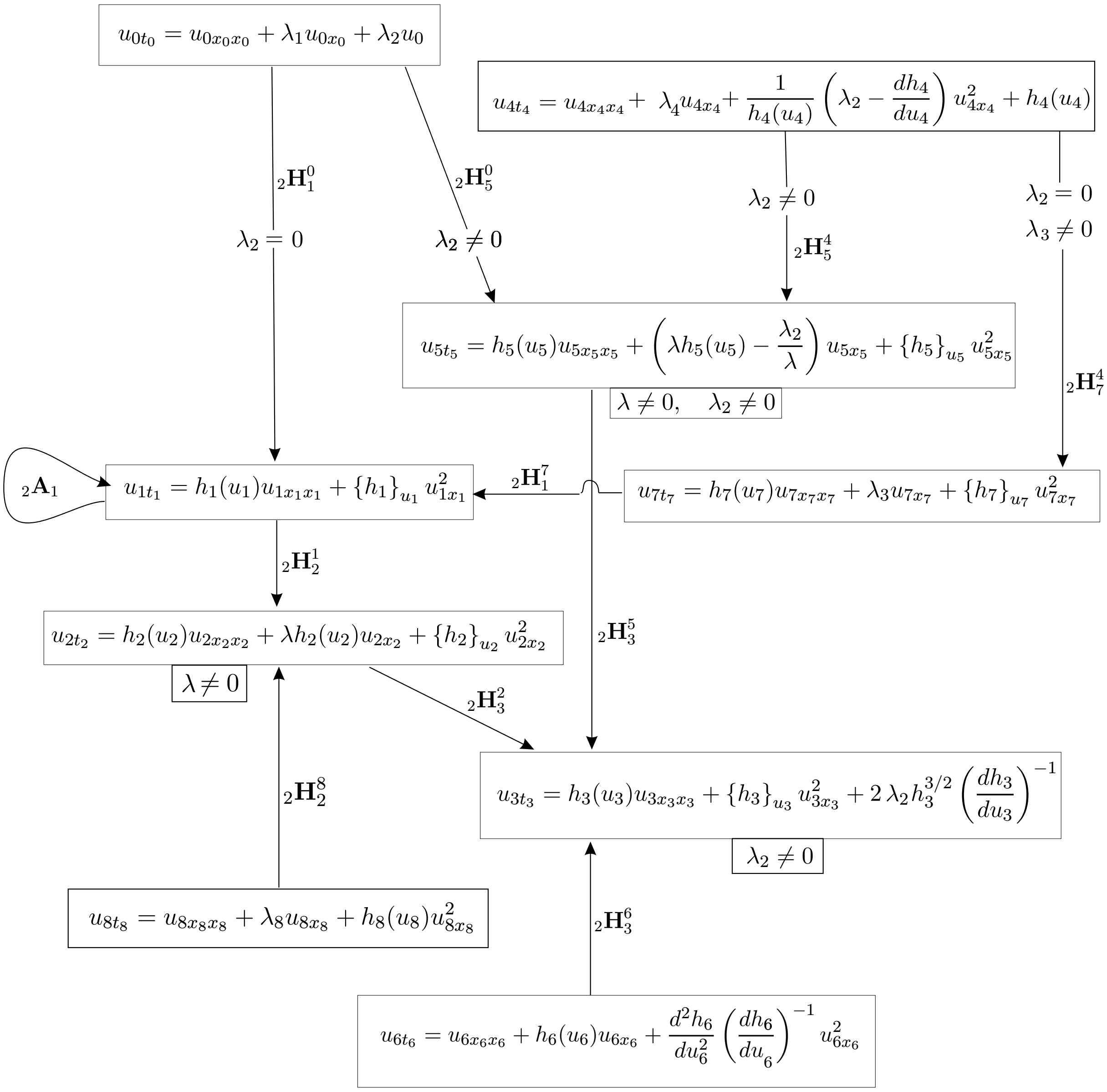
Case VIII: Let $\lambda \in \Re\backslash\{0\}$, $\{\lambda_1, \lambda_8\} \in \Re$ and $h_8 \in C^2(\Re)$. Then

$$u_{8t_8} = u_{8x_8x_8} + \lambda_8 u_{8x_8} + h_8(u_8)u_{8x_8}^2 \tag{3.9}$$

is linearised to $u_{t_0} = u_{0x_0x_0} + \lambda_1 u_{0x_0}$ by the transformation

$$_2\mathbf{L}_0^8 : \begin{cases} dx_8(x_0,t_0) = dx_0 + (\lambda_1 - \lambda_8)dt_0 \\ dt_8(x_0,t_0) = dt_0 \\ \displaystyle\int^{u_8(x_8,t_8)} \left\{\exp\left(\int^{\xi} h_8(\eta)d\eta\right)\left[\lambda\int^{\xi}\exp\left(\int^{\eta} h_8(\eta')d\eta'\right)d\eta\right]^{-1}\right\} d\xi \\ \qquad = \dfrac{1}{2\lambda}\ln\left(u_{0x_0}^2\right). \end{cases}$$

Diagram 1



$$\{h_j\}_{u_j} := -\frac{1}{2}\frac{dh_j}{du_j} + h_j(u_j)\left(\frac{dh_j}{du_j}\right)^{-1}\frac{d^2h_j}{du_j^2}$$

The autohodograph transformation ${}_2\mathbf{A}_1$ which transforms (3.1) into itself, i.e., in

$$\tilde{u}_{1\tilde{t}_1} = h_1(\tilde{u}_1)\tilde{u}_{1\tilde{x}_1\tilde{x}_1} + \{h_1\}_{\tilde{u}_1}\,\tilde{u}_{1\tilde{x}_1}^2,$$

is given by

$$ {}_2\mathbf{A}_1 : \begin{cases} dx_1(\tilde{x}_1,\tilde{t}_1) = (\alpha\tilde{x}_1+\beta)h_1^{-1/2}(\tilde{u}_1)d\tilde{x}_1 \\ \qquad\qquad + \left[\alpha h_1^{1/2}(\tilde{u}_1) - \dfrac{1}{2}(\alpha\tilde{x}_1+\beta)h_1^{-1/2}(\tilde{u}_1)\dfrac{dh_1}{d\tilde{u}_1}\tilde{u}_{1\tilde{x}_1}\right] d\tilde{t}_1 \\ dt_1(\tilde{x}_1,\tilde{t}_1) = d\tilde{t}_1 \\ h_1(u_1(x_1,t_1)) = (\alpha\tilde{x}_1+\beta)^2,\ \alpha\in\Re\backslash\{0\},\ \beta\in\Re. \end{cases}$$

It is noteworthy that (3.1) is the only equation in Diagram 1 that admits an autohodograph transformation.

We consider three examples for the above Cases.

Example 1: We consider Case III with $h_3 = u_3^3$ and $\lambda_1 = \lambda_2 = 1$, i.e.,

$$u_{3t_3} = u_3^3 u_{3x_3x_3} + \frac{1}{2}u_3^2 u_{3x_3}^2 + \frac{2}{3}u_3^{5/2}. \tag{3.10}$$

It follows that (3.10) is linearised to

$$u_{0t_0} = u_{0x_0x_0} + u_{0x_0} \tag{3.11}$$

by the transformation

$$\begin{aligned} x_3(x_0,t_0) &= \frac{2u_{0x_0x_0}}{u_{0x_0}} \\ t_3(x_0,t_0) &= t_0 \\ u_3(x_3,t_3) &= \left[\frac{\partial}{\partial x_0}\left(\frac{2u_{0x_0x_0}}{u_{0x_0}}\right)\right]^{2/3}. \end{aligned} \tag{3.12}$$

By group theoretical methods [9] we obtain the following solution for (3.11):

$$u_0(x_0,t_0) = t_0^{-1/2}\exp\left[-\frac{1}{4}\left(\frac{x_0}{t_0}+1\right)^2 t_0\right].$$

Using (3.12) we transform this solution into a solution for (3.10), namely

$$u_3(x_3,t_3) = (-1)^{2/3}\left[\frac{A+t_3)^2+2t_3}{A^2t_3}\right]^{2/3},$$

where

$$A = \frac{1}{2}\left(x_3{}^2t_3{}^2 + 8t_3\right)^{1/2} - \frac{1}{2}x_3t_3 - t_3.$$

Example 2: In Cases I, II, III we let $\lambda_1 = \lambda_2 = 0$, $\lambda = 1$,

$$h_j(u_j) = u_j^2, \qquad j = 1, 2, 3$$

and derive the transformation ${}_2\mathbf{H}_1^6$, which transforms (3.7) into

$$u_{1t_1} = u_1^2 u_{1x_1x_1}. \tag{3.13}$$

With the above assumptions, (3.2) and (3.3) take the following respective forms:

$$u_{2t_2} = u_2^2 u_{2x_2x_2} + u_2^2 u_{2x_2}, \tag{3.14}$$

$$u_{3t_3} = u_3^2 u_{3x_3x_3} + u_3^2.$$

The transformation into the autohodograph invariant equation (3.13) is obtained by the composition

$${}_2\mathbf{H}_1^6 = \left({}_2\mathbf{H}_3^1\right)^{-1} \circ {}_2\mathbf{H}_3^6,$$

where $\left({}_2\mathbf{H}_3^1\right)^{-1}$ denotes the inverse transformation of ${}_2\mathbf{H}_3^1$. In particular ${}_2\mathbf{H}_3^1 = {}_2\mathbf{H}_3^2 \circ {}_2\mathbf{H}_2^1$. Then

$${}_2\mathbf{H}_3^1 : \begin{cases} u_1^{-1}(x_1, t_1) dx_1(x_3, t_3) = u_3^{-1} dx_3 - \left(u_{3x_3} + \dfrac{1}{2} x_3\right) dt_3 \\ dt_1(x_3, t_3) = dt_3 \\ u_1(x_1, t_1) u_{1x_1x_1}(x_1, t_1) = \dfrac{1}{2} u_3, \end{cases}$$

so that

$$({}_2\mathbf{H}_3^1)^{-1} : \begin{cases} x_3(x_1, t_1) = 2u_{1x_1} \\ dt_3(x_1, t_1) = dt_1 \\ u_3(x_3, t_3) = 2u_1 u_{1x_1x_1} \end{cases}$$

and

$${}_2\mathbf{H}_1^6 : \begin{cases} dx_6(x_1, t_1) = u_1^{-1} dx_1 - u_{1x_1} dt_1 \\ dt_6(x_1, t_1) = dt_1 \\ h_6(u_6(x_6, t_6)) = 2u_{1x_1}. \end{cases}$$

The autohodograph transformation ${}_2\mathbf{A}_1$ transforming (3.13) into

$$\tilde{u}_{1\tilde{t}_1} = \tilde{u}_1^2 \tilde{u}_{1\tilde{x}_1\tilde{x}_1}$$

(with $\alpha = 1$, $\beta = 0$) follows, viz.

$${}_2\mathbf{A}_1 : \begin{cases} dx_1(\tilde{x}_1, \tilde{t}_1) = \tilde{x}_1 \tilde{u}_1^{-1} d\tilde{x}_1 + \left(\tilde{u}_1 - \tilde{x}_1 \tilde{u}_{1\tilde{x}_1}\right) d\tilde{t}_1 \\ dt_1(\tilde{x}_1, \tilde{t}_1) = d\tilde{t}_1 \\ u_1(x_1, t_1) = \tilde{x}_1. \end{cases}$$

The inverse of ${}_2\mathbf{A}_1$ is

$$({}_2\mathbf{A}_1)^{-1}: \begin{cases} \tilde{x}_1 = u_1 \\ d\tilde{t}_1 = dt_1 \\ \tilde{u}_1(\tilde{x}_1, \tilde{t}_1) = u_1 u_{1x_1}. \end{cases}$$

The linearising transformation ${}_2\mathbf{H}_0^6$ of (3.7) into

$$u_{0t_0} = u_{0x_0x_0}$$

is obtained by the composition

$${}_2\mathbf{H}_0^6 = \left({}_2\mathbf{H}_1^0\right)^{-1} \circ \left({}_2\mathbf{H}_3^1\right)^{-1} \circ {}_2\mathbf{H}_3^6 \ ,$$

where

$$\left({}_2\mathbf{H}_1^0\right)^{-1}: \begin{cases} x_1(x_0, t_0) = u_0 \\ dt_1(x_0, t_0) = dt_0 \\ u_1(x_1, t_1) = u_{0x_0} \end{cases}$$

and

$${}_2\mathbf{H}_3^6: \begin{cases} dx_6(x_3, t_3) = u_3^{-1} dx_3 - (u_{3x_3} + x_3) dt_3 \\ dt_6(x_3, t_3) = dt_3 \\ h_6(u_6(x_6, t_6)) = x_3. \end{cases}$$

Note that, with $h_6(u_6) = u_6$ (3.7) reduces to the Burgers' equation and ${}_2\mathbf{H}_0^6$ becomes the Cole-Hopf transformation, i.e.,

$${}_2\mathbf{H}_0^6: \begin{cases} x_6(x_0, t_0) = x_0 \\ t_6(x_0, t_0) = t_0 \\ u_6(x_6, t_6) = 2\phi^{-1}(x_0, t_0) \dfrac{\partial \phi}{\partial x_0}(x_0, t_0), \end{cases}$$

where $\phi(x_0, t_0) = \partial u_0 / \partial x_0$.

Example 3: Consider Cases III, V and let $\lambda = \lambda_1 = \lambda_2 = 1$

$$h_j(u_j) = u_j^2, \qquad j = 3, 5.$$

We derive the transformation ${}_2\mathbf{H}_0^6$ which linearises (3.7), i.e.,

$$u_{6t_6} = u_{6x_6x_6} + h_6(u_6) u_{6x_6} + \frac{d^2 h_6}{du_6^2} \left(\frac{dh_6}{du_6} \right)^{-1} u_{6x_6}^2$$

into

$$u_{0t_0} = u_{0x_0x_0} + u_{0x_0} + u_0.$$

This linearisation is obtained by the following compositions:

$$_2\mathbf{H}_0^6 = ({_2\mathbf{H}_3^0})^{-1} \circ {_2\mathbf{H}_3^6} \quad \text{with} \quad {_2\mathbf{H}_3^0} = {_2\mathbf{H}_3^5} \circ {_2\mathbf{H}_5^0}.$$

Under the above assumptions (3.6) takes the form

$$u_{5t_5} = u_5^2 u_{5x_5x_5} + \left(u_5^2 - 1\right) u_{5x_5} \tag{3.15}$$

and (3.3) is

$$u_{3t_3} = u_3^2 u_{3x_3x_3} + u_3^2. \tag{3.16}$$

The linearising transformation for (3.7) is then

$$_2\mathbf{H}_0^6 : \begin{cases} dx_6(x_0, t_0) = dx_0 + dt_0 \\ dt_6(x_0, t_0) = dt_0 \\ h_6(u_6(x_6, t_6)) = 2u_0^{-1} u_{0x_0}, \end{cases}$$

whereas (3.15) is linearised by

$$({_2\mathbf{H}_5^0})^{-1} : \begin{cases} x_5(x_0, t_0) = \ln |u_0| \\ dt_5(x_0, t_0) = dt_0 \\ u_5(x_5, t_5) = u_0^{-1} u_{0x_0} \end{cases}$$

and (3.16) by

$$({_2\mathbf{H}_3^0})^{-1} : \begin{cases} x_3(x_0, t_0) = 2u_0^{-1} u_{0x_0} \\ dt_3(x_0, t_0) = dt_0 \\ u_3(x_3, t_3) = 2u_0^{-1} u_{0x_0x_0} - 2u_0^{-2} u_{0x_0}^2. \end{cases}$$

Remark: The linearisation of (3.16) is also given in [4].

4 Linearisable nonautonomous second-order equations

Next we list the nonautonomous second-order evolution equations which have been constructed using (2.1), as well as their corresponding linearising transformations. Each equation given in Cases I – VIII above results, by (2.1), in a nonautonomous equation leading to eight further cases. Diagram 2 shows the connection to the autonomous cases.

Case IX: Let $\lambda_1 \in \Re$, $\{h_1, k_1\} \in C^2(\Re)$ with $dh_1/d\tilde{x}_1 \neq 0$ and $dk_1/d\tilde{u}_1 \neq 0$. Then

$$\begin{aligned} \tilde{u}_{1\tilde{t}_1} &= k_1(\tilde{u}_1)\tilde{u}_{1\tilde{x}_1\tilde{x}_1} + \frac{k_1(\tilde{u}_1)\,\{h_1\}_{\tilde{x}_1}}{h_1(\tilde{x}_1)} \tilde{u}_{1\tilde{x}_1} + \{k_1\}_{\tilde{u}_1} \tilde{u}_{1\tilde{x}_1}^2 \\ &\quad + 2k_1^2(\tilde{u}_1) \left(\frac{dk_1}{d\tilde{u}_1}\right)^{-1} \frac{d}{d\tilde{x}_1} \left(\frac{\{h_1\}_{\tilde{x}_1}}{h_1(\tilde{x}_1)}\right) \end{aligned} \tag{4.1}$$

is linearised to $u_{0t_0} = u_{0x_0x_0} + \lambda_1 u_{0x_0}$ by the transformation

$$
{}_2\mathbf{L}_0^{\tilde{1}} : \begin{cases} h_1(\tilde{x}_1(x_0,t_0)) = u_{0x_0}^2 \\ d\tilde{t}_1(x_0,t_0) = dt_0 \\ k_1(\tilde{u}_1(\tilde{x}_1,\tilde{t}_1)) = 4u_{0x_0}^2 u_{0x_0x_0}^2 \left[\left(\dfrac{dh_1}{d\tilde{x}_1} \right)^{-2} \right] \Bigg|_{h_1(\tilde{x}_1)=u_{0x_0}^2} . \end{cases}
$$

Case X: Let $\lambda \in \Re\backslash\{0\}$, $\lambda_1 \in \Re$, $\{h_2, k_2\} \in C^2(\Re)$ with $dh_2/d\tilde{x}_2 \neq 0$ and $dk_2/d\tilde{u}_2 \neq 0$. Then

$$
\begin{aligned}
\tilde{u}_{2\tilde{t}_2} &= k_2(\tilde{u}_2)\tilde{u}_{2\tilde{x}_2\tilde{x}_2} + \{k_2\}_{\tilde{u}_2} \tilde{u}_{2\tilde{x}_2}^2 + 2k_2^2(\tilde{u}_2) \left(\frac{dk_2}{d\tilde{u}_2} \right)^{-1} \frac{d}{d\tilde{x}_2} \left(\frac{\{h_2\}_{\tilde{x}_2}}{h_2(\tilde{x}_2)} \right) \\
&\quad + \frac{k_2(\tilde{u}_2)\{h_2\}_{\tilde{x}_2}}{h_2(\tilde{x}_2)} \tilde{u}_{2\tilde{x}_2} + 2\lambda h_2^{-1/2}(\tilde{x}_2) \frac{dh_2}{d\tilde{x}_2} \left(\frac{dk_2}{d\tilde{u}_2} \right)^{-1} k_2^{3/2}(\tilde{u}_2) \qquad (4.2)
\end{aligned}
$$

is linearised to $u_{0t_0} = u_{0x_0x_0} + \lambda_1 u_{0x_0}$ by the transformation

$$
{}_2\mathbf{L}_0^{\tilde{2}} : \begin{cases} h_2(\tilde{x}_2(x_0,t_0)) = \dfrac{1}{\lambda^2} \left(\dfrac{u_{0x_0x_0}}{u_{0x_0}} \right)^2 \\ d\tilde{t}_2(x_0,t_0) = dt_0 \\ k_2(\tilde{u}_2(\tilde{x}_2,\tilde{t}_2)) = \dfrac{4}{\lambda^4} \left(\dfrac{u_{0x_0x_0}}{u_{0x_0}} \right)^2 \left[\dfrac{\partial}{\partial x_0} \left(\dfrac{u_{0x_0x_0}}{u_{0x_0}} \right) \right]^2 \times \\ \qquad\qquad \times \left[\left(\dfrac{dh_2}{d\tilde{x}_2} \right)^{-2} \right] \Bigg|_{h_2(\tilde{x}_2)=\frac{1}{\lambda^2}\left(\frac{u_{0x_0x_0}}{u_{0x_0}}\right)^2} . \end{cases}
$$

Case XI: Let $\lambda_1 \in \Re$, $\lambda_2 \in \Re\backslash\{0\}$, $\{h_3, k_3\} \in C^2(\Re)$ with $dh_3/d\tilde{x}_3 \neq 0$ and $dk_3/d\tilde{u}_3 \neq 0$. Then

$$
\begin{aligned}
\tilde{u}_{3\tilde{t}_3} &= k_3(\tilde{u}_3)\tilde{u}_{3\tilde{x}_3\tilde{x}_3} + \left(\frac{k_3(\tilde{u}_3)\{h_3\}_{\tilde{x}_3}}{h_3(\tilde{x}_3)} - 2\lambda_2 h_3^{3/2}(\tilde{x}_3) \left(\frac{dh_3}{d\tilde{x}_3} \right)^{-1} \right) \tilde{u}_{3\tilde{x}_3} \\
&\quad + \{k_3\}_{\tilde{u}_3} \tilde{u}_{3\tilde{x}_3}^2 + 2\lambda_2 k_3(\tilde{u}_3) \left(\frac{dk_3}{d\tilde{u}_3} \right)^{-1} \left(4h_3^{1/2}(\tilde{x}_3) - 2h_3^{3/2}(\tilde{x}_3) \left(\frac{dh_3}{d\tilde{x}_3} \right)^{-2} \frac{d^2 h_3}{d\tilde{x}_3^2} \right) \\
&\quad + 2k_3^2(\tilde{u}_3) \left(\frac{dk_3}{d\tilde{u}_3} \right)^{-1} \frac{d}{d\tilde{x}_3} \left(\frac{\{h_3\}_{\tilde{x}_3}}{h_3(\tilde{x}_3)} \right) \qquad (4.3)
\end{aligned}
$$

is linearised to $u_{0t_0} = u_{0x_0x_0} + \lambda_1 u_{0x_0}$ by the transformation

$$
{}_2\mathbf{L}_0^{\tilde{3}}: \begin{cases} h_3(\tilde{x}_3(x_0,t_0)) = \dfrac{4}{\lambda_2^2}\left[\dfrac{\partial}{\partial x_0}\left(\dfrac{u_{0x_0x_0}}{u_{0x_0}}\right)\right]^2 \\ d\tilde{t}_3(x_0,t_0) = dt_0 \\ k_3(\tilde{u}_3(\tilde{x}_3,\tilde{t}_3)) = \dfrac{64}{\lambda_2^4}\left[\dfrac{\partial}{\partial x_0}\left(\dfrac{u_{0x_0x_0}}{u_{0x_0}}\right)\right]^2\left[\dfrac{\partial^2}{\partial {x_0}^2}\left(\dfrac{u_{0x_0x_0}}{u_{0x_0}}\right)\right]^2 \times \\ \qquad \times \left[\left(\dfrac{dh_3}{d\tilde{x}_3}\right)^{-2}\right]\Bigg|_{h_3(\tilde{x}_3)=\frac{4}{\lambda_2^2}\left[\frac{\partial}{\partial x_0}\left(\frac{u_{0x_0x_0}}{u_{0x_0}}\right)\right]^2}. \end{cases}
$$

Case XII.1: Let $\lambda_1 \in \Re$, $\lambda_2 \in \Re\backslash\{0\}$, $\{h_4, k_4\} \in C^2(\Re)$ with $dh_4/d\tilde{x}_4 \neq 0$ and $dk_4/d\tilde{u}_4 \neq 0$. Then

$$
\begin{aligned}
\tilde{u}_{4\tilde{t}_4} &= k_4(\tilde{u}_4)\tilde{u}_{4\tilde{x}_4\tilde{x}_4} + \left[\left(\frac{\lambda_2}{h_4(\tilde{x}_4)} - \frac{1}{h_4(\tilde{x}_4)}\frac{dh_4}{d\tilde{x}_4}\right)k_4(\tilde{u}_4) - h_4(\tilde{x}_4)\right]\tilde{u}_{4\tilde{x}_4} \\
&+ \{k_4\}_{\tilde{u}_4}\tilde{u}_{4\tilde{x}_4}^2 + 2\left[k_4^2(\tilde{u}_4)\left(-\frac{\lambda_2}{h_4^2(\tilde{x}_4)}\frac{dh_4}{d\tilde{x}_4} - \frac{1}{h_4(\tilde{x}_4)}\frac{d^2h_4}{d\tilde{x}_4^2} + \frac{1}{h_4^2(\tilde{x}_4)}\left(\frac{dh_4}{d\tilde{x}_4}\right)^2\right)\right. \\
&\left. + k_4(\tilde{u}_4)\frac{dh_4}{d\tilde{x}_4}\right]\left(\frac{dk_4}{d\tilde{u}_4}\right)^{-1}
\end{aligned} \tag{4.4}
$$

is linearised to $u_{0t_0} = u_{0x_0x_0} + \lambda_1 u_{0x_0} + \lambda_2 u_0$ by the transformation

$$
{}_2\mathbf{L}_0^{\tilde{4}.1}: \begin{cases} \displaystyle\int^{\tilde{x}_4(x_0,t_0)} \frac{d\xi}{h_4(\xi)} = \frac{1}{\lambda_2}\ln|u_0| \\ d\tilde{t}_4(x_0,t_0) = dt_0 \\ k_4(\tilde{u}_4(\tilde{x}_4,\tilde{t}_4)) = \dfrac{1}{\lambda_2^2}\left(\dfrac{u_{0x_0}}{u_0}\right)^2\left[h_4^2(\tilde{x}_4)\right]\Bigg|_{\int^{\tilde{x}_4}\frac{d\xi}{h_4(\xi)}=\frac{1}{\lambda_2}\ln|u_0|}. \end{cases}
$$

Case XII.2: Let $\lambda_1 \in \Re$, $\lambda_3 \in \Re\backslash\{0\}$, $\{h_4, k_4\} \in C^2(\Re)$ with $dh_4/d\tilde{x}_4 \neq 0$ and $dk_4/d\tilde{u}_4 \neq 0$. Then

$$
\begin{aligned}
\tilde{u}_{4\tilde{t}_4} &= k_4(\tilde{u}_4)\tilde{u}_{4\tilde{x}_4\tilde{x}_4} - \left(\frac{k_4(\tilde{u}_4)}{h_4(\tilde{x}_4)}\frac{dh_4}{d\tilde{x}_4} + h_4(\tilde{x}_4)\right)\tilde{u}_{4\tilde{x}_4} + \{k_4\}_{\tilde{u}_4}\tilde{u}_{4\tilde{x}_4}^2 \\
&+2\left[k_4^2(\tilde{u}_4)\left(-\frac{1}{h_4(\tilde{x}_4)}\frac{d^2h_4}{d\tilde{x}_4^2} + \frac{1}{h_4^2(\tilde{x}_4)}\left(\frac{dh_4}{d\tilde{x}_4}\right)^2\right) + k_4(\tilde{u}_4)\frac{dh_4}{d\tilde{x}_4}\right]\left(\frac{dk_4}{d\tilde{u}_4}\right)^{-1}
\end{aligned} \tag{4.5}
$$

is linearised to $u_{0t_0} = u_{0x_0x_0} + \lambda_1 u_{0x_0}$ by the transformation

$$
{}_2\mathbf{L}_0^{\tilde{4}.2}: \begin{cases} h_4^{-1}(\tilde{x}_4)d\tilde{x}_4(x_0,t_0) = -\dfrac{1}{\lambda_3}u_0 dx_0 - \dfrac{1}{\lambda_3}\left(u_{0x_0} + \lambda_1 u_0 - \lambda_3\right)dt_0 - - - (*) \\[2ex] d\tilde{t}_4(x_0,t_0) = dt_0 \\[2ex] k_4(\tilde{u}_4(\tilde{x}_4,\tilde{t}_4)) = \dfrac{u_0^2}{\lambda_3^2}\left[h_4^2(\tilde{x}_4)\right]\Big|_{(*)} \end{cases} .
$$

Remark: By $\Big|_{(*)}$ we mean that the function $h_4(\tilde{x}_4)$ has to be written in terms of x_0, t_0, u_0 and its derivatives with respect to x_0, obtained from the expression $(*)$.

Case XIII: Let $\lambda_1 \in \Re$, $\{\lambda, \lambda_2\} \in \Re\backslash\{0\}$, $\{h_5, k_5\} \in C^2(\Re)$ with $dh_5/d\tilde{x}_5 \neq 0$ and $dk_5/d\tilde{u}_5 \neq 0$. Then

$$
\begin{aligned}
\tilde{u}_{5\tilde{t}_5} &= k_5(\tilde{u}_5)\tilde{u}_{5\tilde{x}_5\tilde{x}_5} + \frac{k_5(\tilde{u}_5)\{h_5\}_{\tilde{x}_5}}{h_5(\tilde{x}_5)}\tilde{u}_{5\tilde{x}_5} + \{k_5\}_{u_5}\tilde{u}_{5\tilde{x}_5}^2 \\
&\quad + 2k_5^2(\tilde{u}_5)\left(\frac{dk_5}{d\tilde{u}_5}\right)^{-1}\frac{d}{d\tilde{x}_5}\left(\frac{\{h_5\}_{\tilde{x}_5}}{h_5(\tilde{x}_5)}\right) + 2\lambda h_5^{-1/2}(\tilde{x}_5)\frac{dh_5}{d\tilde{x}_5}k_5^{3/2}(\tilde{u}_5)\left(\frac{dk_5}{d\tilde{u}_5}\right)^{-1}
\end{aligned} \tag{4.6}
$$

is linearised to $u_{0t_0} = u_{0x_0x_0} + \lambda_1 u_{0x_0} + \lambda_2 u_0$ by the transformation

$$
{}_2\mathbf{L}_0^{\tilde{5}}: \begin{cases} h_5(\tilde{x}_5(x_0,t_0)) = \dfrac{1}{\lambda^2}\left(\dfrac{u_{0x_0}}{u_0}\right)^2 \\[2ex] d\tilde{t}_5(x_0,t_0) = dt_0 \\[2ex] k_5(\tilde{u}_5(\tilde{x}_5,\tilde{t}_5)) = \dfrac{1}{\lambda^2}\left[\dfrac{\partial}{\partial x_0}\left(\dfrac{u_{0x_0}}{u_0}\right)^2\right]^2\left[\left(\dfrac{dh_5}{d\tilde{x}_5}\right)^{-2}\right]\Bigg|_{h_5(\tilde{x}_5)=\frac{1}{\lambda^2}\left(\frac{u_{0x_0}}{u_0}\right)^2} \end{cases} .
$$

Case XIV: Let $\lambda_1 \in \Re$, $\lambda_2 \in \Re\backslash\{0\}$, and $\{h_6, k_6\} \in C^2(\Re)$ with $dh_6/d\tilde{x}_6 \neq 0$ and $dk_6/d\tilde{u}_6 \neq 0$. Then

$$
\begin{aligned}
\tilde{u}_{6\tilde{t}_6} &= k_6(\tilde{u}_6)\tilde{u}_{6\tilde{x}_6\tilde{x}_6} + k_6(\tilde{u}_6)\frac{d^2h_6}{d\tilde{x}_6^2}\left(\frac{dh_6}{d\tilde{x}_6}\right)^{-1}\tilde{u}_{6\tilde{x}_6} + \{h_6\}_{\tilde{u}_6}\tilde{u}_{\tilde{x}_6}^2 \\
&\quad + 2k_6^2(\tilde{u}_6)\left(\frac{dk_6}{d\tilde{u}_6}\right)^{-1}\frac{d}{d\tilde{x}_6}\left[\frac{d^2h_6}{d\tilde{x}_6^2}\left(\frac{dh_6}{d\tilde{x}_6}\right)^{-1}\right] + 2k_6^{3/2}(\tilde{u}_6)\left(\frac{dk_6}{d\tilde{u}_6}\right)^{-1}\frac{dh_6}{d\tilde{x}_6}
\end{aligned} \tag{4.7}
$$

is linearised to $u_{0t_0} = u_{0x_0x_0} + \lambda_1 u_{0x_0}$ by the transformation

$$
{}_2\mathbf{L}_0^{\tilde{6}}: \begin{cases} h_6(\tilde{x}_6(x_0,t_0)) = 2\dfrac{u_{0x_0x_0}}{u_{0x_0}} \\[2ex] d\tilde{t}_6(x_0,t_0) = dt_0 \\[2ex] k_6(\tilde{u}_6(\tilde{x}_6,\tilde{t}_6)) = 4\left[\dfrac{\partial}{\partial x_0}\left(\dfrac{u_{0x_0x_0}}{u_{0x_0}}\right)\right]^2\left[\left(\dfrac{dh_6}{d\tilde{x}_6}\right)^{-2}\right]\Bigg|_{h_6(\tilde{x}_6)=2\frac{u_{0x_0x_0}}{u_{0x_0}}} \end{cases} .
$$

Case XV: Let $\{\lambda_1, \lambda_3\} \in \Re$, $\{h_7, k_7\} \in C^2(\Re)$ with $dh_7/d\tilde{x}_7 \neq 0$ and $dk_7/d\tilde{u}_7 \neq 0$. Then

$$\tilde{u}_{7\tilde{t}_7} = k_7(\tilde{u}_7)\tilde{u}_{7\tilde{x}_7\tilde{x}_7} + \frac{k_7(\tilde{u}_7)\{h_7\}_{\tilde{x}_7}}{h_7(\tilde{x}_7)}\tilde{u}_{7\tilde{x}_7} + \{k_7\}_{\tilde{u}_7}\tilde{u}^2_{7\tilde{x}_7}$$
$$+2k_7^2(\tilde{u}_7)\left(\frac{dk_7}{d\tilde{u}_7}\right)^{-1}\frac{d}{d\tilde{x}_7}\left(\frac{\{h_7\}_{\tilde{x}_7}}{h_7(\tilde{x}_7)}\right) \tag{4.8}$$

is linearised to $u_{0t_0} = u_{0x_0x_0} + \lambda_1 u_{0x_0}$ by the transformation

$${}_2\mathbf{L}_0^{\tilde{7}}: \begin{cases} h_7(\tilde{x}_7(x_0,t_0)) = u_0^2 \\ d\tilde{t}_7(x_0,t_0) = dt_0 \\ k_7(\tilde{u}_7(\tilde{x}_7,\tilde{t}_7)) = 4u_0^2u_{0x_0}^2\left[\left(\dfrac{dh_7}{d\tilde{x}_7}\right)^{-2}\right]\Bigg|_{h_7(\tilde{x}_7)=u_0^2}. \end{cases}$$

Case XVI: Let $\lambda \in \Re\backslash\{0\}$, $\lambda_1 \in \Re$, $\{h_8, k_8\} \in C^2(\Re)$ with $dk_8/d\tilde{u}_8 \neq 0$. Then

$$\tilde{u}_{8\tilde{t}_8} = k_8(\tilde{u}_8)\tilde{u}_{8\tilde{x}_8\tilde{x}_8} + k_8(\tilde{u}_8)h_8(\tilde{x}_8)\tilde{u}_{8\tilde{x}_8} + \{k_8\}_{\tilde{u}_8}\tilde{u}^2_{8\tilde{x}_8}$$
$$+2k_8^2(\tilde{u}_8)\frac{dh_8}{d\tilde{x}_8}\left(\frac{dk_8}{d\tilde{u}_8}\right)^{-1} \tag{4.9}$$

is linearised to $u_{0t_0} = u_{0x_0x_0} + \lambda_1 u_{0x_0}$ by the transformation

$${}_2\mathbf{L}_0^{\tilde{8}}: \begin{cases} \displaystyle\int^{\tilde{x}_8}\left\{\exp\left(\int^{\xi} h_8(\eta)d\eta\right)\left[\lambda\int^{\xi}\exp\left(\int^{\eta} h_8(\eta')d\eta'\right)d\eta\right]^{-1}\right\}d\xi \\ \qquad = \dfrac{1}{2\lambda}\ln\left|u_{0x_0}^2\right| - - - (*) \\ d\tilde{t}_8(x_0,t_0) = dt_0 \\ k_8(\tilde{u}_8(\tilde{x}_8,\tilde{t}_8)) = \dfrac{u^2_{0x_0x_0}}{u^2_{0x_0}}\left[\dfrac{\int^{\tilde{x}_8}\left[\exp\left(\int^{\xi} h_8(\eta)d\eta\right)\right]d\xi}{\exp\left(\int^{\tilde{x}_8} h_8(\xi)d\xi\right)}\right]^2\Bigg|_{(*)}. \end{cases}$$

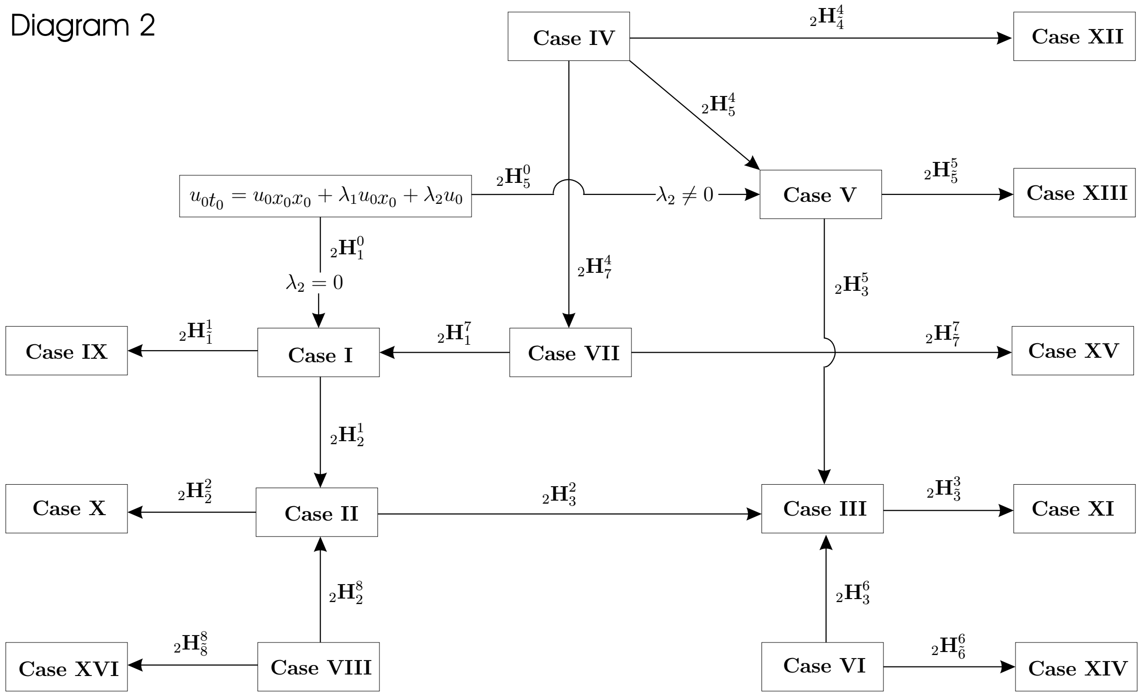
Example 4: We consider Case XI with $h_3(\tilde{x}_3) = \tilde{x}_3^2$, $k_3(\tilde{u}_3) = \tilde{u}_3^2$ and $\lambda_2 = -1$, i.e.,

$$\tilde{u}_{3\tilde{t}_3} = \tilde{u}_3^2\tilde{u}_{3\tilde{x}_3\tilde{x}_3} + \tilde{x}_3^2\tilde{u}_{3\tilde{x}_3} - 3\tilde{x}_3\tilde{u}_3. \tag{4.10}$$

It follows that (4.10) is linearised to

$$u_{0t_0} = u_{0x_0x_0} + \lambda_1 u_{0x_0}$$

Diagram 2



by the transformation

$$\tilde{x}_3(x_0,t_0) = \frac{\partial}{\partial x_0}\left(\frac{-2u_{0x_0x_0}}{u_{0x_0}}\right)$$

$$d\tilde{t}_3(x_0,t_0) = dt_0$$

$$\tilde{u}_3(\tilde{x}_3,\tilde{t}_3) = \frac{\partial^2}{\partial {x_0}^2}\left(\frac{-2u_{0x_0x_0}}{u_{0x_0}}\right).$$

Remark: Equation (4.10) was proposed in [4].

Example 5: We consider Case XVI with $h_8(\tilde{x}_8) = \tilde{x}_8^{-1}$ and $k_8(\tilde{u}_8) = \tilde{u}_8$, i.e.,

$$\tilde{u}_{8\tilde{x}_8} = \tilde{u}_8\tilde{u}_{8\tilde{x}_8\tilde{x}_8} + \tilde{x}_8^{-1}\tilde{u}_8\tilde{u}_{8\tilde{x}_8} - \frac{1}{2}\tilde{u}_{8\tilde{x}_8}^2 - 2\tilde{x}_8^{-2}\tilde{u}_8^2 \tag{4.11}$$

It follows that (4.11) is linearised to

$$u_{0t_0} = u_{0x_0x_0} + \lambda_1 u_{0x_0}$$

by the transformation

$$\tilde{x}_8(x_0,t_0) = u_{0x_0}^{1/2}$$

$$d\tilde{t}_8 = dt_0$$

$$\tilde{u}_8(\tilde{x}_8,\tilde{t}_8) = \frac{1}{4}\frac{u_{0x_0x_0}^2}{u_{0x_0}}.$$

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Appendix

Below we list the x-generalised hodograph transformations ${}_2\mathbf{H}^i_j$ corresponding to Cases I – VIII. Diagram 1 shows the direction in which the transformations act.

$$
{}_2\mathbf{H}^0_1 : \begin{cases} dx_0(x_1,t_1) = h_1^{-1/2}(u_1)dx_1 + \left(-\dfrac{1}{2}h_1^{-1/2}(u_1)\dfrac{dh_1}{du_1}u_{1x_1} - \lambda_1\right)dt_1 \\ dt_0(x_1,t_1) = dt_1 \\ u_0(x_0,t_0) = \alpha_1 x_1 + \beta_1, \qquad \alpha_1 \in \Re\backslash\{0\},\ \beta_1 \in \Re \end{cases}
$$

$$
{}_2\mathbf{H}^0_5 : \begin{cases} dx_0(x_5,t_5) = h_5^{-1/2}(u_5)dx_5 + \Big[-\lambda h_5^{1/2}(u_5) \\ \qquad -\dfrac{\lambda_2}{\lambda}h_5^{-1/2}(u_5) - \dfrac{1}{2}h_5^{-1/2}(u_5)\dfrac{dh_5}{du_5}u_{5x_5} - \lambda_1\Big]dt_5 \\ dt_0(x_5,t_5) = dt_5 \\ u_0(x_0,t_0) = \dfrac{1}{\lambda}e^{\lambda x_5 + \alpha_5}, \qquad \lambda \in \Re\backslash\{0\},\ \alpha_5 \in \Re \end{cases}
$$

$$
{}_2\mathbf{H}^1_2 : \begin{cases} dx_1(x_2,t_2) = e^{\lambda x_2}\sqrt{\beta_2}h_2^{-1/2}(u_2)dx_2 - \dfrac{1}{2}e^{\lambda x_2}\sqrt{\beta_2}h_2^{-1/2}(u_2)\dfrac{dh_2}{du_2}u_{2x_2}dt_2 \\ dt_1(x_2,t_2) = dt_2 \\ h_1\left(u_1(x_1,t_1)\right) = \beta_2 e^{2\lambda x_2}, \qquad \lambda \in \Re\backslash\{0\},\ \beta_2 \in \Re\backslash\{0\} \end{cases}
$$

$$
{}_2\mathbf{H}^2_3 : \begin{cases} dx_2(x_3,t_3) = \left(\dfrac{\lambda_2}{2\lambda}x_3 + \beta_3\right)h_3^{-1/2}(u_3)dx_3 \\ \qquad + \Big[\dfrac{\lambda_2}{2\lambda}h_3^{1/2}(u_3) - \dfrac{1}{2}(\dfrac{\lambda_2}{2\lambda}x_3 + \beta_3)h_3^{-1/2}(u_3)\dfrac{dh_3}{du_3}u_{3x_3} \\ \qquad -\lambda(\dfrac{\lambda_2}{2\lambda}x_3 + \beta_3)^2\Big]dt_3 \\ dt_2(x_3,t_3) = dt_3 \\ h_2\left(u_2(x_2,t_2)\right) = \left(\dfrac{\lambda_2}{2\lambda}x_3 + \beta_3\right)^2, \qquad \{\lambda,\lambda_2\} \in \Re\backslash\{0\},\ \beta_3 \in \Re \end{cases}
$$

$$
{}_2\mathbf{H}_5^4 : \begin{cases}
dx_4(x_5,t_5) = h_5^{-1/2}(u_5)dx_5 + \Bigg[-\lambda h_5^{1/2}(u_5) \\
\qquad -\dfrac{1}{2}h_5^{-1/2}(u_5)\dfrac{dh_5}{du_5}u_{5x_5} - \dfrac{\lambda_2}{\lambda}h_5^{-1/2}(u_5) - \lambda_4\Bigg] dt_5 \\
dt_4(x_5,t_5) = dt_5 \\
\displaystyle\int^{u_4(x_4,t_4)} \frac{d\xi}{h_4(\xi)} = \frac{\lambda}{\lambda_2}\, x_5 + \beta_5, \quad \{\lambda,\lambda_2\} \in \Re\backslash\{0\},\ \{\beta_5,\lambda_4\} \in \Re
\end{cases}
$$

$$
{}_2\mathbf{H}_7^4 : \begin{cases}
dx_4(x_7,t_7) = h_7^{-1/2}(u_7)dx_7 + \left[-\dfrac{1}{2}h_7^{-1/2}(u_7)\dfrac{dh_7}{du_7}u_{7x_7} + \lambda_3 h_7^{-1/2}(u_7) - \lambda_4\right] dt_7 \\
dt_4(x_7,t_7) = dt_7 \\
\displaystyle\int^{u_4(x_4,t_4)} \frac{d\xi}{h_4(\xi)} = -\frac{1}{\lambda_3}\, x_7 + \beta_7, \qquad \lambda_3 \in \Re\backslash\{0\},\ \{\lambda_4,\beta_7\} \in \Re
\end{cases}
$$

$$
{}_2\mathbf{H}_3^5 : \begin{cases}
dx_5(x_3,t_3) = \left(\dfrac{\lambda_2}{2\lambda}x_3 + \tilde{\beta}_3\right) h_3^{-1/2}(u_3)dx_3 + \Bigg[\dfrac{\lambda_2}{2\lambda}h_3^{1/2}(u_3) \\
\qquad -\dfrac{1}{2}\left(\dfrac{\lambda_2}{2\lambda}x_3 + \tilde{\beta}_3\right) h_3^{-1/2}(u_3)\dfrac{dh_3}{du_3}u_{3x_3} - \lambda\left(\dfrac{\lambda_2}{2\lambda}x_3 + \tilde{\beta}_3\right)^2 + \dfrac{\lambda_2}{\lambda}\Bigg] dt_3 \\
dt_5(x_3,t_3) = dt_3 \\
h_5\left(u_5(x_5,t_5)\right) = \left(\dfrac{\lambda_2}{2\lambda}\, x_3 + \tilde{\beta}_3\right)^2, \qquad \{\lambda,\lambda_2\} \in \Re\backslash\{0\},\ \tilde{\beta}_3 \in \Re
\end{cases}
$$

$$
{}_2\mathbf{H}_1^7 : \begin{cases}
dx_7(x_1,t_1) = \left(\alpha x_1 + \tilde{\beta}_1\right) h_1^{-1/2}(u_1)dx_1 \\
\qquad + \left[\alpha h_1^{1/2}(u_1) - \dfrac{1}{2}\left(\alpha x_1 + \tilde{\beta}_1\right) h_1^{-1/2}(u_1)\dfrac{dh_1}{du_1}u_{1x_1} - \lambda_3\right] dt_1 \\
dt_7(x_1,t_1) = dt_1 \\
h_7\left(u_7(x_7,t_7)\right) = \left(\alpha x_1 + \tilde{\beta}_1\right)^2, \qquad \alpha \in \Re\backslash\{0\},\ \{\lambda_3,\tilde{\beta}_1\} \in \Re
\end{cases}
$$

$$
{}_2\mathbf{H}_3^6 : \begin{cases}
dx_6(x_3,t_3) = h_3^{-1/2}(u_3)dx_3 + \left[-\dfrac{1}{2}h_3^{-1/2}(u_3)\dfrac{dh_3}{du_3}u_{3x_3} - \lambda_2 x_3\right] dt_3 \\
dt_6(x_3,t_3) = dt_3 \\
h_6\left(u_6(x_6,t_6)\right) = \lambda_2 x_3, \qquad \lambda_2 \in \Re\backslash\{0\}.
\end{cases}
$$

$$
{}_2\mathbf{H}_2^8 : \begin{cases} dx_8(x_2,t_2) = h_2^{-1/2}(u_2)dx_2 - \left[\lambda h_2^{1/2}(u_2) + \dfrac{1}{2}h_2^{-1/2}(u_2)\dfrac{dh_2}{du_2}u_{2x_2} + \lambda_8\right] dt_2 \\ dt_8(x_2,t_2) = dt_2 \\ \displaystyle\int^{u_8(x_8,t_8)} \left\{ \exp\left(\int^{\xi} h_8(\eta)d\eta\right) \left[\lambda \int^{\xi} \exp\left(\int^{\eta} h_8(\eta')d\eta'\right) d\eta\right]^{-1} \right\} d\xi = x_2 \end{cases}
$$

Below we list the x-generalised hodograph transformations ${}_2\mathbf{H}_{\tilde{n}}^n$ by which autonomous equation n is transformed in the nonautonomous equation $\tilde{n}$. This refers to Cases IX – XVI. Diagram 2 shows the direction in which the transformations act.

$$
{}_2\mathbf{H}_{\tilde{1}}^1 : \begin{cases} dx_1(\tilde{x}_1,\tilde{t}_1) = h_1^{1/2}(\tilde{x}_1)k_1^{-1/2}(\tilde{u}_1)d\tilde{x}_1 - \left[\dfrac{k_1^{1/2}(\tilde{u}_1)\{h_1\}_{\tilde{x}_1}}{h_1^{1/2}(\tilde{x}_1)}\right. \\ \qquad\qquad \left. - \dfrac{1}{2}h_1^{-1/2}(\tilde{x}_1)\dfrac{dh_1}{d\tilde{x}_1}k_1^{1/2}(\tilde{u}_1) + \dfrac{1}{2}h_1^{1/2}(\tilde{x}_1)k_1^{-1/2}(\tilde{u}_1)\dfrac{dk_1}{d\tilde{u}_1}\tilde{u}_{1\tilde{x}_1}\right] d\tilde{t}_1 \\ dt_1(\tilde{x}_1,\tilde{t}_1) = d\tilde{t}_1 \\ u_1(x_1,t_1) = \tilde{x}_1 \end{cases}
$$

$$
{}_2\mathbf{H}_{\tilde{2}}^2 : \begin{cases} dx_2(\tilde{x}_2,\tilde{t}_2) = h_2^{1/2}(\tilde{x}_2)k_2^{-1/2}(\tilde{u}_2)d\tilde{x}_2 - \left[\dfrac{k_2^{1/2}(\tilde{u}_2)\{h_2\}_{\tilde{x}_2}}{h_2^{1/2}(\tilde{x}_2)} + \lambda h_2(\tilde{x}_2)\right. \\ \qquad\qquad \left. - \dfrac{1}{2}h_2^{-1/2}(\tilde{x}_2)\dfrac{dh_2}{d\tilde{x}_2}k_2^{1/2}(\tilde{u}_2) + \dfrac{1}{2}h_2^{1/2}(\tilde{x}_2)k_2^{-1/2}(\tilde{u}_2)\dfrac{dk_2}{d\tilde{u}_2}\tilde{u}_{2\tilde{x}_2}\right] d\tilde{t}_2 \\ dt_2(\tilde{x}_2,\tilde{t}_2) = d\tilde{t}_2 \\ u_2(x_2,t_2) = \tilde{x}_2 \qquad \lambda \in \Re\backslash\{0\} \end{cases}
$$

$$
{}_2\mathbf{H}_{\tilde{3}}^3 : \begin{cases} dx_3(\tilde{x}_3,\tilde{t}_3) = h_3^{1/2}(\tilde{x}_3)k_3^{-1/2}(\tilde{u}_3)d\tilde{x}_3 \\ \qquad + \left[\left(h_3^{-1/2}(\tilde{x}_3)\dfrac{dh_3}{d\tilde{x}_3} - h_3^{1/2}(\tilde{x}_3)\left(\dfrac{dh_3}{d\tilde{x}_3}\right)^{-1}\dfrac{d^2h_3}{d\tilde{x}_3^2}\right)k_3^{1/2}(\tilde{u}_3)\right. \\ \qquad -\dfrac{1}{2}h_3^{1/2}(\tilde{x}_3)k_3^{-1/2}(\tilde{u}_3)\dfrac{dk_3}{d\tilde{u}_3}\tilde{u}_{3\tilde{x}_3} \\ \qquad \left. -2\lambda_2 h_3^2(\tilde{x}_3)\left(\dfrac{dh_3}{d\tilde{x}_3}\right)^{-1}k_3^{-1/2}(\tilde{u}_3)\right] d\tilde{t}_3 \\ dt_3(\tilde{x}_3,\tilde{t}_3) = d\tilde{t}_3 \\ u_3(x_3,t_3) = \tilde{x}_3 \qquad \lambda_2 \in \Re\backslash\{0\} \end{cases}
$$

$$
{}_2\mathbf{H}^4_{\tilde{4}}:\begin{cases}
dx_4(\tilde{x}_4,\tilde{t}_4) = k_4^{-1/2}(\tilde{u}_4)d\tilde{x}_4 - \left[\dfrac{k_4^{1/2}(\tilde{u}_4)}{h_4(\tilde{x}_4)}\left(\lambda_2 - \dfrac{dh_4}{d\tilde{x}_4}\right)\right.\\
\qquad\qquad \left. + \dfrac{1}{2}k_4^{-1/2}(\tilde{u}_4)\dfrac{dk_4}{d\tilde{u}_4}\tilde{u}_{4\tilde{x}_4} + \lambda_4 + k_4^{-1/2}(\tilde{u}_4)h_4(\tilde{x}_4)\right] d\tilde{t}_4\\
dt_4(\tilde{x}_4,\tilde{t}_4) = d\tilde{t}_4\\
u_4(x_4,t_4) = \tilde{x}_4, \qquad \{\lambda_2,\lambda_4\}\in\Re
\end{cases}
$$

$$
{}_2\mathbf{H}^5_{\tilde{5}}:\begin{cases}
dx_5(\tilde{x}_5,\tilde{t}_5) = h_5^{1/2}(\tilde{x}_5)k_5^{-1/2}(\tilde{u}_5)d\tilde{x}_5 - \left[\dfrac{\{h_5\}_{\tilde{x}_5}k_5^{1/2}(\tilde{u}_5)}{h_5^{1/2}(\tilde{x}_5)}\right.\\
\qquad\qquad - \dfrac{1}{2}h_5^{-1/2}(\tilde{x}_5)\dfrac{dh_5}{d\tilde{x}_5}k_5^{1/2}(\tilde{u}_5) + \dfrac{1}{2}h_5^{1/2}(\tilde{x}_5)k_5^{-1/2}(\tilde{u}_5)\dfrac{dk_5}{d\tilde{u}_5}\tilde{u}_{5\tilde{x}_5}\\
\qquad\qquad \left. + \lambda h_5(\tilde{x}_5) - \dfrac{\lambda_2}{\lambda}\right] d\tilde{t}_5\\
dt_5(\tilde{x}_5,\tilde{t}_5) = d\tilde{t}_5\\
u_5(x_5,t_5) = \tilde{x}_5, \qquad \lambda\in\Re\backslash\{0\},\ \lambda_2\in\Re
\end{cases}
$$

$$
{}_2\mathbf{H}^6_{\tilde{6}}:\begin{cases}
dx_6(\tilde{x}_6,\tilde{t}_6) = k_6^{-1/2}(\tilde{u}_6)d\tilde{x}_6 - \left[k_6^{1/2}(\tilde{u}_6)\left(\dfrac{dh_6}{d\tilde{x}_6}\right)^{-1}\dfrac{d^2h_6}{d\tilde{x}_6^2}\right.\\
\qquad\qquad \left. + \dfrac{1}{2}k_6^{-1/2}(\tilde{u}_6)\dfrac{dk_6}{d\tilde{u}_6}\tilde{u}_{6\tilde{x}_6} + h_6(\tilde{x}_6)\right] d\tilde{t}_6\\
dt_6(\tilde{x}_6,\tilde{t}_6) = d\tilde{t}_6\\
u_6(x_6,t_6) = \tilde{x}_6
\end{cases}
$$

$$
{}_2\mathbf{H}^7_{\tilde{7}}:\begin{cases}
dx_7(\tilde{x}_7,\tilde{t}_7) = h_7^{1/2}(\tilde{x}_7)k_7^{-1/2}(\tilde{u}_7)d\tilde{x}_7 - \left[\dfrac{k_7^{1/2}(\tilde{u}_7)\{h_7\}_{\tilde{x}_7}}{h_7^{1/2}(\tilde{x}_7)} + \lambda_3\right.\\
\qquad\qquad \left. - \dfrac{1}{2}h_7^{-1/2}(\tilde{x}_7)\dfrac{dh_7}{d\tilde{x}_7}k_7^{1/2}(\tilde{u}_7) + \dfrac{1}{2}h_7^{1/2}(\tilde{x}_7)k_7^{-1/2}(\tilde{u}_7)\dfrac{dk_7}{d\tilde{u}_7}\tilde{u}_{7\tilde{x}_7}\right] d\tilde{t}_7\\
dt_7(\tilde{x}_7,\tilde{t}_7) = d\tilde{t}_7\\
u_7(x_7,t_7) = \tilde{x}_7, \qquad \lambda_3\in\Re
\end{cases}
$$

$$
{}_2\mathbf{H}^8_{\tilde{8}}: \begin{cases} dx_8(\tilde{x}_8, \tilde{t}_8) = k_8^{-1/2}(\tilde{u}_8) d\tilde{x}_8 \\ \qquad\qquad - \left[h_8(\tilde{x}_8) k_8^{1/2}(\tilde{u}_8) + \dfrac{1}{2} k_8^{-1/2}(\tilde{u}_8) \dfrac{dk_8}{d\tilde{u}_8} \tilde{u}_{8\tilde{x}_8} + \lambda_8 \right] d\tilde{t}_8 \\ dt_8(\tilde{x}_8, \tilde{t}_8) = d\tilde{t}_8 \\ u_8(x_8, t_8) = \tilde{x}_8 \qquad \lambda_8 \in \Re \end{cases}
$$

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