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Asymptotic Approximation of Hyperbolic Weakly Nonlinear Systems

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Abstract

An averaging method for getting uniformly valid asymptotic approximations of the solution of hyperbolic systems of equations is presented. The averaged system of equations disintegrates into independent equations for non-resonance systems. We consider the resonance conditions for some classes of solutions. The averaged system can be solved numerically in the resonance case. The shallow water problem is considered as an example of the resonance system. Results of numerical experiments are presented.

1 Introduction

In this paper we consider a system of weakly nonlinear equations with a small positive parameter ε:

$$U_t + A(U)U_x = \varepsilon B(t, x, \varepsilon t, \varepsilon x, U, U_x, U_{xx}, U_{xxx}), \tag{1.1}$$

where $U = (u_1, u_2, \dots, u_n)^T$ is a column vector, $A = \|a_{ij}\|$ is an $n \times n$ matrix, and $B = (b_1, b_2, \dots, b_n)^T$ is a column vector. We assume that all coefficients are sufficiently smooth functions.

Many physical problems are described by such systems. We mention only some examples: dispersive waves in plasma, problems of weakly nonlinear optics, one-dimensional gas dynamic equations, shallow water waves. The construction of uniform asymptotic approximations of the solution of (1.1) becomes a nontrivial task if secular terms arise in the expansion. We can illustrate this situation by considering the following linear problem

$$\begin{aligned} &u_t + u_x = \varepsilon u, \\ &u(0, x) = \sin x. \end{aligned} \tag{1.2}$$

We find easily the exact solution of problem (1.2)

$$u(t, x, \varepsilon) = e^{\varepsilon t} \sin(x - t).$$

Using a Taylor series expansion of the function $e^{\varepsilon t}$ we get that

$$u(t, x, \varepsilon) = sin(x - t) + \varepsilon t \sin(x - t) + \frac{\varepsilon^2 t^2}{2} \sin(x - t) + \cdots . \tag{1.3}$$

This expansion has secular terms εt, $\varepsilon^2 t^2$, ..., and therefore the formula (1.3) is asymptotical only if $\varepsilon t \ll 1$. In this case we have that

$$u(t, x, \varepsilon) = \sin(x - t) + O(\varepsilon t).$$

For $\varepsilon t = O(1)$ this expansion is not asymptotical. On the other hand the problem (1.2) has a classical solution in any domain

$$0 < t + |x| < c_0/\varepsilon,$$

here c_0 is a constant. It is easy to prove that

$$\lim_{\varepsilon \to 0} u(t, x, \varepsilon) \neq \sin(x - t).$$

Therefore it is a nontrivial task to construct an asymptotical approximation, which is uniformly valid in the region $t + |x| = O\left(\varepsilon^{-1}\right)$.

The basic idea of all asymptotic methods is to introduce new "slow" variables, e.g., $\tau = \varepsilon t$, $\xi = \varepsilon x$, and to define explicitly the dependence on "fast" variables. For example in [21] the solution is obtained in the following form:

$$u_j = \psi_j(\zeta, \eta_j) + O(\varepsilon), \qquad \zeta = \varepsilon^{1+a} t, \qquad \eta_j = \varepsilon^a \left(x - \lambda_j t + \varepsilon^{1-a} \varphi(t, x)\right).$$

Substituting these expressions into system (1.1), using a Taylor expansion with respect to ε, and equating coefficients at ε^j we get equations for new unknown functions ψ_j

$$\psi_{j\zeta} + (\alpha_j \psi_j + \beta_j)\psi_{j\eta_j} + \gamma_j \psi_{j\eta_j\eta_j\eta_j} + \delta_j \psi_j = 0.$$

Our goal is to approximate the initial nonlinear problem by some fundamental equations, for which the analytical solution exists. The Burgers' equation and the Korteweg-de Vries equations are examples of such problems. Most asymptotical methods are based on physical assumptions and no strict mathematical proofs of the validity of these approximations are given [2, 11]. Important applications of asymptotic methods are given in [6, 8, 19]. A survey of mathematical results on asymptotic expansion methods is presented by Kalyakin in [12], see also Keworkian and Cole [13]. Similar problems arise not only for systems of equations, but also for semilinear perturbed wave equations, the telegraph equation and weakly nonlinear beam equation [3, 7, 18].

Usually application of formal asymptotic method reduces the initial problem to a single nonlinear wave equation [21]. Our method reduces problem (1.1) to the integro-differential system of averaged equations. A new problem can be seen as a more difficult problem, than the initial formulation. However, we will show that the averaged system disintegrates into independent equations in the non-resonance case. Moreover, our method allows to describe accurately the resonance interaction of waves, as well. Our internal averaging method is close to the multiple-scale method, which is used for similar problems in [7]. The application of internal averaging method for solving gas dynamic equations is presented in [5].

The rest of the paper is organized as follows. In Section 2, we formulate the general averaging scheme. We propose the averaging operators and compare our scheme with the other similar methods. In Section 3, we consider the resonance case, when the averaged system is connected. Section 4 deals with the application of the proposed algorithm for a shallow water model. It is proved that the initial system is ill-posed, hence the regularized model is formulated. The averaged system also gives a nontrivial regularization of the shallow water model. Finally we give the results of numerical experiments.

2 Method of averaging

Let U_0 be a constant solution of equation (1.1):

$$B(t,x,\varepsilon t,\varepsilon x,U_0,0,0,0)=0. \tag{2.1}$$

We assume that problem (1.1) is hyperbolic in the neighborhood of U_0, i.e. there exists an $n\times n$ matrix $R=\|r_{ij}\|$ $\det R\neq 0$ such that

$$\Lambda\equiv \operatorname{diag}\{\lambda_1,\lambda_2,\dots,\lambda_n\}=RA(U_0)R^{-1}. \tag{2.2}$$

We are interested in finding a small-amplitude wave solution

$$U(t,x,\varepsilon)=U_0+\varepsilon U_1(t,x,\varepsilon). \tag{2.3}$$

Linearizing equations (1.1) with respect to U_0 yields the system of equations

$$V_t+\Lambda V_z=\varepsilon F(t,x,\varepsilon t,\varepsilon x,V,V_x,V_{xx},V_{xxx})+o(\varepsilon), \tag{2.4}$$

where $V=RU_1$, and the function F is given by

$$\begin{aligned}
&F=-RA_1\left[R^{-1}V\right]R^{-1}V_x+R\left(B_0\left[R^{-1}V\right]+B_1\left[R^{-1}V_x\right]\right.\\
&\qquad\left.+B_2\left[R^{-1}V_{xx}\right]+B_3\left[R^{-1}V_{xxx}\right]\right),\\
&A_1[U_1]\equiv\frac{dA(U_0)}{dU}U_1=\left\|\sum_{k=1}^{n}\left[\frac{\partial}{\partial u_k}a_{ij}(U_0)\right]u_{1k}\right\|,\\
&B_m[U_1]=(b_{m1},b_{m2},\dots,b_{mn})^T,\qquad m=0,1,2,3,\\
&b_{0j}=\sum_{k=1}^{n}\frac{\partial b_j(t,x,\varepsilon t,\varepsilon x,U_0,0,0,0)}{\partial u_{0k}}u_{1k},\qquad b_{1j}=\sum_{k=1}^{n}\frac{\partial b_j}{\partial u_{0kx}}u_{1kx},\\
&b_{2j}=\sum_{k=1}^{n}\frac{\partial b_j}{\partial u_{0kxx}}u_{1kxx},\qquad b_{3j}=\sum_{k=1}^{n}\frac{\partial b_j}{\partial u_{0kxxx}}u_{1kxxx}.
\end{aligned}$$

If $\varepsilon=0$, then system (2.4) with the initial condition

$$V(0,x,\varepsilon)=V_0(x) \tag{2.5}$$

describes n independent linear waves $v_j=v_{0j}(x-\lambda_j t)$.

If $t+|x|\sim\varepsilon^{-1}$, then the exact solution of initial-value problem (2.4), (2.5) (and therefore also of problem (1.1)) is not close to the simple wave. For example, the nonlinear equation

$$v_t+v_x=\varepsilon vv_x$$

describes a nonlinear wave, which is given by the implicit relation

$$v(t,x,\varepsilon)=v_0(x-t+\varepsilon tv(t,x)).$$

Obviously it can not be approximated by a simple wave $v=v_0(x-t)$ for $\varepsilon t=O(1)$.

We first rewrite system (2.4), (2.5) in a coordinate form

$$\frac{\partial v_j}{\partial t} + \lambda_j \frac{\partial v_j}{\partial x} = \varepsilon f_j \left(t, x, \varepsilon t, \varepsilon x, V, \frac{\partial V}{\partial x}, \frac{\partial^2 V}{\partial x^2}, \frac{\partial^3 V}{\partial x^3} \right),$$
$$v_j(0, x, \varepsilon) = v_{0j}(\varepsilon x, x), \qquad j = 1, 2, \dots, n. \tag{2.6}$$

Let $\tau = \varepsilon t$, $\xi = \varepsilon x$ be "slow" variables, and $y_j = x - \lambda_j t$, $j = 1, 2, \dots, n$ be "fast" characteristic variables. The operator of averaging along the j-th characteristic of the non-perturbed (i.e., $\varepsilon = 0$) system (2.6) is given by

$$M_j[g(t, x, \tau, \xi, v_1, v_2, \dots, v_n)] \equiv \lim_{T \to \infty} \frac{1}{T} \int_0^T g\Big(s, y_j + \lambda_j s, \tau, \xi,$$
$$v_1(y_j + (\lambda_j - \lambda_1)s), \dots, v_n(y_j + (\lambda_j - \lambda_n)s)\Big) ds. \tag{2.7}$$

We define the following averaged system of equations

$$\frac{\partial w_j}{\partial \tau} + \lambda_j \frac{\partial w_j}{\partial \xi} = M_j \left[f_j \left(t, x, \tau, \xi, W, \frac{\partial W}{\partial y_k}, \frac{\partial^2 W}{\partial y_k^2}, \frac{\partial^3 W}{\partial y_k^3} \right) \right],$$
$$w_j(0, \xi, y_j) = v_{0j}(\xi, y_j), \qquad j = 1, 2, \dots, n. \tag{2.8}$$

If the operator $B \equiv 0$, then averaged system (2.8) takes the form

$$\frac{\partial w_j}{\partial \tau} + \lambda_j \frac{\partial w_j}{\partial \xi} = \sum_{k=1}^{n} \sum_{m=1}^{n} f_{jkm} M_j \left[w_k \frac{\partial w_m}{\partial y_m} \right]. \tag{2.9}$$

Without a loss of generality we can assume, that

$$\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} v_{0j}(\xi, x) dx = 0. \tag{2.10}$$

It is easy to prove that the following properties of the averaging operators M_j are valid:

$$M_j \left[v_j \frac{\partial v_j}{\partial y_j} \right] \equiv v_j \frac{\partial v_j}{\partial y_j}, \qquad M_j \left[v_i \frac{\partial v_j}{\partial y_j} \right] \equiv M_j[v_i] \frac{\partial v_j}{\partial y_j},$$
$$M_j \left[v_j \frac{\partial v_i}{\partial y_i} \right] \equiv 0, \qquad i \neq j. \tag{2.11}$$

For $n = 2$ using (2.9)–(2.11) we get that the averaged system of equations reduces to two independent problems:

$$\frac{\partial w_j}{\partial \tau} + \lambda_j \frac{\partial w_j}{\partial \xi} = f_{jjj} w_j \frac{\partial w_j}{\partial y_j},$$
$$w_j(0, \xi, y_j) = v_{0j}(\xi, y_j). \tag{2.12}$$

If the operator B is a linear function of U_{xx} or U_{xxx}, then averaged equations (2.12) are described by the Burgers' or the Korteweg-de Vries equations.

In most cases equations of averaged system (2.8) are connected and they describe the interaction of waves. Asymptotic analysis is finished at this stage, but some numerical analysis is still needed in order to get the solution.

3 Approximation accuracy analysis

Let $\mathcal{M}$ be a class of functions, for which averages (2.7) exist uniformly for all variables. Then the solution of averaged system (2.6) exists and we can analyze the approximation properties of this solution.

The case of periodical initial conditions $u_{0j}(x) \in C^1_{2\pi}(\mathbb{R})$ was considered in [14, 20]. For sufficiently smooth functions $f_j(u_1, \dots, u_n, u_{1x}, \dots, u_{nx})$ it was proved that if $(v_1, v_2, \dots, v_n)$ is the solution of system (2.6) and $(w_1, w_2, \dots, w_n)$ is the solution of averaged system (2.8), then there exists a constant $c_0 > 0$ such that

$$\lim_{\varepsilon\to 0} \max_j \sup_{0\le t+|x|\le \frac{c_0}{\varepsilon}} |u_j(t,x,\varepsilon) - v_j(\varepsilon t, \varepsilon x, x-\lambda_j t)| = 0. \tag{3.1}$$

This result is analogous to the First Bogoliubov's theorem for ordinary differential equations [4].

The essence of the proposed averaging method is the following: the average $M_j[g(\tau, \xi, t, x, v)]$ along the j-th characteristic $y_j = x - \lambda_j t$ is the limit of the integral

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T g(\tau,\xi,s,y_j+\lambda_j s, v(\tau,\xi,s,y_j+\lambda_j s))ds.$$

We integrate function v, which is still unknown, that is why our method can be called the *internal* averaging. For comparison the classical averaging along characteristics [16] is defined by:

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T g(\tau,\xi,s,y_j+\lambda_j s, v(\tau,\xi,t,x))ds$$

and it can be called the *external* averaging.

We illustrate the difference between these two techniques, by considering the model system with the internal resonance

$$\begin{aligned} &u_t + u_x = \varepsilon v \sin x, \qquad u(0,x,\varepsilon) = 0,\\ &v_t = 0, \qquad v(0,x) = \sin x. \end{aligned}$$

The *external* averaging gives the problem

$$U_t + U_x = 0, \qquad U(0,x) = 0$$

and $U \equiv 0$ does not approximate the exact solution

$$u(t,x,\varepsilon) = \frac{\varepsilon}{4}(2t + \sin 2(x-t) - \sin 2x)$$

if $t \sim \varepsilon^{-1}$. The *internal* averaging of this system gives the problem

$$\begin{aligned} &U_\tau = \lim_{T\to\infty}\frac{1}{T}\int_0^T V(\tau, y+s)\sin(y+s)ds, \qquad U(0,y) = 0,\\ &V_\tau = 0, \qquad V(0,x) = \sin x, \qquad y = x - t. \end{aligned}$$

After simple computations we get

$$V(\tau,x)=\sin x,$$
$$U_\tau=\lim_{T\to\infty}\frac{1}{T}\int_0^T \sin^2(y+s)ds=\frac{1}{2}.$$

Hence $U=\frac{\varepsilon t}{2}$ and the equality $u=U+o(1)$ is satisfied uniformly for $t\in\left[0,O\left(\varepsilon^{-1}\right)\right]$.

Using Fourier series we can write the resonance conditions for system (2.6). Let $f_j(t,x,\tau,\xi,v_1,\dots v_n)$ be periodic functions with respect to t:

$$f_j\left(t+\Lambda_j^t,x+\Lambda_j^x,\tau,\xi,v_1(\tau,\xi,y_1+\Lambda_1),\dots,v_n(\tau,\xi,y_n+\Lambda_n)\right)$$
$$=f_j(t,x,\tau,\xi,v_1(\tau,\xi,y_1),\dots,v_n(\tau,\xi,y_n)).$$

Integrating these functions along characteristics we get that

$$f_j(t,x,\tau,\xi,v_1(\tau,\xi,y_1),\dots,v_n(\tau,\xi,y_n))$$
$$=\sum_{(l^t,l^x,l_1,\dots,l_n)\neq 0} f_{jl}(\tau,\xi)\,e^{2\pi i\left(\frac{l^t t}{\Lambda_j^t}+\frac{l^x x}{\Lambda_j^x}+\frac{l_1y_1}{\Lambda_1}+\dots+\frac{l_ny_n}{\Lambda_n}\right)}.$$

Then using the substitution

$$t=x,\quad x=y_j+\lambda_j s,\quad y_i=y_j+(\lambda_j-\lambda_i)s$$

we obtain that the condition

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T f_j(s,y_j+\lambda_j s,\tau,\xi,\dots y_j+(\lambda_j-\lambda_i)s\dots)ds=0 \tag{3.2}$$

is satisfied if and only if

$$\frac{l^t}{\Lambda_j^t}+\frac{\lambda_j l^x}{\Lambda_j^x}+\sum_{k\neq j}\frac{\lambda_j-\lambda_k}{\Lambda_k}l_k\neq 0,$$
$$\forall\, l^t,l^x,l_k\in\mathbb{Z}\quad \&\quad |l^t|+|l^x|+\sum_{k\neq j}|l_k|\neq 0. \tag{3.3}$$

The condition (3.2) specifies the absence of the resonance in system (2.6). If it is satisfied, then averaged system (2.8) disintegrates into independent equations, which are similar to (2.12).

It was proved in [15], that the class of functions $\mathcal{M}$, for which the *internal* averaging can be applied, includes almost periodic functions. For example, let assume that

$$f_j=\rho_j(\tau,\xi,t,x)u_1u_2\cdots u_n,\qquad \rho_j=\sum_{l=(l^t,l^x)\neq 0}\rho_j(\tau,\xi)e^{i(\nu_{jl^t}^t t+\nu_{jl^x}^x x)},$$
$$u_{0j}\sim\sum_{k\in\mathbb{Z}}u_{jk}(\xi)e^{i\nu_{jk}^0 x}.$$

Then system (2.6) is non-resonance if

$$\begin{aligned} &\forall\, l^t, l^x, l_k \in \mathbb{Z} \qquad |l^t| + |l^x| + \sum_{k \neq j} |l_k| \neq 0 : \\ &\nu^t_{jl^t} + \nu^x_{jl^x}\lambda_j + \sum_{k \neq j} \nu^0_{kl_k}(\lambda_j - \lambda_k) \neq 0. \end{aligned} \tag{3.4}$$

Thus our method treats uniformly both the resonance and non-resonance problems. The mathematical justification of the method was given only in the case, when the operator B in system (1.1) does not depend on U_{xx} and U_{xxx}. But our results for non-resonance systems coincide with the results given by some other methods. Therefore we expect that asymptotic solution approximates uniformly the exact solution in more general cases too. The numerical analysis of such problems will be given in the next section.

4 Shallow water waves

In this section we consider the system of shallow water equations [1]

$$\begin{aligned} &Z_t + (HU)_x = \varepsilon\left(\frac{1}{6}\left(H^3 U_{xx}\right)_x - \frac{1}{2}(HU)_{xxx} - HH_x(HU)_{xx} - (ZU)_x\right), \\ &U_t + Z_x = -\varepsilon U U_x, \end{aligned} \tag{4.1}$$

where $z = \varepsilon Z$ is the water surface level, $u = \varepsilon U$ is the horizontal velocity of the fluid, H is the normalized bottom equation. All variables are normalized by some typical horizontal (L_*) and vertical (H_*) sizes:

$$x = \frac{x_1}{L_*}, \qquad z = \frac{z_1}{H_*}, \qquad t = \frac{\sqrt{gH_*}}{L_*}t_1, \qquad H = \frac{H_1}{H_*}, \quad \varepsilon = \left(\frac{H_*}{L_*}\right)^2 \ll 1,$$

where x_1 is the horizontal coordinate, t_1 the time, z_1 the water surface equation, g the acceleration due to gravity, $H_1(x)$ the bottom equation. We assume, that

$$H = 1 + \varepsilon h(x).$$

Then we can simplify the first equation of the system:

$$\begin{aligned} &Z_t + U_x = \varepsilon\left(-\frac{1}{3}U_{xxx} - (hU)_x - (ZU)_x\right), \\ &U_t + Z_x = -\varepsilon U U_x. \end{aligned} \tag{4.2}$$

We define new functions v^+ and v^-, which are related to U and Z in the following way:

$$U = v^+ - v^-, \qquad Z = v^+ + v^-.$$

Then problem (4.2) can be reduced to the system

$$v_t^+ + v_x^+ = -\frac{\varepsilon}{2}\Bigg(\frac{1}{3}\left(v_{xxx}^+ - v_{xxx}^-\right) + \left(h(x)\left(v^+ - v^-\right)\right)_x$$

$$+ \left(\left(v^+\right)^2 - \left(v^-\right)^2\right)_x + \left(v^+ - v^-\right)\left(v_x^+ - v_x^-\right)\Bigg),$$

$$v_t^- - v_x^- = -\frac{\varepsilon}{2}\Bigg(\frac{1}{3}\left(v_{xxx}^+ - v_{xxx}^-\right) + \left(h(x)\left(v^+ - v^-\right)\right)_x$$

$$- \left(\left(v^+\right)^2 - \left(v^-\right)^2\right)_x - \left(v^+ - v^-\right)\left(v_x^+ - v_x^-\right)\Bigg). \tag{4.3}$$

The asymptotic solution $V^\pm(\tau, y^\pm) = v^\pm(t, x, \varepsilon) + o(1)$, $y^\pm = x \mp t$ satisfies the averaged system

$$\frac{\partial V^+}{\partial \tau} = -\frac{1}{2}\left(\frac{1}{3}V^+_{y^+y^+y^+} + \left(\left\langle h(x)V^-\right\rangle_+\right)_{y^+} + 3V^+V^+_{y^+}\right),$$

$$\frac{\partial V^-}{\partial \tau} = \frac{1}{2}\left(\frac{1}{3}V^-_{y^-y^-y^-} + \left(\left\langle h(x)V^+\right\rangle_-\right)_{y^-} + 3V^-V^-_{y^-}\right), \tag{4.4}$$

here we have denoted the averaging operators:

$$\left\langle h(x)V^+\right\rangle_- = \lim_{T\to\infty}\frac{1}{T}\int_0^T h(y^- - s)V^+(\tau, y^- - 2s)ds,$$

$$\left\langle h(x)V^-\right\rangle_+ = \lim_{T\to\infty}\frac{1}{T}\int_0^T h(y^+ + s)V^-(\tau, y^+ + 2s)ds.$$

In [1] the analysis of the same problem is based on the assumption that $U = Z$, i.e. only one wave is considered. After simple computations we get the following Korteweg-de Vries problem for Z:

$$Z_t + Z_x + \frac{3}{2}\varepsilon Z Z_x + \frac{1}{6}\varepsilon Z_{xxx} = 0.$$

We will prove that in the non-resonance case system (4.4) describes two independent waves.

Let assume, that (2.10) holds for the initial condition, then we get

$$\left\langle V^\pm\right\rangle_\mp = \lim_{t\to\infty}\frac{1}{T}\int_o^T V^\pm(\tau, y)dy = 0.$$

System (4.3) is non-resonance if the following equalities

$$\langle h(x)V^-\rangle_+ = 0, \qquad \langle h(x)V^+\rangle_- = 0 \tag{4.5}$$

hold. Then averaged system (4.4) reduces to two Korteweg-de Vries equations:

$$\frac{\partial V^\pm}{\partial \tau} \pm \frac{3}{2}V^\pm\frac{\partial V^\pm}{\partial y} \pm \frac{1}{6}\frac{\partial^3 V^\pm}{\partial y^3} = 0.$$

Let consider the following initial conditions:

$$U(0,x)=0, \qquad Z(0,x)\sim\sum_{k\in\mathbb{Z}} Z_k e^{i\nu_k x}.$$

We also assume that

$$h(x)\sim\sum_{k\in\mathbb{Z}} h_k e^{i\mu_k x},$$

then the non-resonance condition can be written as

$$\mu_k\neq\pm2\nu_l, \qquad \forall\, k,l\in\mathbb{Z} \quad \& \quad |k|+|l|\neq 0. \tag{4.6}$$

In the periodic case, when the periods of functions $h(x)$ and $Z(0,x)$ are equal to 2π, we have that $\mu_k=\nu_k=k$. Thus, if $h(x)\neq\text{const}$ and $Z(0,x)\neq\text{const}$, then condition (4.6) is not satisfied and system (4.3) has a resonance.

4.1 Finite difference scheme

We define the space ω_h and time ω_τ meshes and assume that the space mesh size h and time mesh size τ are uniform. We denote by $v_j^n=v(t^n,y_j)$ a discrete function defined on $\omega_h\times\omega_\tau$. The following common notation of difference derivatives is used in our paper

$$v_\tau=\frac{v^{n+1}-v^n}{\tau}, \qquad v_{\bar y}=\frac{v_j-v_{j-1}}{h},$$
$$v_y=\frac{v_{j+1}-v_j}{h}, \qquad v_{\overset{\circ}{y}}=\frac{v_{j+1}-v_{j-1}}{2h}.$$

The finite difference approximation of system (4.4) is defined as follows (see also [10]):

$$\begin{aligned} V_\tau &= -\frac{1}{6}\left(\frac{V^{n+1}+V^n}{2}\right)_{\bar y y\overset{\circ}{y}}-\frac{3}{4}\left(\frac{\left(V^{n+1}\right)^2+V^{n+1}V^n+\left(V^n\right)^2}{3}\right)_{\overset{\circ}{y}}\\ &\quad-\frac{F_+\left(W^{n+1},W^n,j+1\right)-F_+\left(W^{n+1},W^n,j-1\right)}{4h},\\ W_\tau &= \frac{1}{6}\left(\frac{W^{n+1}+W^n}{2}\right)_{\bar y y\overset{\circ}{y}}+\frac{3}{4}\left(\frac{\left(W^{n+1}\right)^2+W^{n+1}W^n+\left(W^n\right)^2}{3}\right)_{\overset{\circ}{y}}\\ &\quad+\frac{F_-\left(V^{n+1},V^n,j+1\right)-F_-\left(V^{n+1},V^n,j-1\right)}{4h}, \end{aligned} \tag{4.7}$$

where the integrals are approximated as follows:

$$F_\pm\left(V^{n+1},V^n,j\right)=\frac{1}{2\pi}\sum_{i=1}^{N} h(y_j\mp i\ h)\frac{V_{j\mp2i}^{n+1}+V_{j\mp2i}^{n}}{2}\,h.$$

The approximation error of this finite difference scheme is estimated as $O\left(\tau^2+h^2\right)$. Numerical methods for solving the Korteweg-de Vries equation are investigated in [9, 17].

4.2 Linear dispersion problem

In this section we consider a linear problem

$$\begin{aligned} &Z_t + (HU)_x = -\frac{\varepsilon}{3}\, U_{xxx}, \\ &U_t + Z_x = 0. \end{aligned} \tag{4.8}$$

First we will prove that system (4.8) defines an ill-posed problem. Let consider the case $H = 1$. After simple computations we get the equation for U:

$$U_{tt} - U_{xx} = \frac{\varepsilon}{3}\, U_{xxxx}. \tag{4.9}$$

Considering the k-th Fourier mode we get that the solution of (4.9) is unstable for $k^2\varepsilon \geq 3$. In order to define a stable solution we use the following regularized problem

$$\begin{aligned} &Z_t + (HU)_x = -\frac{\varepsilon}{3}\, U_{xxx} - \frac{\varepsilon^2}{20}\, U_{xxxxx}, \\ &U_t + Z_x = 0. \end{aligned} \tag{4.10}$$

We note that the averaged system (4.4) also gives a nontrivial regularization of this ill-posed problem.

The accuracy of asymptotic solution is illustrated by solving problem (4.10) with following initial conditions

$$U(x,0) = 0, \qquad Z(x,0) = \cos x + \sin 2x, \qquad h(x) = 5\sin 2x. \tag{4.11}$$

Fig. 1 shows the solution of system (4.10) and the asymptotic solution at $t = 1/\varepsilon$, for $\varepsilon = 0.1$ and $\varepsilon = 0.01$. For $\varepsilon = 0.001$ the difference between the asymptotic solution and the exact solution is too small to be illustrated on the figure. We also note, that the averaged system must be solved numerically only once and then a solution can be computed using simple interpolation procedure for any parameters ε.

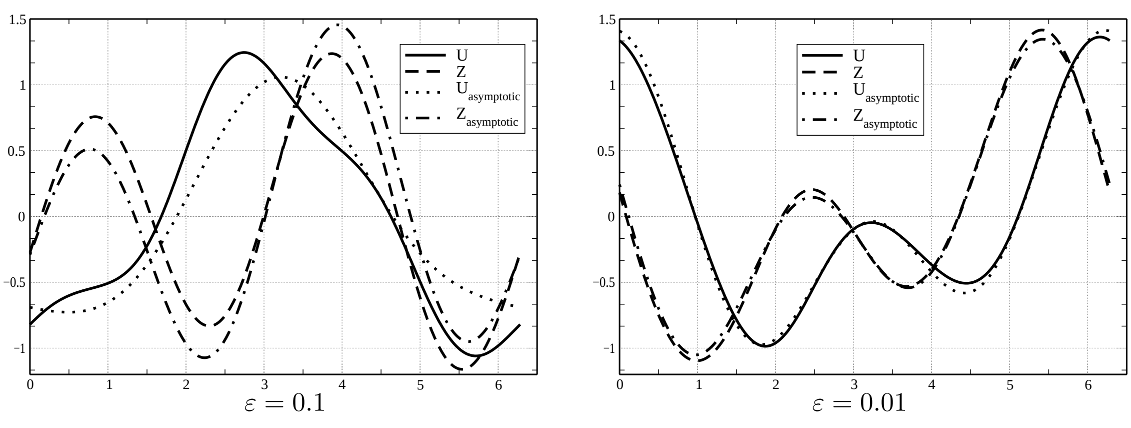


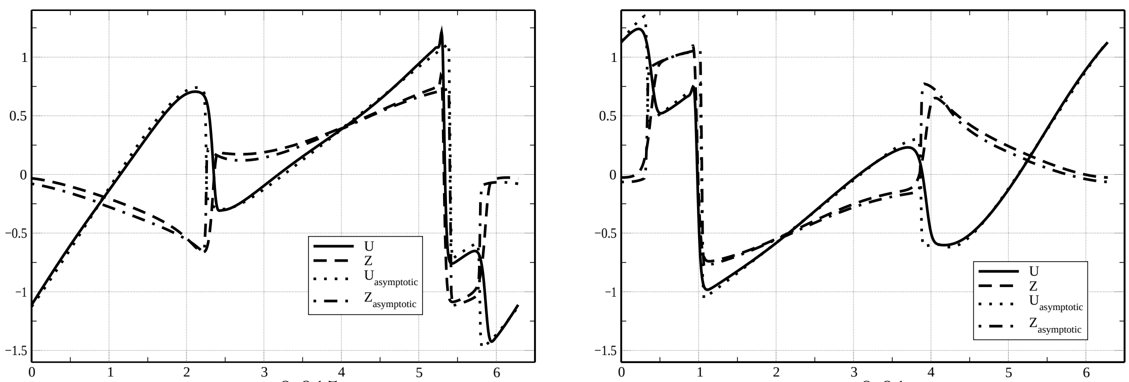
Figure 1. Asymptotical solution for linear dispersion problem (4.8).

4.3 Nonlinear nondispersive problem

In this section we consider a nonlinear problem

$$
\begin{aligned}
&Z_t + (HU)_x = -\varepsilon (ZU)_x, \\
&U_t + Z_x = -\varepsilon U U_x
\end{aligned}
\tag{4.12}
$$

with the same initial conditions (4.11). As it follows from results given in previous sections, we have the resonance case. Fig. 2 shows the solution of system (4.12) and the asymptotic solution at $t = 1/\varepsilon$.

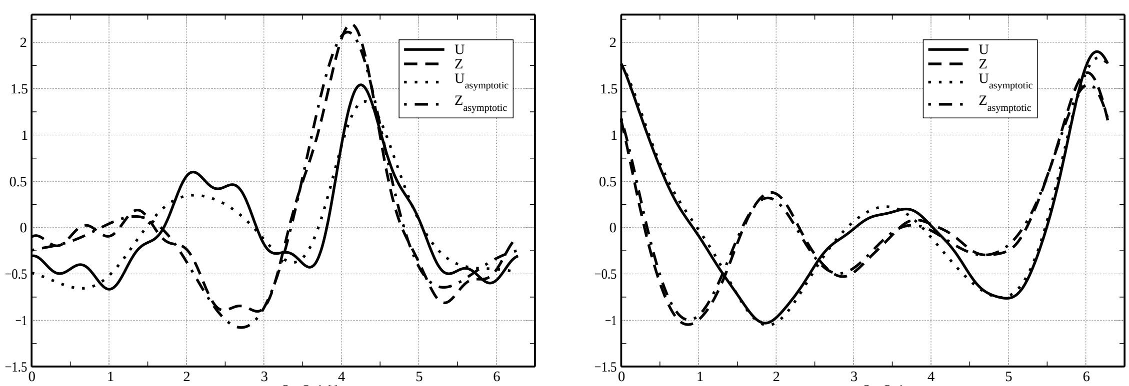


$\varepsilon = 0.015$ $\varepsilon = 0.01$

Figure 2. Asymptotical solution for nonlinear problem (4.12).

4.4 Shallow water waves

Here we solved the full nonlinear system (4.1). The regularization of section 4.2 is used in order to define the stable solution. Fig. 3 shows the solution of system (4.1) and the asymptotic solution at $t = 1/\varepsilon$.



$\varepsilon = 0.015$ $\varepsilon = 0.01$

Figure 3. Asymptotical solution for the shallow water problem.

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