Journal of Nonlinear Mathematical Physics 2002, Volume 9, Supplement 1, 106–126 Proceedings: Hong Kong

Adjoint Symmetry Constraints Leading to Binary Nonlinearization

Wen-Xiu MA † *and Ruguang ZHOU* ‡

† *Department of Mathematics, City University of Hong Kong, Hong Kong, P. R. China*
E-mail: mawx@math.cityu.edu.hk

‡ *Department of Mathematics, Xuzhou Normal University, Xuzhou 221009, P. R. China*
E-mail: rgzhou@public.xz.js.cn

Received July 25, 2001; Accepted August 16, 2001

Abstract

Adjoint symmetry constraints are presented to manipulate binary nonlinearization, and shown to be a slight weaker condition than symmetry constraints in the case of Hamiltonian systems. Applications to the multicomponent AKNS system of nonlinear Schrödinger equations and the multi-wave interaction equations, associated with 3×3 matrix spectral problems, are made for establishing their integrable decompositions.

1 Introduction

Symmetry constraints are of particular significance in the theory of binary nonlinearization [1, 2]. It is due to symmetry constraints that the so-called constrained flows, both spatial [3] and temporal [4], possess beautiful properties such as separated variables [5, 6] and the Liouville integrability [7]. Therefore, a pair of spatial and temporal constrained flows provides a good example of integrable decompositions [8] and the resulting potential constraints offer the Bäcklund transformations [9] for the system of evolution equations under consideration. Applications of symmetry constraints have been successfully made for various soliton hierarchies [7]-[12].

A non-Lie type symmetry, generated from the variational derivative of the spectral parameter, is essential in making symmetry constraints for systems of evolution equations associated with spectral problems [11, 12]. However, for non-Hamiltonian systems of evolution equations, it is not natural or even impossible to generate such a symmetry from the associated spectral problems. Therefore, we need to put forward a good theory for exposing intrinsic characteristics of binary nonlinearization.

In this article, we will present an exceptional explanation for binary nonlinearization from the adjoint symmetry point of view. The variational derivative of the spectral parameter will be shown to be an adjoint symmetry of the underlying system of evolution equations in the isospectral case. Moreover, we will show that all we need in the process of nonlinearization is just a kind of adjoint symmetry constraints. This allows us to manipulate nonlinearization without having to compute any non-Lie type symmetry. Therefore, adjoint symmetry constraints provide a good basis of the theory of binary nonlinearization for both Hamiltonian and non-Hamiltonian systems of evolution equations. Moreover, for a Hamiltonian system of evolution equations, adjoint symmetries automatically yield symmetries of the same system under the Hamiltonian transformation, and thus, adjoint symmetry constraints generate symmetry constraints which lead to the same potential decomposition. In this sense, adjoint symmetry constraints have broader applicability than symmetry constraints.

More specifically, we are going to present a general scheme to carry out adjoint symmetry constraints for soliton equations associated with square matrix spectral problems. Two applications will be made for the multicomponent AKNS system of nonlinear Schrödinger equations and the multi-wave interaction equations associated with two 3×3 matrix spectral problems, along with their integrable decompositions. Some concluding remarks are given in the last section.

2 Adjoint symmetry constraints

2.1 Preliminaries

Let us assume that we have a system of evolution equations

$$u_t = K(t,x,u),\ u=(u_1,\cdots,u_q)^T. \tag{2.1}$$

For the sake of simplicity of exposition, we restrict our discussion to the $1+1$ dimensional case, i.e., the case of the time and space variables t and x being scalar. Moreover, the inner product for r-dimensional vector functions is assumed to be taken as

$$\langle Y,Z\rangle_r = \int_\Omega \sum_{i=1}^r Y_i Z_i\,dx,\ Y=(Y_1,\cdots,Y_r)^T,\ Z=(Z_1,\cdots,Z_r)^T, \tag{2.2}$$

where $\Omega=(0,T)$ if u is supposed to be periodic with period T, or $\Omega=(-\infty,\infty)$ if u is supposed to belong to the Schwartz space. For convenience, we often use the notation

$$\langle Y,Z\rangle = \langle Y,Z\rangle_q,\ \int F\,dx = \int_\Omega F\,dx,$$

if there is no confusion, and the existence is assumed to be guaranteed for any required objects.

Definition 1. For any object $X=X(u)$ depending on u, the Gateaux derivative X' of X at a direction Y with respect to the potential u is defined as follows

$$(X'Y)(u) = X'(u)[Y(u)] = \left.\frac{\partial X(u+\varepsilon Y(u))}{\partial \varepsilon}\right|_{\varepsilon=0}. \tag{2.3}$$

Definition 2. Let Φ be an operator transforming r-dimensional vector functions to s-dimensional vector functions. The adjoint operator of Φ, denoted by $\Phi^\dagger$, transforming s-dimensional vector functions to r-dimensional vector functions is defined through

$$\langle \Phi^\dagger Y, Z\rangle_r = \langle Y,\Phi Z\rangle_s, \tag{2.4}$$

where $\langle\cdot,\cdot\rangle_r$ and $\langle\cdot,\cdot\rangle_s$ are the corresponding inner products determined by (2.2).

For any r-dimensional vector function $X = X(u)$, its Gateaux derivative X' can be viewed as an operator transforming q-dimensional vector functions to r-dimensional vector functions: $X' : Y \mapsto X'Y$, and the adjoint operator $(X')^\dagger$ of X' can be viewed as an operator transforming r-dimensional vector functions to q-dimensional vector functions.

If we have $X = X(t, x, u, u_x, \cdots, u^{(n)})$, where $u^{(n)}$ denotes the nth order spatial derivative of u, i.e., X only depends on t, x and spatial derivatives of u up to some finite order, then X is called to be local, and we often write $X = X[u]$. It is well-known that for any local vector function $X = X[u] = (X_1, \cdots, X_r)^T$, its Gateaux operator can be computed as follows

$$X' = (V_j(X_i))_{r\times q} = \begin{bmatrix} V_1(X_1) & V_2(X_1) & \cdots & V_q(X_1) \\ V_1(X_2) & V_2(X_2) & \cdots & V_q(X_2) \\ \vdots & \vdots & \ddots & \vdots \\ V_1(X_r) & V_2(X_r) & \cdots & V_q(X_r) \end{bmatrix}, \ V_i(X_j) = \sum_{k=0}^{\infty} \frac{\partial X_j}{\partial u_i^{(k)}} \partial^k, \tag{2.5}$$

where $\partial = \partial/\partial x$ and $u_i^{(k)} = \partial^k u_i$, and thus, the adjoint operator $(X')^\dagger$ of X' can be determined by

$$(X')^\dagger = (X')^\dagger(u) = (V_i^\dagger(X_j))_{q\times r}, \ V_i^\dagger(X_j) = \sum_{k=0}^{\infty} (-\partial)^k \frac{\partial X_j}{\partial u_i^{(k)}}. \tag{2.6}$$

For example, if $X = u_1 u_2 + u_{1,x}^2$, then we have

$$X' = (u_2 + 2u_{1,x}\partial\,, u_1), \ (X')^\dagger = (u_2 - 2u_{1,xx} - 2u_{1,x}\partial\,, u_1)^T.$$

Definition 3. For a functional $\tilde H = \tilde H(u)$, its variational derivative $\frac{\delta \tilde H}{\delta u}$ is defined by

$$\langle \frac{\delta \tilde H}{\delta u}, X\rangle = \tilde H'[X], \ (\tilde H'[X])(u) = \tilde H'(u)[X(u)]. \tag{2.7}$$

A q-dimensional vector function is called gradient if it can be written as the variational derivative of a functional, and the functional is called a Lagrangian of the vector function.

It is known that a q-dimensional vector function $X = X(u)$ is gradient iff its Gateaux derivative operator X' is symmetric, i.e., $(X')^\dagger = X'$. If $\tilde H = \int H\, dx$ and H is local, then the variational derivative $\frac{\delta \tilde H}{\delta u}$ can be computed as

$$\frac{\delta \tilde H}{\delta u} = \Big(\sum_{k=0}^{\infty} (-\partial)^k \frac{\partial H}{\partial u_1^{(k)}}, \cdots, \sum_{k=0}^{\infty} (-\partial)^k \frac{\partial H}{\partial u_q^{(k)}} \Big)^T. \tag{2.8}$$

If $X = (X_1(u), \cdots, X_q(u))^T$ is local and gradient, then its Lagrangian can be given by

$$\tilde H = \int H\, dx, \ H = \int_0^1 \langle X(\lambda u), u\rangle\, d\lambda.$$

In particular, given $\tilde{H}=\int H\,dx$, $H=2u_1u_{2,x}+u_2^2+u_{3,x}^4$, we have

$$\frac{\delta \tilde{H}}{\delta u}=(2u_{2,x}\,,-2u_{1,x}+2u_2\,,-12u_{3,x}^2u_{3,xx})^T.$$

If we are given $X=(6u_1^2+u_{2,x}\,,-u_{1,x})^T$, then

$$X'=\begin{bmatrix} 12u_1 & \partial \\ -\partial & 0 \end{bmatrix},$$

and thus $(X')^\dagger=X'$ and its Lagrangian is computed as

$$\tilde{H}=\int H\,dx,\ H=\int_0^1\langle X(\lambda u),u\rangle\,d\lambda=2u_1^3+\frac{1}{2}u_1u_{2,x}-\frac{1}{2}u_{1,x}u_2.$$

Definition 4. A q-dimensional vector function $X=X(u)$ is called an adjoint symmetry of the system of evolution equations (2.1), if it satisfies the adjoint linearized system

$$(X(u))_t=-(K'(u))^\dagger X(u) \tag{2.9}$$

when u solves $u_t=K[u]$. A conservation law of the system of evolution equations (2.1) is given by

$$H_t+F_x=0,\ H=H(u),\ F=F(u), \tag{2.10}$$

where H and F are scalar functions and u solves $u_t=K[u]$, of which H is called a conserved density of (2.1) and F, a conserved flux of (2.1) associated with H.

Any linear combination of adjoint symmetries (or conserved densities) of (2.1) is again an adjoint symmetry (or a conserved density) of (2.1). The following result just needs a direct computation.

Proposition 1. Let $\tilde{H}(u)=\int H(u)\,dx$. Then the function $H=H(u)$ is a conserved density of the system (2.1) iff $\frac{\delta\tilde{H}}{\delta u}$ is an adjoint symmetry of the same system (2.1).

2.2 General scheme

Let us now show the structure of soliton equations which associate with two square matrix spectral problems

$$\phi_x=U\phi=U(u,\lambda)\phi, \tag{2.11a}$$
$$\phi_{t_n}=V^{(n)}\phi=V^{(n)}(u,u_x,\cdots;\lambda)\phi, \tag{2.11b}$$

where $n\ge 0$, λ is a spectral parameter, and U and $V^{(n)}$ are two square matrices, called spectral matrices. If the Gateaux derivative U' of U is injective, then under the isospectral condition

$$\lambda_{t_n}=0, \tag{2.12}$$

zero curvature equations

$$U_{t_n} - V_x^{(n)} + [U, V^{(n)}] = 0 \tag{2.13}$$

will usually determine a hierarchy of soliton equations with a Hamiltonian structure:

$$u_{t_n} = K_n(u),\ K_n = JG_n = J\frac{\delta \tilde{H}_n}{\delta u},\ \tilde{H}_n = \int H_n\, dx, \tag{2.14}$$

where $J(u)$ is a Hamiltonian operator and $\tilde{H}_n(u)$ is a Hamiltonian functional. Obviously, the compatability conditions of the adjoint spectral problem and the adjoint associated spectral problems

$$\psi_x = -U^T(u,\lambda)\psi,\ \psi_{t_n} = -V^{(n)T}(u,\lambda)\psi, \tag{2.15}$$

determine the same soliton hierarchy (2.14).

It is known (for example, see [12]) that

$$\frac{\delta\lambda}{\delta u} = E^{-1}\psi^T\frac{\partial U(u,\lambda)}{\partial u}\phi,\ E = -\int_\Omega \psi^T\frac{\partial U(u,\lambda)}{\partial \lambda}\phi\, dx,$$

where E is called the normalized constant. Therefore, based on the above proposition, if the functionals $\tilde{H}_m$ are assumed to be conserved, we have the following common adjoint symmetries:

$$G_m,\ \psi^T\frac{\partial U(u,\lambda)}{\partial u}\phi \tag{2.16}$$

for the soliton hierarchy (2.14), since $\lambda = \lambda(u)$ is conserved.

Let us go on to introduce N distinct eigenvalues $\lambda_1, \cdots, \lambda_N$, and so we have

$$\phi_x^{(s)} = U(u,\lambda_s)\phi^{(s)},\ \psi_x^{(s)} = -U^T(u,\lambda_s)\psi^{(s)},\ 1 \le s \le N, \tag{2.17}$$

$$\phi_{t_n}^{(s)} = V^{(n)}(u,\lambda_s)\phi^{(s)},\ \psi_{t_n}^{(s)} = -V^{(n)T}(u,\lambda_s)\psi^{(s)},\ 1 \le s \le N, \tag{2.18}$$

where the corresponding eigenfunctions and adjoint eigenfunctions are denoted by $\phi^{(s)}$ and $\psi^{(s)}$, $1 \le s \le N$. Fix an adjoint symmetry G_{m_0}, and then we can define the so-called binary adjoint symmetry constraint

$$G_{m_0} = \sum_{s=1}^N E_s\mu_s\frac{\delta\lambda_s}{\delta u} = \sum_{s=1}^N \mu_s\psi^{(s)T}\frac{\partial U(u,\lambda_s)}{\partial u}\phi^{(s)}, \tag{2.19}$$

where μ_s, $1 \le s \le N$, are arbitrary nonzero constants, and E_s, $1 \le s \le N$, are N normalized constants. The right-hand side of the binary adjoint symmetry constraint (2.19) is a general linear combination of N adjoint symmetries $\delta\lambda_s/\delta u$, $1 \le s \le N$. Such an adjoint symmetry is not Lie type, since $\phi^{(s)}$ and $\psi^{(s)}$ can not be expressed in terms of x, u and spatial derivatives of u to some finite order. But, usually G_{m_0} is a Lie type adjoint symmetry. According to the property of the Lie-type symmetry G_{m_0}, all adjoint symmetry constraints (2.19) can be divided into the following three categories:

- **Neumann type:** (2.19) does not depend on any spatial derivative of u and it is impossible to solve (2.19) for u.

- **Bargmann type:** (2.19) does not depend on any spatial derivative of u but it is possible to solve (2.19) for u.

- **Ostrogradsky type:** (2.19) depends on spatial derivatives of u.

In a soliton hierarchy, usually the first conserved functional corresponds to the Neumann type constraint, the second conserved functional corresponds to the Bargmann type constraint, and the other conserved functionals correspond to the Ostrogradsky type constraints.

Let us focus on the Bargmann type adjoint symmetry constraints. Upon solving the adjoint symmetry constraint (2.19) for u, we are assumed to have

$$u = \widetilde{u}(\phi^{(1)}, \cdots, \phi^{(N)}; \psi^{(1)}, \cdots, \psi^{(N)}). \tag{2.20}$$

Make the replacement of u with $\widetilde{u}$ in the Lax systems (2.17) and (2.18), and then we obtain the so-called spatial binary constrained flow:

$$\phi^{(s)}_x = U(\widetilde{u}, \lambda_s)\phi^{(s)},\ \psi^{(s)}_x = -U^T(\widetilde{u}, \lambda_s)\psi^{(s)},\ 1 \le s \le N, \tag{2.21}$$

and the so-called temporal binary constrained flow:

$$\phi^{(s)}_{t_n} = V^{(n)}(\widetilde{u}, \lambda_s)\phi^{(s)},\ \psi^{(s)}_{t_n} = -V^{(n)T}(\widetilde{u}, \lambda_s)\psi^{(s)},\ 1 \le s \le N. \tag{2.22}$$

These two constrained flows still require the nth system of evolution equations $u_{t_n} = K_n(u)$ as their compatability condition. Therefore, $u = \widetilde{u}$ gives rise to an integrable decomposition of the system $u_{t_n} = K_n(u)$, if (2.21) and (2.22) are Liouville integrable.

Note that the constrained flows (2.21) and (2.22) are nonlinear, although the original Lax systems (2.17) and (2.18) are linear with the eigenfunctions and adjoint eigenfunctions. In view of this property and the involvement of the original spectral problems and the adjoint ones, the above process of carrying out the Bargmann adjoint symmetry constraints is called binary nonlinearization [7, 13].

The whole manipulation above shows us that binary nonlinearization can also result from adjoint symmetry constraints. Actually, if the underlying system of evolution equations possesses a Hamiltonian structure, then adjoint symmetries generate symmetries under the Hamiltonian transformation and thus adjoint symmetry constraints also yield symmetry constraints which are required in binary nonlinearization. However, noting that adjoint symmetry constraints do not require the Hamiltonian structure for systems of evolution equations, they can be applied to non-Hamiltonian systems of evolution equations, for example, the Burgers type systems of evolution equations, for which symmetry constraints don't succeed.

3 Application to multicomponent AKNS equations

3.1 Multicomponent AKNS hierarchy

Let us consider the following 3×3 matrix spectral problem

$$\phi_x = U(u,\lambda)\phi,\ U(u,\lambda) = \begin{bmatrix} -2\lambda & q_1 & q_2 \\ r_1 & \lambda & 0 \\ r_2 & 0 & \lambda \end{bmatrix} = U_0\lambda + U_1,\ \frac{\partial U_0}{\partial \lambda} = \frac{\partial U_1}{\partial \lambda} = 0, \tag{3.1}$$

where λ is a spectral parameter and

$$\phi = (\phi_1,\phi_2,\phi_3)^T,\ u = \rho(U_1) = (q_1,q_2,r_1,r_2)^T,\ q = (q_1,q_2),\ r = (r_1,r_2)^T. \tag{3.2}$$

Since U_0 has multiple eigenvalues, (3.1) is degenerate. Under the special reduction of $q_2 = r_2 = 0$, (3.1) is equivalent to the AKNS spectral problem [14], and thus (3.1) is called a multicomponent AKNS spectral problem.

To derive an associated soliton hierarchy, we first solve the adjoint equation $W_x = [U,W]$ of (3.1) following the generalized Tu scheme [15]. We suppose that a solution W is given by

$$W = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \tag{3.3}$$

where a is a scalar, b^T and c are two-dimensional columns, and d is a 2×2 matrix. Then the adjoint equation $W_x = [U,W]$ is equivalent to

$$a_x = qc - br,\ b_x = -3\lambda b + qd - aq, \tag{3.4a}$$

$$c_x = 3\lambda c + ra - dr,\ d_x = rb - cq. \tag{3.4b}$$

We seek a formal solution as

$$W = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \sum_{k=0}^{\infty} W_k \lambda^{-k} = \sum_{k=0}^{\infty} \begin{bmatrix} a^{(k)} & b^{(k)} \\ c^{(k)} & d^{(k)} \end{bmatrix} \lambda^{-k} \tag{3.5}$$

with $b^{(k)}, c^{(k)}$ and $d^{(k)}$ being assumed to be

$$b^{(k)} = (b_1^{(k)}, b_2^{(k)}),\ c^{(k)} = (c_1^{(k)}, c_2^{(k)})^T,\ d^{(k)} = (d_{ij}^{(k)})_{2\times 2}.$$

Thus, the condition (3.4) becomes the following recursion relation:

$$b^{(0)} = 0,\ c^{(0)} = 0,\ a_x^{(0)} = 0,\ d_x^{(0)} = 0, \tag{3.6a}$$

$$b^{(k+1)} = \frac{1}{3}(-b_x^{(k)} + qd^{(k)} - a^{(k)}q),\ k \geq 0, \tag{3.6b}$$

$$c^{(k+1)} = \frac{1}{3}(c_x^{(k)} - ra^{(k)} + d^{(k)}r),\ k \geq 0, \tag{3.6c}$$

$$a_x^{(k)} = qc^{(k)} - b^{(k)}r,\ d_x^{(k)} = rb^{(k)} - c^{(k)}q,\ k \geq 1. \tag{3.6d}$$

We choose the initial values as follows

$$a^{(0)}=-2,\ d^{(0)}=I_2, \tag{3.7}$$

where I_2 is the second-order identity matrix, and require that

$$W_k|_{u=0}=0,\ k\geq 1. \tag{3.8}$$

This requirement (3.8) means to identify all constants of integration to be zero while using (3.6) to determine W, and thus, with $a^{(0)}$ and $d^{(0)}$ given by (3.7), all matrices $W_k,\ k\geq 1$, will be uniquely determined. For example, it follows from (3.6) that

$$\begin{aligned}
&b_i^{(1)}=q_i,\ c_i^{(1)}=r_i,\ a^{(1)}=0,\ d_{ij}^{(1)}=0,\\
&b_i^{(2)}=-\frac{1}{3}q_{i,x},\ c_i^{(2)}=\frac{1}{3}r_{i,x},\ a^{(2)}=\frac{1}{3}(q_1r_1+q_2r_2),\ d_{ij}^{(2)}=-\frac{1}{3}r_iq_j,\\
&b_i^{(3)}=\frac{1}{9}[q_{i,xx}-2(q_1r_1+q_2r_2)q_i],\ c_i^{(3)}=\frac{1}{9}[r_{i,xx}-2(q_1r_1+q_2r_2)r_i].
\end{aligned}$$

Noting (3.6d), we can obtain a recursion relation for $b^{(k)}$ and $c^{(k)}$:

$$\begin{bmatrix} c^{(k+1)} \\ b^{(k+1)T} \end{bmatrix}=\Psi\begin{bmatrix} c^{(k)} \\ b^{(k)T} \end{bmatrix},\ k\geq 1, \tag{3.10}$$

where Ψ is a 4×4 matrix operator

$$\Psi=\frac{1}{3}\begin{bmatrix} (\partial-\sum_{k=1}^{2}r_k\partial^{-1}q_k)I_2-r\partial^{-1}q & r\partial^{-1}r^T+(r\partial^{-1}r^T)^T \\ -q^T\partial^{-1}q-(q^T\partial^{-1}q)^T & (-\partial+\sum_{k=1}^{2}q_k\partial^{-1}r_k)I_2+q^T\partial^{-1}r^T \end{bmatrix}. \tag{3.11}$$

As usual, for any integer $n\geq 0$, choose

$$V^{(n)}=(\lambda^nW)_+=\sum_{j=0}^{n}W_j\lambda^{n-j}, \tag{3.12}$$

and then introduce the time evolution law for the eigenfunction ϕ:

$$\phi_{t_n}=V^{(n)}\phi=V^{(n)}(u,u_x,\cdots,u^{(n-1)};\lambda)\phi. \tag{3.13}$$

The compatibility condition of (3.1) and (3.13) leads to a system of evolution equations

$$u_{t_n}=\begin{bmatrix} q^T \\ r \end{bmatrix}_{t_n}=K_n=\begin{bmatrix} -3b^{(n+1)T} \\ 3c^{(n+1)} \end{bmatrix}. \tag{3.14}$$

The first nonlinear system in this soliton hierarchy (3.14) is given by

$$q_{i,t_2}=-\frac{1}{3}[q_{i,xx}-2(q_1r_1+q_2r_2)q_i],\ 1\leq i\leq 2, \tag{3.15a}$$

$$r_{i,t_2}=\frac{1}{3}[r_{i,xx}-2(q_1r_1+q_2r_2)r_i],\ 1\leq i\leq 2, \tag{3.15b}$$

which is the multicomponent version of the AKNS system of nonlinear Schrödinger equations. Therefore, the soliton hierarchy (3.14) is called the multicomponent AKNS soliton hierarchy.

In order to generate the Hamiltonian structure of the multicomponent AKNS hierarchy (3.14), we apply the trace identity [16]:

$$\frac{\delta}{\delta u}\int \mathrm{tr}(W\frac{\partial U}{\partial \lambda})dx = \lambda^{-\gamma}\frac{\partial}{\partial \lambda}\left[\lambda^{\gamma}\mathrm{tr}(W\frac{\partial U}{\partial u})\right],$$

with γ being a constant to be found, which yields

$$\frac{\delta \tilde{H}_{n+1}}{\delta u} = G_n,\ \tilde{H}_n = \int (-2a^{(n)} + d_{11}^{(n)} + d_{22}^{(n)})\,dx,\ G_{n-1} = \left[\begin{array}{c} c^{(n)} \\ b^{(n)T} \end{array}\right],\ n \geq 0. \quad (3.16)$$

Actually, we have

$$\mathrm{tr}(W\frac{\partial U}{\partial \lambda}) = -2a + \mathrm{tr}(d) = \sum_{k=0}^{\infty}(-2a^{(k)} + d_{11}^{(k)} + d_{22}^{(k)})\lambda^{-k},$$

and

$$\mathrm{tr}(W\frac{\partial U}{\partial u}) = \left[\begin{array}{c} c \\ b^T \end{array}\right] = \sum_{k\geq 0} G_{k-1}\lambda^{-k}.$$

Inserting these into the trace identity and considering the case of $k = 2$, we get $\gamma = 0$ and thus we have (3.16). Now it follows from (3.16) that the multicomponent AKNS equations (3.14) have the following bi-Hamiltonian formulation

$$u_{t_n} = K_n = JG_n = J\frac{\delta \tilde{H}_{n+1}}{\delta u} = M\frac{\delta \tilde{H}_n}{\delta u}, \quad (3.17)$$

where the Hamiltonian pair $(J, M = J\Psi)$ reads as

$$J = \left[\begin{array}{cc} 0 & -3I_2 \\ 3I_2 & 0 \end{array}\right], \quad (3.18a)$$

$$M = \left[\begin{array}{cc} q^T\partial^{-1}q + (q^T\partial^{-1}q)^T & (\partial - \sum_{k=1}^{2} q_k\partial^{-1}r_k)I_2 - q^T\partial^{-1}r^T \\ (\partial - \sum_{k=1}^{2} r_k\partial^{-1}q_k)I_2 - r\partial^{-1}q & r\partial^{-1}r^T + (r\partial^{-1}r^T)^T \end{array}\right]. \quad (3.18b)$$

3.2 Adjoint symmetry constraint

Let us go on to consider the problem of adjoint symmetry constraints for the multicomponent AKNS equations $u_{t_2} = K_2$ defined by (3.14) or (3.15). A direct computation shows us that the Gateaux derivative operator of K_2 is given by

$$K_2' = \frac{2}{3}(-\frac{1}{2}\partial^2 + qr)\left[\begin{array}{cc} I_2 & 0 \\ 0 & -I_2 \end{array}\right] + \frac{2}{3}\left[\begin{array}{cc} q^Tr^T & q^Tq \\ -rr^T & -rq \end{array}\right], \quad (3.19)$$

and thus its adjoint operator is given by

$$(K_2')^\dagger = \frac{2}{3}(-\frac{1}{2}\partial^2 + qr)\begin{bmatrix} I_2 & 0 \\ 0 & -I_2 \end{bmatrix} + \frac{2}{3}\begin{bmatrix} rq & -rr^T \\ q^Tq & -q^Tr^T \end{bmatrix} = (K')^T. \tag{3.20}$$

We choose a constant diagonal matrix:

$$\Gamma = \mathrm{diag}(\gamma_1, \gamma_2, \gamma_3),\ \ \gamma_i \neq \gamma_j,\ \ 1 \le i \neq j \le 3. \tag{3.21}$$

Then the commutator

$$[\Gamma, U_1] = \begin{bmatrix} 0 & (\gamma_1-\gamma_2)q_1 & (\gamma_1-\gamma_3)q_2 \\ (\gamma_2-\gamma_1)r_1 & 0 & 0 \\ (\gamma_3-\gamma_1)r_2 & 0 & 0 \end{bmatrix},$$

will give rise to a symmetry of the multicomponent AKNS equations (3.15):

$$\bar K_0 := \rho([\Gamma, U_1]) = \begin{bmatrix} (\gamma_1-\gamma_2)q_1 \\ (\gamma_1-\gamma_3)q_2 \\ (\gamma_2-\gamma_1)r_1 \\ (\gamma_3-\gamma_1)r_2 \end{bmatrix}. \tag{3.22}$$

Through the Hamiltonian structure in (3.17), we obtain an adjoint symmetry of (3.15):

$$\bar G_0 := J^{-1}\bar K_0 = \begin{bmatrix} 0 & \frac{1}{3}I_2 \\ -\frac{1}{3}I_2 & 0 \end{bmatrix}\bar K_0 = \frac{1}{3}\begin{bmatrix} (\gamma_2-\gamma_1)r_1 \\ (\gamma_3-\gamma_1)r_2 \\ (\gamma_2-\gamma_1)q_1 \\ (\gamma_3-\gamma_1)q_2 \end{bmatrix}, \tag{3.23}$$

which contains three arbitrary distinct constants γ_1, γ_2 and γ_3. This also can be shown by directly checking

$$\bar G_{0,t_2} = -(K_2')^\dagger \bar G_0,$$

while u solves $u_{t_2} = K_2(u)$. Now the Bargmann adjoint symmetry constraint reads as

$$\bar G_0 = \sum_{s=1}^N \mu_s \psi^{(s)T}\frac{\partial U(u,\lambda_s)}{\partial u}\phi^{(s)}, \tag{3.24}$$

where for a later use, $\phi^{(s)}$ and $\psi^{(s)}$ are assumed to be

$$\phi^{(s)} = (\phi_{1s}, \phi_{2s}, \phi_{3s})^T,\ \ \psi^{(s)} = (\psi_{1s}, \psi_{2s}, \psi_{3s})^T,\ \ 1 \le s \le N. \tag{3.25}$$

Upon introducing the matrix

$$B = \mathrm{diag}(\mu_1, \mu_2, \cdots, \mu_N), \tag{3.26}$$

and through solving (3.24) for q and r, we are led to the potential constraints

$$q_i = \tilde{q}_i := \frac{3}{\gamma_{i+1}-\gamma_1}\langle \Phi_1, B\Psi_{i+1}\rangle,\ r_i = \tilde{r}_i := \frac{3}{\gamma_{i+1}-\gamma_1}\langle \Phi_{i+1}, B\Psi_1\rangle, \tag{3.27}$$

where $1 \le i \le 2$, Φ_i and Ψ_i are defined by

$$\Phi_i = (\phi_{i1}, \phi_{i2}, \cdots, \phi_{iN})^T,\ \Psi_i = (\psi_{i1}, \psi_{i2}, \cdots, \psi_{iN})^T,\ 1 \le i \le 3, \tag{3.28}$$

and $\langle \cdot, \cdot \rangle$ refers to the standard inner product of the Euclidian space $\mathbb{R}^N$. Now the spatial and temporal constrained flows of the multicomponent AKNS equations (3.15) read as

$$\phi_x^{(s)} = U(\widetilde{u}, \lambda_s)\phi^{(s)},\ \psi_x^{(s)} = -U^T(\widetilde{u}, \lambda_s)\psi^{(s)},\ 1 \le s \le N, \tag{3.29}$$

and

$$\phi_{t_2}^{(s)} = V^{(2)}(\widetilde{u}, \lambda_s)\phi^{(s)},\ \psi_{t_2}^{(s)} = -V^{(2)T}(\widetilde{u}, \lambda_s)\psi^{(s)},\ 1 \le s \le N, \tag{3.30}$$

where $\widetilde{u} = (\widetilde{q}_1, \widetilde{q}_2, \widetilde{r}_1, \widetilde{r}_2)^T$ and

$$\begin{aligned} V^{(2)}(u,\lambda) = {} & \begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\lambda^2 + \begin{bmatrix} 0 & q_1 & q_2 \\ r_1 & 0 & 0 \\ r_2 & 0 & 0 \end{bmatrix}\lambda \\ & +\frac{1}{3}\begin{bmatrix} q_1r_2 + q_2r_2 & -q_{1,x} & -q_{2,x} \\ r_{1,x} & -r_1q_1 & -r_1q_2 \\ r_{2,x} & -r_2q_1 & r_2q_2 \end{bmatrix}. \end{aligned} \tag{3.31}$$

As usual, let us denote by $\widetilde{V}^{(2)}(\widetilde{u}, \lambda)$ the transformed matrix of $V^{(2)}(\widetilde{u}, \lambda)$ under (3.29), i.e.,

$$\widetilde{V}^{(2)}(\widetilde{u}, \lambda) = V^{(2)}(\widetilde{u}, \lambda)|_{\text{spatial constrained flow (3.29)}}. \tag{3.32}$$

Since $\widetilde{V}^{(2)}(\widetilde{u}, \lambda)$ just depends on ϕ_{is} and ψ_{is} but not on any spatial derivative of ϕ_{is} and ψ_{is}, the transformed temporal constrained flow (3.30) under (3.29) becomes the following system of ordinary differential equations

$$\phi_{t_2}^{(s)} = \widetilde{V}^{(2)}(\widetilde{u}, \lambda_s)\phi^{(s)},\ \psi_{t_2}^{(s)} = -\widetilde{V}^{(2)T}(\widetilde{u}, \lambda_s)\psi^{(s)},\ 1 \le s \le N. \tag{3.33}$$

In order to derive the Liouville integrability of the resulting two constrained flows, let us define a constant coefficient symplectic structure

$$\omega^2 = \sum_{i=1}^{3} Bd\Phi_i \wedge d\Psi_i = \sum_{i=1}^{3}\sum_{s=1}^{N} \mu_s d\phi_{is} \wedge d\psi_{is} \tag{3.34}$$

over the Euclidian space $\mathbb{R}^{6N}$, and then the corresponding Poisson bracket is given by

$$\begin{aligned} \{f, g\} = \omega^2(Idg, Idf) &= \sum_{i=1}^{3}(\langle \frac{\partial f}{\partial \Psi_i}, B^{-1}\frac{\partial g}{\partial \Phi_i}\rangle - \langle \frac{\partial f}{\partial \Phi_i}, B^{-1}\frac{\partial g}{\partial \Psi_i}\rangle) \\ &= \sum_{i=1}^{3}\sum_{s=1}^{N}\mu_s^{-1}\Big(\frac{\partial f}{\partial \psi_{is}}\frac{\partial g}{\partial \phi_{is}} - \frac{\partial f}{\partial \phi_{is}}\frac{\partial g}{\partial \psi_{is}}\Big),\ f, g \in C^\infty(\mathbb{R}^{6N}), \end{aligned} \tag{3.35a}$$

where the vector field Idf is defined by

$$\omega^2(X, Idf) = df(X), \; X \in T(\mathbb{R}^{6N}).$$

A general Hamiltonian system with a Hamiltonian H defined over the symplectic manifold $(\mathbb{R}^{6N}, \omega^2)$ is given by

$$\Phi_{i,t} = \{\Phi_i, H\} = -B^{-1}\frac{\partial H}{\partial \Psi_i}, \; \Psi_{i,t} = \{\Psi_i, H\} = B^{-1}\frac{\partial H}{\partial \Phi_i}, \; 1 \le i \le 3, \tag{3.36}$$

where t is taken as the evolution variable. For presenting Lax representations of the constrained flows, we need a square matrix Lax operator

$$L(\lambda) = \Gamma + D(\lambda), \tag{3.37}$$

with Γ being defined by (3.21) and $D(\lambda)$, by

$$D(\lambda) = (D_{ij}(\lambda))_{3\times 3}, \; D_{ij}(\lambda) = \sum_{s=1}^{N} \frac{\mu_s}{\lambda - \lambda_s}\phi_{is}\psi_{js}, \; 1 \le i, j \le 3. \tag{3.38}$$

Theorem 1. *Under the symplectic structure (3.34), the spatial constrained flow (3.29) and the temporal constrained flow (3.33) of the multicomponent AKNS equations (3.15) are Hamiltonian systems with the evolution variables x and t_2, and the Hamiltonians*

$$H^x = 2\langle A\Phi_1, B\Psi_1\rangle - \sum_{i=1}^{2}\langle A\Phi_{i+1}, B\Psi_{i+1}\rangle + \sum_{j=1}^{2}\frac{3}{\gamma_1 - \gamma_{j+1}}\langle \Phi_1, B\Psi_{j+1}\rangle\langle \Phi_{j+1}, B\Psi_1\rangle, \tag{3.39}$$

$$\begin{aligned} H^{t_2} = {} & -2\langle A^2\Phi_1, B\Psi_1\rangle + \sum_{i=1}^{2}\langle A^2\Phi_{i+1}, B\Psi_{i+1}\rangle + \sum_{i=1}^{2}\widetilde{q}_i\langle A\Phi_{i+1}, B\Psi_1\rangle \\ & + \sum_{j=1}^{2}\widetilde{r}_j\langle A\Phi_1, B\Psi_{j+1}\rangle + \frac{1}{3}\sum_{i=1}^{2}\widetilde{q}_i\widetilde{r}_i\langle \Phi_1, B\Psi_1\rangle - \frac{1}{3}\sum_{i,j=1}^{2}\widetilde{q}_i\widetilde{r}_j\langle \Phi_{i+1}, B\Psi_{j+1}\rangle, \end{aligned} \tag{3.40}$$

respectively, where $\widetilde{q}_i$ and $\widetilde{r}_i$ are given by (3.27) and A is defined by

$$A = diag(\lambda_1, \lambda_2, \cdots, \lambda_N). \tag{3.41}$$

Moreover, the constrained flows (3.29) and (3.33) admit the Lax representations

$$(L(\lambda))_x = [U(\widetilde{u}, \lambda), L(\lambda)], \; (L(\lambda))_{t_2} = [\widetilde{V}^{(2)}(\widetilde{u}, \lambda), L(\lambda)], \tag{3.42}$$

respectively, where $L(\lambda)$ is given by (3.37) and (3.38), and $U(\widetilde{u}, \lambda)$ and $\widetilde{V}^{(2)}(\widetilde{u}, \lambda)$ are two constrained spectral matrices generated from U and $V^{(2)}$.

Proof: A direct but long calculation can show the Hamiltonian structures of the spatial constrained flow (3.29) and the temporal constrained flow (3.33) with H^x and H^{t_2} defined by (3.39) and (3.40).

Let us then check the Lax representations. By use of the spatial constrained flow (3.29), we can make the following computation:

$$
\begin{aligned}
(L(\lambda))_x &= \sum_{s=1}^{N} \frac{\mu_s}{\lambda-\lambda_s}(\phi_x^{(s)}\psi^{(s)T} + \phi^{(s)}\psi_x^{(s)T}) \\
&= \sum_{s=1}^{N} \frac{\mu_s}{\lambda-\lambda_s}(U(\widetilde{u},\lambda_s)\phi^{(s)}\psi^{(s)T} - \phi^{(s)}\psi^{(s)T}U(\widetilde{u},\lambda_s)) \\
&= \sum_{s=1}^{N} \frac{\mu_s}{\lambda-\lambda_s}[U(\widetilde{u},\lambda_s), \phi^{(s)}\psi^{(s)T}] \\
&= [U(\widetilde{u},\lambda), L(\lambda)-\Gamma] - [U_0, \sum_{s=1}^{N}\mu_s\phi^{(s)}\psi^{(s)T}] \\
&= [U(\widetilde{u},\lambda), L(\lambda)] + [\Gamma, U(\widetilde{u},\lambda)] - [U_0, \sum_{s=1}^{N}\mu_s\phi^{(s)}\psi^{(s)T}] \\
&= [U(\widetilde{u},\lambda), L(\lambda)] + [\Gamma, U_1(\widetilde{u})] - [U_0, \sum_{s=1}^{N}\mu_s\phi^{(s)}\psi^{(s)T}].
\end{aligned}
$$

In the last step above, we have used $[\Gamma, U_0] = 0$. Now it follows that $(L(\lambda))_x = [U(\widetilde{u},\lambda), L(\lambda)]$ if and only if

$$
[\Gamma, U_1(\widetilde{u})] = [U_0, \sum_{s=1}^{N}\mu_s\phi^{(s)}\psi^{(s)T}].
$$

This equality equivalently requires the potential constraints shown in (3.27). Therefore, the spatial constrained flow (3.29) admit the Lax representation defined as in (3.42).

It is also direct to prove the other Lax representation

$$
(L(\lambda))_{t_2} = [\widetilde{V}^{(2)}(\widetilde{u},\lambda), L(\lambda)],
$$

and so we do not go to the detail. Thus, the proof of the theorem is completed. ∎

Associated with the Lax operator $L(\lambda)$ defined by (3.37), there is a functionally independent and involutive system of polynomial functions [8]: $\{F_{is} \,|\, 1 \le i \le 3,\ 1 \le s \le N\}$, defined as follows

$$
\mathcal{F}_i(\lambda) = \sum_{l=0}^{\infty} F_{il}\lambda^{-l}, \ 1 \le i \le 3, \tag{3.43a}
$$

$$
\det(\nu I_3 - L(\lambda)) = \nu^3 - \mathcal{F}_1(\lambda)\nu^2 + \mathcal{F}_2(\lambda)\nu - \mathcal{F}_3(\lambda). \tag{3.43b}
$$

It follows from Theorem 1 that we can have the following result on the Liouville integrability of the constrained flows (3.29) and (3.33).

Theorem 2. *The spatial constrained flow (3.29) and the temporal constrained flow (3.33) of the multicomponent AKNS equations (3.15) are Liouville integrable. Moreover, they possess an involutive system of integrals of motion being functionally independent:* $\{F_{is} \,|\, 1 \le i \le 3,\ 1 \le s \le N\}$*, defined by (3.43).*

It follows that the potential constraints (3.27) present an integrable decomposition and thus show the integrability by quadratures for the multicomponent AKNS equations (3.15). Furthermore, the resulting solutions from (3.27) are involutive solutions to the multicomponent AKNS equations (3.15), because we can prove that $\{H^x, H^{t_2}\} = 0$.

4 Application to multi-wave interaction equations

4.1 Multi-wave interaction hierarchy

Let us consider the 3×3 matrix AKNS spectral problem:

$$\phi_x = U(u,\lambda)\,\phi,\ U = \begin{bmatrix} \alpha_1\lambda & u_{12} & u_{13} \\ u_{21} & \alpha_2\lambda & u_{23} \\ u_{31} & u_{32} & \alpha_3\lambda \end{bmatrix} = \lambda U_0 + U_1,\ \frac{\partial U_0}{\partial\lambda} = \frac{\partial U_1}{\partial\lambda} = 0, \tag{4.1}$$

where α_1, α_2 and α_3 are distinct constants, and the eigenfunction ϕ and the potential u are defined by

$$\phi = (\phi_1, \phi_2, \phi_3)^T,\ u = \rho(U_1) = (u_{21}, u_{12}, u_{31}, u_{13}, u_{32}, u_{23})^T. \tag{4.2}$$

To construct a soliton hierarchy associated with the spectral problem (4.1). Similarly, we first solve the adjoint equation for W:

$$W_x = [U, W],\ W = (W_{ij})_{3\times 3}. \tag{4.3}$$

We look for a formal solution of the form

$$W = \sum_{l=0}^{\infty} W_l \lambda^{-l},\ W_l = (W_{ij}^{(l)})_{3\times 3}, \tag{4.4}$$

and then the adjoint equation (4.3) is equivalent to

$$[U_0, W_0] = 0,\ W_{l,x} = [U_1, W_l] + [U_0, W_{l+1}],\ l \ge 0, \tag{4.5}$$

which gives us the following recursion relation

$$W_{ii,x}^{(0)} = 0,\ W_{ij}^{(0)} = 0,\ i \ne j, \tag{4.6a}$$

$$W_{ij,x}^{(l)} + u_{ij}(W_{ii}^{(l)} - W_{jj}^{(l)}) + \sum_{\substack{k=1 \\ k\ne i,j}}^{3} (u_{kj}W_{ik}^{(l)} - u_{ik}W_{kj}^{(l)}) - (\alpha_i - \alpha_j)W_{ij}^{(l+1)} = 0,\ i \ne j, \tag{4.6b}$$

$$W_{ii,x}^{(l+1)} = \sum_{\substack{k=1 \\ k\ne i}}^{3} (u_{ik}W_{ki}^{(l+1)} - u_{ki}W_{ik}^{(l+1)}), \tag{4.6c}$$

where $1 \le i, j \le 3$ and $l \ge 0$.

We set the initial values

$$W_{ii}^{(0)} = \beta_i = \text{const.},\ 1 \le i \le 3, \tag{4.7}$$

where β_i, $1 \le i \le 3$, are three arbitrary constants, and require that

$$W_{ij}^{(l)}|_{u=0} = 0,\ 1 \le i, j \le 3,\ l \ge 1. \tag{4.8}$$

This condition (4.8) means to take all constants of integration to be zero while using (4.6) to determine W, and thus all matrices $W_l,\ l \ge 1$, will be uniquely determined. In particular, we can have that

$$W_{ii}^{(1)} = 0,\ 1 \le i \le 3,\ W_{ij}^{(1)} = \frac{\beta_i - \beta_j}{\alpha_i - \alpha_j} u_{ij},\ 1 \le i \ne j \le 3;$$

$$W_{ii}^{(2)} = \sum_{\substack{k=1\\k\ne i}}^{3} \frac{\beta_k - \beta_i}{(\alpha_k - \alpha_i)^2} u_{ik} u_{ki},\ 1 \le i \le 3,$$

$$W_{ij}^{(2)} = \frac{\beta_i - \beta_j}{(\alpha_i - \alpha_j)^2} u_{ij,x} + \frac{1}{\alpha_i - \alpha_j} \sum_{\substack{k=1\\k\ne i,j}}^{3} \left(\frac{\beta_k - \beta_i}{\alpha_k - \alpha_i} - \frac{\beta_k - \beta_j}{\alpha_k - \alpha_j}\right) u_{ik} u_{kj},\ 1 \le i \ne j \le 3.$$

It is easy to see that the recursion relation (4.6) can lead to

$$\begin{aligned} &2u_{ij}\partial^{-1}u_{ij}W_{ji}^{(l)} + (\partial - 2u_{ij}\partial^{-1}u_{ji})W_{ij}^{(l)} \\ &\quad + \sum_{\substack{k=1\\k\ne i,j}}^{3} \left[u_{ij}\partial^{-1}u_{ik}W_{ki}^{(l)} + (u_{kj} - u_{ij}\partial^{-1}u_{ki})W_{ik}^{(l)}\right] \\ &\quad + \sum_{\substack{k=1\\k\ne i,j}}^{3} \left[u_{ij}\partial^{-1}u_{kj}W_{jk}^{(l)} - (u_{ik} + u_{ij}\partial^{-1}u_{jk})W_{kj}^{(l)}\right] = (\alpha_i - \alpha_j)W_{ij}^{(l+1)}, \end{aligned}$$

where $1 \le i \ne j \le 3$ and $l \ge 1$. This can be written as the Lenard form

$$MG_{l-1} = JG_l,\ l \ge 1, \tag{4.9}$$

where J is a constant operator

$$J = \operatorname{diag}\Big((\alpha_1 - \alpha_2)\sigma_0, (\alpha_1 - \alpha_3)\sigma_0, (\alpha_2 - \alpha_3)\sigma_0\Big),\ \sigma_0 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \tag{4.10}$$

and $G_l = \rho(W_{l+1})$ is given by

$$G_l = (W_{21}^{(l+1)}, W_{12}^{(l+1)}, W_{31}^{(l+1)}, W_{13}^{(l+1)}, W_{32}^{(l+1)}, W_{23}^{(l+1)})^T,\ l \ge 0. \tag{4.11}$$

These two operators J and M can be shown to be a Hamiltonian pair (see [17] for definition).

We go on to introduce the associated spectral problems with the spectral problem (4.1):

$$\phi_{t_n} = V^{(n)}\phi,\ V^{(n)} = V^{(n)}(u,\lambda) = (\lambda^n W)_+ = \sum_{i=0}^{n} W_i\lambda^{n-i},\ n \ge 1. \tag{4.12}$$

Noting (4.5), we can compute that

$$\begin{aligned} &[U, V^{(n)}] = [\lambda U_0 + U_1, \sum_{l=0}^{n} \lambda^{n-l} W_l] \\ &= \sum_{l=0}^{n} [U_0, W_l]\lambda^{n+1-l} + \sum_{l=0}^{n} [U_1, W_l]\lambda^{n-l} \\ &= \sum_{l=0}^{n-1} [U_0, W_{l+1}]\lambda^{n-l} + \sum_{l=0}^{n} [U_1, W_l]\lambda^{n-l}, \end{aligned}$$

where we have used $[U_0, W_0] = 0$. Therefore, under the isospectral conditions

$$\lambda_{t_n} = 0,\ n \ge 1, \tag{4.13}$$

the compatibility conditions of the spectral problem (4.1) and the associated spectral problems (4.12) become

$$U_{1t_n} = W_{nx} - [U_1, W_n] = [U_0, W_{n+1}].$$

This gives rise to the so-called 3×3 AKNS soliton hierarchy

$$u_{t_n} = K_n := JG_n,\ n \ge 1, \tag{4.14}$$

where J and $G_n = \rho(W_{n+1})$ are defined by (4.10) and (4.11), respectively.

Similarly, applying the trace identity [16], the soliton hierarchy (4.14) has a bi-Hamiltonian formulation

$$u_{t_n} = K_n = J\frac{\delta \tilde{H}_{n+1}}{\delta u} = M\frac{\delta \tilde{H}_n}{\delta u},\ n \ge 1, \tag{4.15}$$

where the Hamiltonian functionals $\tilde{H}_l$ are defined by

$$\tilde{H}_l := -\frac{1}{l}\int (\alpha_1 W_{11}^{(l+1)} + \alpha_2 W_{22}^{(l+1)} + \alpha_3 W_{33}^{(l+1)})dx,\ l \ge 1. \tag{4.16}$$

The first nonlinear system in this soliton hierarchy (4.15) is the multi-wave interaction equations [18]

$$u_{ij,t_1} = \frac{\beta_i - \beta_j}{\alpha_i - \alpha_j} u_{ij,x} + \sum_{\substack{k=1 \\ k \ne i,j}}^{3} (\frac{\beta_i - \beta_k}{\alpha_i - \alpha_k} - \frac{\beta_k - \beta_j}{\alpha_k - \alpha_j}) u_{ik} u_{kj},\ 1 \le i \ne j \le 3, \tag{4.17}$$

which contain three-wave interaction equations arising in fluid dynamics and plasma physics [19], if U is chosen to be an anti-Hermitian matrix.

4.2 Adjoint symmetry constraint

Let us assume, for simplicity, that

$$c_{ij} = \frac{\beta_i - \beta_j}{\alpha_i - \alpha_j},\ d_{ijkl} = c_{ij} - c_{kl},\ 1 \le i \ne j \le 3,\ 1 \le k \ne l \le 3. \tag{4.18}$$

A direct computation shows

$$K_1' = \begin{bmatrix} c_{21}\partial_x & 0 & d_{3231}u_{23} & 0 & 0 & d_{3231}u_{31} \\ 0 & c_{12}\partial_x & 0 & d_{3132}u_{32} & d_{3132}u_{13} & 0 \\ d_{2321}u_{32} & 0 & c_{31}\partial_x & 0 & d_{2321}u_{21} & 0 \\ 0 & d_{2123}u_{23} & 0 & c_{13}\partial_x & 0 & d_{2123}u_{12} \\ 0 & d_{1312}u_{31} & d_{1312}u_{12} & 0 & c_{32}\partial_x & 0 \\ d_{1213}u_{13} & 0 & 0 & d_{1213}u_{21} & 0 & c_{23}\partial_x \end{bmatrix} \tag{4.19}$$

and thus its adjoint operator reads as

$$(K_1')^\dagger = \begin{bmatrix} -c_{21}\partial_x & 0 & d_{2321}u_{32} & 0 & 0 & d_{1213}u_{13} \\ 0 & -c_{12}\partial_x & 0 & d_{2123}u_{23} & d_{1312}u_{31} & 0 \\ d_{3231}u_{23} & 0 & -c_{31}\partial_x & 0 & d_{1312}u_{12} & 0 \\ 0 & d_{3132}u_{32} & 0 & -c_{13}\partial_x & 0 & d_{1213}u_{21} \\ 0 & d_{3132}u_{13} & d_{2321}u_{21} & 0 & -c_{32}\partial_x & 0 \\ d_{3231}u_{31} & 0 & 0 & d_{2123}u12 & 0 & -c_{23}\partial_x \end{bmatrix}, \tag{4.20}$$

where K_1 is defined by (4.14).

To carry out binary nonlinearization, we need to present an adjoint symmetry for the multi-wave interaction equations (4.17). It is easy to see by using (4.20) that the first vector function,

$$G_0 = (c_{12}u_{21}, c_{12}u_{12}, c_{13}u_{31}, c_{13}u_{13}, c_{23}u_{32}, c_{23}u_{23})^T,$$

among the vector functions G_n, $n \ge 0$, defined by (4.11), is an adjoint symmetry of the multi-wave interaction equations (4.17). However, for the multi-wave interaction equations (4.17), we can introduce a more general Lie point adjoint symmetry:

$$\bar{G}_0 := J^{-1}\rho([\Gamma, U_1]),\ \Gamma = \mathrm{diag}(\gamma_1, \gamma_2, \gamma_3), \tag{4.21}$$

where γ_1, γ_2 and γ_3 are arbitrary distinct constants. Noting

$$\bar{G}_0 = (\bar{c}_{12}u_{21}, \bar{c}_{12}u_{12}, \bar{c}_{13}u_{31}, \bar{c}_{13}u_{13}, \bar{c}_{23}u_{32}, \bar{c}_{23}u_{23})^T,\ \bar{c}_{ij} = \frac{\gamma_i - \gamma_j}{\alpha_i - \alpha_j}, \tag{4.22}$$

this can be directly proved by checking

$$\bar{G}_{0,t_1} = -(K_1')^\dagger \bar{G}_0,$$

while u solves $u_{t_1} = K_1$. In particular, G_0 is an example of $\bar{G}_0$ with $\Gamma = W_0$, if $\beta_i \neq \beta j$, $1 \le i \neq j \le 3$. Now make the following Bargmann type adjoint symmetry constraint

$$\bar{G}_0 = \sum_{s=0}^{N} \mu_s \psi^{(s)T} \frac{\partial U(u, \lambda_s)}{\partial u} \phi^{(s)}, \tag{4.23}$$

where μ_s, $1 \le s \le N$, are arbitrary nonzero constants, and the eigenfunctions and adjoint functions are assumed to be

$$\phi^{(s)} = (\phi_{1s}, \phi_{2s}, \phi_{3s})^T,\ \psi^{(s)} = (\psi_{1s}, \psi_{2s}, \psi_{3s})^T,\ 1 \le s \le N. \tag{4.24}$$

The above constraint (4.23) is equivalent to

$$[\Gamma, U_1] = [U_0, \sum_{s=1}^{N} \mu_s \phi^{(s)} \psi^{(s)T}].$$

When N and μ_s vary, (4.23) provides us with a set of adjoint symmetry constraints of the multi-wave interaction equations (4.17).

We still use two diagonal matrices:

$$A = \mathrm{diag}(\lambda_1, \cdots, \lambda_N),\ B = \mathrm{diag}(\mu_1, \cdots, \mu_N). \tag{4.25}$$

Solving the Bargmann adjoint symmetry constraint (4.23) for u, we obtain

$$u_{ij} = \widetilde{u}_{ij} := \frac{\alpha_i - \alpha_j}{\gamma_i - \gamma_j} \langle \Phi_i, B\Psi_j \rangle,\ 1 \le i \neq j \le 3, \tag{4.26}$$

where B is given by (4.25), and Φ_i and Ψ_i are defined by

$$\Phi_i = (\phi_{i1}, \phi_{i2}, \cdots, \phi_{iN})^T,\ \Psi_i = (\psi_{i1}, \psi_{i2}, \cdots, \psi_{iN})^T,\ 1 \le i \le 3, \tag{4.27}$$

Two constrained flows for the multi-wave interaction equations (4.17) read as

$$\phi_x^{(s)} = U(\widetilde{u}, \lambda_s)\phi^{(s)},\ \psi_x^{(s)} = -U^T(\widetilde{u}, \lambda_s)\psi^{(s)},\ 1 \le s \le N, \tag{4.28}$$

and

$$\phi_{t_1}^{(s)} = V^{(1)}(\widetilde{u}, \lambda_s)\phi^{(s)},\ \psi_{t_1}^{(s)} = -V^{(1)T}(\widetilde{u}, \lambda_s)\psi^{(s)},\ 1 \le s \le N, \tag{4.29}$$

where $\widetilde{u}$ is defined by

$$\widetilde{u} = \rho((\widetilde{u}_{ij})_{3\times 3}) = (\widetilde{u}_{21}, \widetilde{u}_{12}, \widetilde{u}_{31}, \widetilde{u}_{13}, \widetilde{u}_{32}, \widetilde{u}_{23})^T,$$

and $V^{(1)}$ is defined by

$$V^{(1)}(u, \lambda) = \lambda W_0 + W_1 = \begin{bmatrix} \beta_1 \lambda & W_{12}^{(1)} & W_{13}^{(1)} \\ W_{21}^{(1)} & \beta_2 \lambda & W_{23}^{(1)} \\ W_{31}^{(1)} & W_{32}^{(1)} & \beta_3 \lambda \end{bmatrix},\ W_{ij}^{(1)} = \frac{\beta_i - \beta_j}{\alpha_i - \alpha_j} u_{ij}. \tag{4.30}$$

Similarly to Theorem 1, we can prove the following result for the multi-wave interaction equations (4.17).

Theorem 3. *Under the symplectic structure (3.34), the spatial constrained flow (4.28) and the temporal constrained flow (4.29) for the multi-wave interaction equations (4.17) are Hamiltonian systems with the evolution variables x and t_1, and the Hamiltonians*

$$H^x = -\sum_{k=1}^{3} \alpha_k \langle A\Phi_k, B\Psi_k\rangle - \sum_{1\le k<l\le 3} \frac{\alpha_k - \alpha_l}{\gamma_k - \gamma_l} \langle \Phi_k, B\Psi_l\rangle\langle \Phi_l, B\Psi_k\rangle, \tag{4.31}$$

$$H^{t_1} = -\sum_{k=1}^{3} \beta_k \langle A\Phi_k, B\Psi_k\rangle - \sum_{1\le k<l\le 3} \frac{\beta_k - \beta_l}{\gamma_k - \gamma_l} \langle \Phi_k, B\Psi_l\rangle\langle \Phi_l, B\Psi_k\rangle, \tag{4.32}$$

respectively, where A and B are defined by (4.25), and Φ_i and Ψ_i, $1 \le i \le n$, are defined by (4.27). Moreover, they admit the Lax representations:

$$(L(\lambda))_x = [U(\widetilde{u}, \lambda), L(\lambda)],\ (L(\lambda))_{t_1} = [V^{(1)}(\widetilde{u}, \lambda), L(\lambda)], \tag{4.33}$$

respectively, where $L(\lambda)$, $U(\lambda)$, and $V^{(1)}(\lambda)$ are given by (3.37) and (3.38), (4.1) and (4.30).

Now it is a direct computation to verify the following theorem on the Liouville integrability of the spatial constrained flows (4.28) and the temporal constrained flow (4.29).

Theorem 4. *The spatial constrained flows (4.28) and the temporal constrained flow (4.29) of the multi-wave interaction equations have the common involutive integrals of motion: F_{il}, $1 \le i \le 3$, $l \ge 1$, defined by (3.43), of which the functions F_{is}, $1 \le i \le 3$, $1 \le s \le N$, are functionally independent. Therefore, the constrained flows (4.28) and (4.29) are Liouville integrable.*

The potential constraints (4.26) present an integrable decomposition

$$u_{ij}(x,t) = \frac{\alpha_i - \alpha_j}{\gamma_i - \gamma_j} \langle \Phi_i(x,t), B\Psi_j(x,t)\rangle,\ 1 \le i \ne j \le 3, \tag{4.34}$$

for the multi-wave interaction equations (4.17). This also shows the integrability by quadratures for the multi-wave interaction equations (4.17), since Φ_i and Ψ_i can be determined by quadratures. There are many arbitrary constants involved in the resulting solutions. Moreover, the solutions generated above are all involutive solutions to the multi-wave interaction equations (4.17), since two Hamiltonian flows of (4.28) and (4.29) commute, due to $\{H^x, H^{t_1}\} = 0$. The results generated from the adjoint symmetry constraints (4.23) are the same as those generated from the symmetry constrains [2, 8, 20].

5 Concluding remarks

We remark that for the multicomponent AKNS equations (3.15), the introduction of the adjoint symmetry $\bar{G}_0$ is very crucial for the success of making integrable decompositions. If we choose G_0 as the required adjoint symmetry, then we can not show that the resulting constrained flows are Liouville integrable. Therefore, adjoint symmetry constraints generalizes the idea of carrying out nonlinearization by symmetry constraints, which was also seen in the case of the multi-wave interaction equations (4.17).

We also mention that for the Neumann type adjoint symmetry constraints, we can often use the Moser constraint technique to show the Liouville integrability for the resulting constrained flows, although there are some exceptions which can not lead to the Liouville integrability of the Neumann problem associated with soliton equations. The Ostrogradsky type adjoint symmetry constraints with involved Lie-Bäcklund symmetries having non-degenerate Hamiltonians can also result in the Liouville integrable constrained flows, under the help of the Ostrogradsky coordinates, but the case of degenerate Hamiltonians needs particular consideration for introducing canonical variables for the resulting constrained flows [21]. Like symmetry constraints [22], the whole theory of our adjoint symmetry constraints also can be applied to the perturbation systems, which are associated with higher-order matrix spectral problems.

Acknowledgments

This work was supported by the Research Grants Council of Hong Kong (CERG 9040466), the City University of Hong Kong (SRGs 7001041 and 7001178), the National Science Foundation of China (Project no. 19801031), the special grant of Excellent PhD Thesis (Project No. 200013), the Special Funds for Major State Basic Research Projects (Project No. G2000077301), and the 'Qinglan Gongcheng' of Jiangsu Province, China. One of the authors (Zhou) also acknowledges warm hospitality at the City University of Hong Kong.

References

[1] Ma W X and Strampp W, An Explicit Symmetry Constraint for the Lax Pairs and the Adjoint Lax Pairs of AKNS Systems, *Phys. Lett.* A **185** (1994), 277–286.

[2] Ma W X, Binary Bargmann Symmetry Constraints of Soliton Equations, in Proceedings of the Third World Congress of Nonlinear Analysts, *Nonlinear Analysis* **47** (2001), 5199–5211.

[3] Cao C W and Geng X G, Classical integrable systems generated through nonlinearization of eigenvalue problems, in Nonlinear Physics, Editors Gu C H, Li Y S and Tu G Z, Springer - Berlin, 1990, 68–78.

[4] Ma W X, The Confocal Involutive System and the Integrability of the Nonlinearized Lax Systems of AKNS Hierarchy, in Nonlinear Physics, Editors Gu C H, Li Y S and Tu G Z, Springer - Berlin, 1990, 79–84.

[5] Zeng Y B and Ma W X, Separation of Variables for Soliton Equations via Their Binary Constrained Flows, *J. Math. Phys.* **40** (1999) 6526–6557.

[6] Ma W X and Zeng Y B, Binary Constrained Flows and Separation of Variables for Soliton Equations, in Proceedings of the Conference on Integrable Systems, in celebration of Martin D. Kruskal's 75th Birthday, Adelaide, Australia, 2000, to appear

[7] Ma W X, Symmetry Constraint of MKdV Equations by Binary Nonlinearization, *Physica A* **219** (1995) 467–481.

[8] Ma W X and Zhou Z X, Binary Symmetry Constraints of $\mathcal{N}$-wave Interaction Equations in $1+1$ and $2+1$ Dimensions, *J. Math. Phys.* **42** (2001) 4345–4382.

[9] Ma W X and Geng X G, Bäcklund Transformations of Soliton Systems from Symmetry Constraints, in Proceedings of the Workshop on Bäcklund & Darboux Transformations, Halifax, Canada, 1999, to appear

[10] Xu X X, New Finite-dimensional Integrable Systems by Binary Nonlinearization of WKI Equations, *Chinese Phys. Lett.* **12** (1995) 513–516.

[11] Ma W X, Ding Q, Zhang W G and Lu B G, Binary Nonlinearization of Lax Pairs of Kaup-Newell Soliton Hierarchy, *Il Nuovo Cimento B* **111** (1996) 1135–1149.

[12] Ma W X, Fuchssteiner B and Oevel W, A 3×3 Matrix Spectral Problem for AKNS Hierarchy and Its Binary Nonlinearization, *Physica A* **233** (1996) 331–354.

[13] Ma W X and Fuchssteiner B, Binary Nonlinearization of Lax Pairs, in Nonlinear Physics: Theory and Experiment, Editors Alfinito E, Boiti M, Martina L and Pempinelli F, World Scientific - River Edge, NJ, 1996, 217–224.

[14] Ablowitz M J, Kaup D J, Newell A C and Segur H, The Inverse Scattering Transform-Fourier Analysis for Nonlinear Problems, *Stud. Appl. Math.* **53** (1974) 249–315.

[15] Ma W X, Some aspects of nonlinear integrable systems, Ph.D. dissertation, Academia Sinica, Beijing, China, 1990

[16] Tu G Z, The Trace Identity, a Powerful Tool for Constructing the Hamiltonian Structure of Integrable Systems, *J. Math. Phys.* **30** (1989) 330–338.

[17] Fuchssteiner B and Fokas A C, Symplectic Structures, Their Bäcklund Transformations and Hereditary Symmetries, *Physica D* **4** (1981/82) 47–66.

[18] Ablowitz M J and Haberman R, Resonantly Coupled Nonlinear Evolution Equations, *J. Math. Phys.* **16** (1975) 2301–2305.

[19] Kaup D J, The Three-wave Interaction - a Nondispersive Phenomenon, *Stud. Appl. Math.* **55** (1976) 9–44.

[20] Wu Y T and Geng X G, A Finite-dimensional Integrable System Associated with the Three-wave Interaction Equations, *J. Math. Phys.* **40** (1999) 3409–3430.

[21] Li Y S and Ma W X, Binary Nonlinearization of AKNS Spectral Problem under Higher-order Symmetry Constraints, *Chaos, Solitons and Fractals* **11** (2000) 697–710.

[22] Ma W X and Zhou R G, Nonlinearization of Spectral Problems for the Perturbation KdV Systems, *Physica A* **296** (2001) 60–74.