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On a q-Analog of ADHMN Construction for Self-Dual Yang-Mills

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Abstract

It is known that many integrable systems can be reduced from self-dual Yang-Mills equations. The formal solution space to the self-dual Yang-Mills equations is given by the so called ADHM construction, in which the solution space are graded by vector spaces with dimensionality concerning topological index. When we consider a reduced self-dual system such as the Bogomol'nyi equations, in terms of ADHM construction, we need to incorporate an infinite dimensional vector space, in general. In this paper, we reformulate the ADHM construction by introducing various infinite dimensional vector spaces taking into account the reduction of self-dual system.

1 Introduction

There are many integrable systems which can be obtained by a reduction of the self-dual Yang-Mills (SDYM) equations, defined on $\mathbb{R}^4$. In case of the reduction to 2 (or $1 + 1$) dimensions, we may regard the SDYM a simple zero curvature condition, so that there appear standard soliton systems such as Korteweg-de Vries (KdV), Nonlinear Schrödinger, Sine-Gordon equation, and so on [1, 2]. Another interesting case is the reduction to 3 (or $2 + 1$) dimensions, in which there emerge the Bogoyavlenski (($2 + 1$)-dimensional KdV)[3], the Bogomol'nyi equation [4], *etc.*, which are considered as integrable. This is not so surprising fact because the SDYM equations are shown to be completely solvable by defining their formal solution space graded by topological index, the ADHM construction [5, 6].

Although the solution space of SDYM equations are found completely, our knowledge of analytic descriptions to exact solutions is insufficient, since the ADHM method does not give an explicit procedure to fix functional forms of instantons in general. In fact the well-known analytic solutions, 'tHooft instantons [7], are not the most general solutions, they are obtained through imposing some symmetries in their configurations. Hence, to understand more about the SDYM equations, and/or the reduced systems, we need to investigate further concrete examples of exact analytic solutions. In this context the author and the collaborator obtained a new family of solutions to SDYM through constructing a q-analog of the ADHM method [8].

The ADHM construction is, as we shall see later, designed so that each solution with topological index (instanton number) k, say, is assigned by a vector in an $(n+k)$-dimensional linear vector space, where the gauge group is fixed $Sp(n)$, and the vector is determined by a certain finite dimensional linear algebraic equation. The advantage of the method is that we can treat linear system rather than the original nonlinear PDE.

Now, when we consider the dimensionally reduced SDYM systems, how does the ADHM construction work? For this subject, Nahm introduced an infinite dimensional $\mathcal{L}^2[I]$ vector space to find a solution to the Bogomol'nyi equation [9, 10], where I is an interval on which the $\mathcal{L}^2$ functions defined. Hereafter we call these formulations the ADHMN construction. We can intuitively understand the necessity for the infinite dimensional vector space as follows, see also section 2. If we perform a reduction to 3 dimensions by requiring "time" translational invariance, then the solutions to the reduced system have infinite number of instantons aligned on some time-axes, except for the vacuum configuration. Hence we have to assign an infinite dimensional vector space to the solutions in the reduced system, since the dimensionality of the vector space associated to each instanton solution is concerned with the instanton number, as mentioned above. As a result of the fact that the vector is in $\mathcal{L}^2[I]$, the linear equation determining it turns out to be a linear differential equation instead of an algebraic one.

In this paper we consider how to merge infinite dimensional vector spaces into the ADHMN construction. Of course the $\mathcal{L}^2[I]$ vector space introduced by Nahm is a standard example. Another example is given in ref.[8] in which a function space defined on a multiplicatively located discrete point set such as $\{\pm 1/2, \pm q/2, \pm q^2/2, \pm q^3/2, \dots\} =: I_q$, where $q \in (0,1)$, is considered instead of $\mathcal{L}^2[I]$. We call this infinite dimensional function space an $\ell^2[I_q]$ vector space, say. In this "q-analog" formulation of the ADHMN construction, a vector associated to SD solution is determined by a linear q-difference equation. This paper deals with further extension of a q-analogous formulation, a generalized $\hat{\mathcal{L}}^2[I]$ formulation, where the same q-difference equation in the $\ell^2[I_q]$ formulation appears, however we will see a degrees of freedom of "pseudo-constant" coming out in the solution. In another point of view, a difference analog of the Nahm equations is known, the discrete Nahm equations [11], whose solutions correspond to a monopole in hyperbolic 3-space, a hyperbolic monopole [12]. We will comment on the relation between these distinct difference analogs in the concluding remarks.

This paper is organized as follows. In the next section we review the ADHMN construction briefly. In section 3 we present its q-analog formulation with focusing on the linear q-difference equation. Finally, section 4 is given for concluding remarks.

2 The ADHMN construction

In this section, we give a brief review to the ADHMN construction. In subsection 2.1 we consider the ADHM formulation for instanton solutions [5, 6] to SDYM (or anti-SD one, hereafter (A)SD in short) equations in $\mathbb{R}^4$, and in subsection 2.2, the Nahm formulation for monopole solutions [10] to Bogomol'nyi equations in $\mathbb{R}^3$.

2.1 The ADHM formulation for SDYM in $\mathbb{R}^4$

The standard variational equations in classical $Sp(n)$ Yang-Mills theories in $\mathbb{R}^4$, $D_\mu F_{\mu\nu} = 0$, are automatically satisfied by the (A)SD equations $F_{\mu\nu} = \pm\tilde{F}_{\mu\nu}$ due to the Bianchi identity $D_\mu \tilde{F}_{\mu\lambda} = 0$. The ADHM construction gives (A)SD configurations with a finite instanton number k, which are obtained through a vector v of an $(n+k)$-dimensional quaternion vector space V^{n+k} with inner product $< w, v >:= w^\dagger v$. The connection one-form is given by

$$A(x) = i < v, dv >= iv^\dagger(x)dv(x) \tag{2.1}$$

and is (A)SD due to the theorem:

Theorem 1. [5] For $Sp(n)(\supset U(n),\ O(n))$ gauge group, the $(n+k)\times n$ matrix v enjoying a linear equation $\Delta^\dagger v = 0$ and normalization $v^\dagger v = 1_n$ yield (A)SD gauge fields, if the matrix $\Delta^\dagger\Delta$ is quaternionic real and invertible. Here the $(n+k)\times k$ matrix Δ is assumed to be linear in x, *i.e.*, $\Delta = a + bx$.

We can trace the proof of this theorem in the following way[1]. First of all, we notice that the $n+k$ column vectors of the matrices v and Δ span V^{n+k}, which can be understood from the normalization $v^\dagger v = 1_n$, the invertibility of the matrix $\Delta^\dagger\Delta$, and the linear equation $\Delta^\dagger v =< \Delta, v >= 0$ which implies that the column vectors of v and those of Δ are orthogonal to each other. Then we can find the completeness condition, $1_{n+k} = v(v^\dagger v)^{-1}v^\dagger + \Delta(\Delta^\dagger\Delta)^{-1}\Delta^\dagger$. This gives the following two projection operators, $P := v(v^\dagger v)^{-1}v^\dagger = vv^\dagger$ and $P' := \Delta(\Delta^\dagger\Delta)^{-1}\Delta^\dagger = \Delta\mathcal{F}\Delta^\dagger$, where $\mathcal{F} := (\Delta^\dagger\Delta)^{-1}$, onto the n and k dimensional subspaces spanned by the column vectors of v and Δ, respectively. We obtain the curvature two-form from (2.1) expressed in terms of P and v as

$$\begin{aligned} F &= dA - iA\wedge A \\ &= iv^\dagger dP \wedge dP\, v \\ &= iv^\dagger b\, dx\, \mathcal{F} \wedge dx^\dagger\, b^\dagger v, \end{aligned} \tag{2.2}$$

provided that the matrices Δ is linear in quaternion x, *i.e.*, $\Delta = a + bx$. Since $\Delta^\dagger\Delta$ is quaternionic real, *i.e.*, each entry is proportional to 1_2, $\mathcal{F}$ is also quaternionic real, hence commutes with the quaternion coordinate x. We find that F is ASD, because

$$dx \wedge dx^\dagger = i\bar{\eta}^j_{\mu\nu}\sigma_j dx_\mu \wedge dx_\nu, \tag{2.3}$$

where $\bar{\eta}^j_{\mu\nu}$ is the 'tHooft ASD tensor [7]. On the other hand, a SD curvature two-form is also derived by exchanging x and $x^\dagger$, which yields the 'tHooft SD tensor $\eta^j_{\mu\nu}$ [7]

$$dx^\dagger \wedge dx = i\eta^j_{\mu\nu}\sigma_j dx_\mu \wedge dx_\nu. \tag{2.4}$$

Note that the gauge group acts on v on the right, $v \to v' = vg$ where $g \in Sp(n)$, then the connection one-form transforms correctly, $A \to A' = g^\dagger A g + g^\dagger i dg$. And that we can

[1] Hereafter, $\dagger$ denotes hermitian conjugation, $\tau_\mu = (1_2, i\sigma_1, i\sigma_2, i\sigma_3)$ and $x_\mu = (x_0, x_1, x_2, x_3)$ are quaternion elements and spacetime coordinates, respectively, $x := \sum_{\mu=0}^3 x_\mu\tau_\mu$, $|x|^2 := \sum_{\mu=0}^3 x_\mu x_\mu$, and $\hat{x} := \sum_{j=1}^3 x_j\sigma_j/r$.

prove that the integral $(16\pi^2)^{-1}\int \operatorname{tr}(F\wedge F)$ gives the instanton number k, hence the action becomes finite as long as $k<\infty$.

To be concrete, we hereafter restrict ourselves to the gauge group $SU(2)\simeq Sp(1)$, that is, $n=1$. In this case the matrix v reduces to a $(1+k)\times 1$ matrix, *i.e.*, an $(1+k)$ column vector with quaternion entries. For the $k=1$ case, we can derive one-instanton solution, known as the BPST solution [16]. We set the matrix Δ as follows,

$$\Delta=\begin{pmatrix} x-i\lambda 1_2 \\ x+i\lambda 1_2 \end{pmatrix}, \tag{2.5}$$

without loss of generality. This leads to the well known connection one-form in regular gauge,

$$A=\bar{\eta}^j_{\mu\nu}\frac{\sigma_j x_\nu}{|x|^2+\lambda^2}dx_\mu. \tag{2.6}$$

For $k>1$ cases, the canonical form [5, 17] of Δ is known to be,

$$\Delta=\begin{pmatrix} \lambda_1 1_2 & \lambda_2 1_2 & \cdots & \lambda_k 1_2 \\ x+\alpha_1 1_2 & 0 & \cdots & 0 \\ 0 & x+\alpha_2 1_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & x+\alpha_k 1_2 \end{pmatrix}. \tag{2.7}$$

These yield the multi-instanton solution with $5k-3$ parameters in singular gauge [7, 5], firstly obtained by 'tHooft,

$$A=\frac{1}{2}\eta^j_{\mu\nu}\sigma_j\partial_\mu \ln\left(1+\sum_{i=1}^{k}\frac{\lambda_i^2}{|x+\alpha_i|^2}\right)dx_\nu. \tag{2.8}$$

The most general multi-instanton solutions admits $8k-3$ $(k>1)$ parameters [5, 6], however no explicit formula for these solutions has never been known.

2.2 The Nahm formulation in $\mathbb{R}^3$

The Bogomol'nyi equations [4], which governs minimum energy configurations in static Yang-Mills-Higgs system, are $D_j\Phi=\pm\frac{1}{2}\epsilon_{jkl}F_{kl}$. This equations can be regarded as a reduction of SDYM equations into Euclidean three dimensions, if we identify $\Phi=A_0$. As mentioned in introduction, Nahm [10] applied the ADHM construction to constructing monopoles, localized energy configurations in $\mathbb{R}^3$ to the Bogomol'nyi equations, by bringing in an infinite dimensional vector space $\mathcal{L}^2[I]$. Intuitively, we can recognize the necessity of the infinite dimensional $\mathcal{L}^2$ space through the following argument [19]. We consider a single monopole as a superposition of instantons putting densely on a time axis at a definite location in $\mathbb{R}^3$. To compose this configuration we firstly put the instantons periodically on each time axis, this situation is called caloron solution [20, 21], and next we take the limit of infinitesimal periodicity to restore the time translation invariance. Obviously, the instanton number of monopoles is infinite, so that we need an infinite dimensional vector space in the language of ADHM construction.

In Nahm's construction, we assume a linear equation $\Delta^\dagger v = 0$ for the vector $v(z) \in \mathcal{L}^2[I] \otimes V_N \otimes \mathbb{H}$ which defines the monopole configurations, where V_N is an additional N-dimensional vector space representing a multi-monopole configuration. The matrices Δ and $\Delta^\dagger$ of the ADHM construction turn out to be differential operators here. The connection one-form is given by the formula (2.1) with the $\mathcal{L}^2[I]$ inner product,

$$< w, v >= \int_I w(z)^\dagger v(z)\, dz. \tag{2.9}$$

We find that the conditions on $\mathcal{F}$ become,

Theorem 2. [10, 18] If

$$\Delta^\dagger = i\frac{d}{dz} \otimes 1_N \otimes 1_2 + 1 \otimes 1_N \otimes x^\dagger + \sum_{j=1}^{3} 1 \otimes T_j(z) \otimes \tau_j^\dagger, \tag{2.10}$$

the quaternionic reality and invertibility of $\Delta^\dagger\Delta$ are equivalent to the differential equations for the matrices T_j (the Nahm equation),

$$\frac{dT_j}{dz} = \frac{1}{2}\epsilon_{jkl}[T_k, T_l]. \tag{2.11}$$

In practice we need some additional conditions on T_j to impose correct boundary conditions which guarantee the finiteness of energy in the monopole configurations. Note that the Nahm equation also appears when we perform the reduction of SDYM into 1-dimension.

The simplest, however, non-trivial example of solutions to the Nahm equation (2.11) is given by the one-dimensional sector of the matrices $T_j(z)$. The Nahm equation yields all $T_j(z), (j = 1, 2, 3)$ are constants, which are natural to choose zeroes since they lead to the single BPS monopole at the origin. The linear equation $\Delta^\dagger v = 0$ with $\Delta^\dagger = i\frac{d}{dz} + x^\dagger$ gives $v(z) = N(x_\mu)e^{ix^\dagger z}$, where $N(x_\mu)$ is a normalization function, hence we arrive at the following connection one-form of the BPS monopole [4, 13],

$$A_{BPS} = -\frac{1}{2}\Big(\coth r - \frac{1}{2}\Big)\hat{x}dx_0 - \frac{1}{2}\Big(1 - \frac{r}{\sinh r}\Big)\epsilon_{ijk}\frac{x_i}{r^2}dx_j\sigma_k. \tag{2.12}$$

3 A q-analog of ADHMN construction

In the ADHMN construction reviewed in the last section, we associated a certain linear equation, $\Delta^\dagger v = 0$, to each solution to SDYM equations. At this point, the author and the collaborator considered whether one can obtain a SD solution through a q-difference equation or not, and indeed obtained an analytic SD configuration, see the following table.

linear eqn.		**SD solutions**
algebraic (fin. dim.)	$\leftrightarrow$	instantons
q-difference eqn.	$\leftrightarrow$	??
differential eqn.	$\leftrightarrow$	monopoles

In this q-analog construction to the ADHMN formalism, the authors were inspired by the fact that the Knizhnik-Zamolodchikov (KZ) equation in conformal field theory allows meaningful q-analog, the $q-$KZ equation, a difference equation on a q-interval I_q, *e.g.*,

$$I_q = \left\{ \pm\frac{1}{2}, \pm\frac{1}{2}q, \pm\frac{1}{2}q^2, \pm\frac{1}{2}q^3, \dots \right\}, \ (0 < q < 1). \tag{3.1}$$

Here we consider the q-analog formulation in ref.[8] and its generalization, in which we need appropriate inner products $< w, v >_q$ compatible with SD condition on the q-derivative operator in new Δ's. As we will see the q-analog formulations approach the Nahm formulation for monopoles when $q \to 1$, since then q-derivative tends to ordinary derivative.

3.1 Definition of Δ associated with q-derivative

We introduce the following q-analysis operations, the q-derivative and the q-integration ("Thomae-Jackson integral") [14],

$$D_q f(z) : \ = \ \frac{f(z) - f(qz)}{(1-q)z} \xrightarrow[q\to 1]{} \frac{df}{dz}(z), \tag{3.2}$$

$$\int_0^a f(z)\, d_q z : \ = \ a(1-q)\sum_{n=0}^{\infty} f(aq^n)q^n \xrightarrow[q\to 1]{} \int_0^a f(z)\, dz, \tag{3.3}$$

respectively. If we define a Δ operator associated with q-derivative similarly to the monopole construction, that is,

$$\Delta = iD_q \otimes 1_N \otimes 1_2 + 1 \otimes 1_N \otimes x + \sum_{j=1}^{3} 1 \otimes T_j(z) \otimes \tau_j, \tag{3.4}$$

then we need, at least, the self-adjointness for q-derivative,

$$< iD_q w, v >_q = < w, iD_q v >_q \ . \tag{3.5}$$

This is critical for the SD condition in ADHMN construction, *i.e.*, the reality and invertibility for the operator $\Delta^*\Delta$, where the symbol * is the involution associated with an inner product of vector space under consideration, in the construction given in the last section it is simply the hermitian conjugation †. We can see this explicitly for the simplest case (one-dimensional sector of the matrices $T_j(z)$), which brings a q-analog of the BPS monopole, *i.e.*, if we define the linear operator Δ^* as,

$$\Delta^* = iD_q + x^\dagger, \tag{3.6}$$

then we have

$$\Delta = iD_q + x, \tag{3.7}$$

due to the self-adjointness of iD_q, this leads to the SD conditions[1],

[1] As for the case $k > 1$ we need further consideration, see concluding remarks

Proposition 3 (The reality). The product $\Delta^*\Delta$ is quaternionic real, that is,

$$\begin{aligned}\Delta^*\Delta &= -D_q^2 + 2ix_0 D_q + |x|^2\\ &= (iD_q + \rho_+)(iD_q + \rho_-),\end{aligned}$$

where $\rho_\pm := x_0 \pm ir$.

Proposition 4 (The invertibility). The function:

$$\begin{aligned}F(x_\mu; z, z'; q) = \tfrac{1}{4r}\epsilon(z,z')\{&E_q(-i\rho_+(1-q)qz')e_q(i\rho_+(1-q)z)\\ &-E_q(-i\rho_-(1-q)qz')e_q(i\rho_-(1-q)z)\},\end{aligned}$$

where e_q and E_q are the so called q-exponential functions defined below. We can see F satisfies the equation,

$$\Delta^*\Delta F(x_\mu; z, z'; q) = \frac{1}{(1-q)|z'|}\delta_{z,z'}, \tag{3.8}$$

We easily find the expected limit,

$$\begin{aligned}F \quad &\xrightarrow[q\to 1]{} \quad \frac{1}{4r}\epsilon(z-z')\left(e^{i\rho_+(z-z')} - e^{i\rho_-(z-z')}\right)\\ &= \quad -\frac{1}{2r}e^{ix_0(z-z')}\sinh r|z-z'|, \text{(the BPS limit)}\end{aligned} \tag{3.9}$$

3.2 Inner product compatible with Δ

Now we determine the inner products $< w, v >_q$ with which the q-derivative enjoys the self-adjointness (3.5). According to the functional vector space on which the vector v is defined, we have the following choice:

(I) For functions defined on a q-interval I_q, we can define an inner product by using the q-integral,

Definition 1 (The $\ell^2[I_q]$-inner product).

$$< w, v >_q = \int_{-1/2}^{1/2} w^* v \, d_q z := \int_0^{1/2} w^* v \, d_q z - \int_0^{-1/2} w^* v \, d_q z, \tag{3.10}$$

where I_q is fixed $\{\pm 1/2, \pm q/2, \pm q^2/2, \pm q^3/2, \dots\}$ and $q \in (0,1)$ as usual.

(II) For functions defined on an interval I, we can also define another inner product by ordinary integral,

Definition 2 ($\hat{\mathcal{L}}^2[I]$-inner product).

$$< w, v >_q = \int_{-1/2}^{1/2} w^* v \, dz \tag{3.11}$$

provided that outside the defining region I (here we fixed $I = [-1/2, 1/2]$) functions does not take values.

In both cases, we define the conjugate vector v^* as,

Definition 3 (the conjugate vector).

$$v^* = [f(x_\mu, z; q)]^* \;=\; f^\dagger(x_\mu, qz; q^{-1}), \tag{3.12}$$

where † is hermitian conjugation of quaternion. This definition of the conjugate vector distinguishes the functional inner product in the original Nahm formulation and the case (II). Note that $(v^*)^* = v$.

3.3 The ASD solution in the case (I)

In this subsection we determine the ℓ^2 vector v defined on I_q by solving q-difference equation. We have already known that the operator $\Delta^*\Delta$ satisfies the SD condition for the one-dimensional sector of $T_j(z)$, so that the equations we must solve are,

$$\Delta^* v = (iD_q + x^\dagger)v = 0 \tag{3.13}$$

$$< v, v >_q = 1. \tag{3.14}$$

By introducing the q-exponential functions familiar in q-analysis,

$$e_q(z) \;=\; \sum_{n=0}^{\infty} \frac{z^n}{(q;q)_n}, \tag{3.15}$$

$$E_q(z) \;=\; \sum_{n=0}^{\infty} \frac{q^{n(n-1)/2}}{(q;q)_n} z^n, \tag{3.16}$$

where $(a;q)_n = \prod_{j=0}^{n}(1 - aq^j)$ (q-shifted factorial). We can easily find the solution to the linear q-difference equation (3.13),

$$v = e_q(ix^\dagger(1-q)z)N(x_\mu; q), \tag{3.17}$$

$N(x_\mu; q)$ being a "normalization function" taking a value in quaternion. Next we fix the functional form of N as follows,

$$< e_q(ix^\dagger(1-q)z), e_q(ix^\dagger(1-q)z) > = \Lambda_+(x_0, r; q)1 + \Lambda_-(x_0, r; q)\hat{x}, \tag{3.18}$$

where

$$\Lambda_\pm(x_0, r; q) = \frac{1-q}{2}\left\{\sum_{n=0}^{\infty} \frac{(\frac{\rho_+}{\rho_-}; q)_{2n}}{(q;q)_{2n+1}}\left(i\frac{(1-q)\rho_-}{2}\right)^{2n} \pm (\rho_+ \leftrightarrow \rho_-)\right\}, \tag{3.19}$$

then we find the normalization condition is,

$$N^*(x_\mu; q)(\Lambda_+ 1 + \Lambda_- \hat{x})N(x_\mu; q) = 1. \tag{3.20}$$

Finally, we obtain the solution by power series expansion in $\hat{x}$,

Proposition 5 (The normalization function).

$$\begin{aligned} N(x_\mu; q) \;&= \tfrac{1}{2}\{(\Lambda_+ + \Lambda_-)^{-\frac{1}{2}} + (\Lambda_+ - \Lambda_-)^{-\frac{1}{2}}\}1 \\ &+ \tfrac{1}{2}\{(\Lambda_+ + \Lambda_-)^{-\frac{1}{2}} - (\Lambda_+ - \Lambda_-)^{-\frac{1}{2}}\}\hat{x} \end{aligned} \tag{3.21}$$

The next step is to find the connection one-form determined by the vector v fixed above. A straightforward calculation leads to,

$$A = \frac{1}{4}(-\frac{\partial\Omega}{\partial r}dx_0 + \frac{\partial\Omega}{\partial x_0}dr) + f d\hat{x} + g\epsilon_{ijk}\frac{x_i}{r^2}dx_j\sigma_k \tag{3.22}$$

where

$$\Omega(x_\mu) = \log\frac{L^+}{L^-}\cdot 1 + \log(L^+L^-)\cdot\hat{x} \tag{3.23}$$

$$f(x_0,r) = -\frac{i}{4}(M_+ - M_-)(L^+L^-)^{-1/2} \tag{3.24}$$

$$g(x_0,r) = -\frac{1}{2}\{1 - \frac{M_+ + M_-}{2}(L^+L^-)^{-1/2}\}. \tag{3.25}$$

Here $L^\pm$ and $M_\pm$ are the functions of $\rho_\pm := x_0 \pm ir$

$$L^\pm := \Lambda_+ \pm \Lambda_- = \sum_{n=0}^{\infty}\frac{(q\frac{\rho_\mp}{\rho_\pm};q)_{2n}}{(q^2;q)_{2n}}\left\{-\frac{(1-q)^2\rho_\pm^2}{4}\right\}^n \tag{3.26}$$

$$M_\pm = \sum_{n=0}^{\infty}\frac{1-q}{1-q^{2n+1}}\left\{-\frac{(1-q)^2\rho_\pm^2}{4}\right\}^n. \tag{3.27}$$

We should observe another expression to the solution,

$$L^\pm = {}_2\phi_1[q\frac{\rho_\mp}{\rho_\pm}, q^2\frac{\rho_\mp}{\rho_\pm}; q^3; -\frac{(1-q)^2\rho_\pm^2}{4}], \tag{3.28}$$

where ${}_2\phi_1$ is well-known basic hypergeometric series (with base q^2).

We have $A_\mu^* = A_\mu$ instead of the hermiticity $A_\mu^\dagger = A_\mu$. However, we can obtain $su(2)$-valued connection through a certain "gauge transformation" with $g^* = g^{-1}$

The components of curvature two-form are

$$F_{0r} = (\partial_0^2 + \partial_r^2)\log(L^+L^-)\cdot\hat{x} \tag{3.29}$$

$$F_{\theta\phi} = -\frac{1}{2}\left(1 - \frac{M_+M_-}{L^+L^-}\right)\sin\theta\hat{x} \tag{3.30}$$

$$\begin{aligned} F_{0\theta} = {} & \{\partial_0 f + \frac{1}{4}(1+2g)\partial_r\log(L^+L^-)\}\frac{\partial\hat{x}}{\partial\theta} \\ & +\{\partial_0 g - \frac{1}{2}f\partial_r\log(L^+L^-)\}\frac{1}{\sin\theta}\frac{\partial\hat{x}}{\partial\phi} \end{aligned} \tag{3.31}$$

$$\begin{aligned} F_{r\phi} = {} & \{\partial_r f - \frac{1}{4}(1+2g)\partial_0\log(L^+L^-)\}\frac{\partial\hat{x}}{\partial\phi} \\ & -\{\partial_r g + \frac{1}{2}f\partial_0\log(L^+L^-)\}\sin\theta\frac{\partial\hat{x}}{\partial\theta} \end{aligned} \tag{3.32}$$

$$\begin{aligned} F_{r\theta} = {} & \partial_r f - \frac{1}{4}(1+2g)\partial_0\log(L^+L^-)\}\frac{\partial\hat{x}}{\partial\theta} \\ & +\{\partial_r g + \frac{1}{2}f\partial_0\log(L^+L^-)\}\frac{1}{\sin\theta}\frac{\partial\hat{x}}{\partial\phi} \end{aligned} \tag{3.33}$$

$$\begin{aligned} F_{0\phi} = {} & \{-\partial_0 g + \frac{1}{2}f\partial_r\log(L^+L^-)\}\sin\theta\frac{\partial\hat{x}}{\partial\theta} \\ & +\{\partial_0 f + \frac{1}{4}(1+2g)\partial_r\log(L^+L^-)\}\frac{\partial\hat{x}}{\partial\phi} \end{aligned} \tag{3.34}$$

We can explicitly confirm the (A)SD condition,

$$F_{\mu\nu} = -\tilde{F}_{\mu\nu}. \tag{3.35}$$

We can find two extreme cases, in which the solutions turn out to be elementary functions, one is the BPS monopole limit as $q \to 1$,

$$L^{\pm} \xrightarrow[q\to 1]{} \frac{\sinh r}{r}, \quad M_{\pm} \xrightarrow[q\to 1]{} 1 \tag{3.36}$$

then

$$A \xrightarrow[q\to 1]{} -\frac{1}{2}(\coth r - \frac{1}{2})\hat{x}dx_0 \quad - \quad \frac{1}{2}(1 - \frac{r}{\sinh r})\epsilon_{ijk}\frac{x_i}{r^2}dx_j\sigma_k \tag{3.37}$$

$$= \quad A_{BPS} \tag{3.38}$$

as expected. The other is obtained by taking the $q \to 0$ limit, which brings a zero curvature configuration (vacuum),

$$L^{\pm}, M_{\pm} \xrightarrow[q\to 0]{} \frac{1}{1+\rho_{\pm}^2/4} \Rightarrow A \simeq 0. \tag{3.39}$$

Hence we find that our solution is interpolating vacuum and the BPS monopole by the parameter q.

3.4 The ASD solution in the case (II)

In the case of the $\hat{\mathcal{L}}^2[I]$ inner product (3.11), the vector v must be considered as an analytic function on the continuous region $I = [-1/2, 1/2]$. The defining relations are formally the same as in the $\ell^2[I_q]$ case,

$$\Delta^* v = (iD_q + x^\dagger)v = 0 \tag{3.40}$$

$$< v, v >_q = 1. \tag{3.41}$$

It is a little bit surprising that the self-adjointness (3.5) for the q-derivative holds even in this case. Here we should pay attention on the fact that the solution to the linear equation (3.40) has ambiguity of so called pseudo-constants $C(z)$; if v is a solution to (3.40) then $C(z)v$ is also. The pseudo-constants are defined as invariance under the q-shift,

$$C(qz) = C(z), \tag{3.42}$$

which are familiar in the q-analysis and we can easily give an example with a new parameter α,

$$C(z) = z^{\alpha}\frac{\Theta(q^{\alpha}z)}{\Theta(z)}, \ (\alpha \neq \mathbb{Z}) \tag{3.43}$$

where $\Theta(z)$ is Jacobi's elliptic theta function defined as,

$$\Theta(z) = (z;q)_\infty(q/z;q)_\infty(q;q)_\infty. \tag{3.44}$$

Introducing the pseudo-constant (3.43) results in the dependence on the special values of theta function for the normalization functions and also the connection one-form. Further consideration is needed for the consequence of the pseudo-constants.

4 Concluding remarks

The configurations (3.22) obtained by a q-analog formulation of the ADHMN construction are deeply concerned with the axisymmetric instantons derived by Witten [15] in his early work. The axisymmetric instantons are the solutions to SD equations in $\mathbb{R}^4$ with the ansatz that the configurations are spherically symmetric in three space $\mathbb{R}^3$. Hence the solutions are depending only on x_0 and r, "time" and radial coordinate, respectively. With this ansatz, the SD equations in $\mathbb{R}^4$ can be reduced to YM-Higgs system in $\mathbb{R}^2$, in which the field equation turns out to be a Liouville equation,

$$\partial_+\partial_-\phi = 2e^{-\phi}. \tag{4.1}$$

Thus we can easily find the general solution to (4.1),

$$-\phi = \log \frac{g'_+ g'_-}{(g_+ - g_-)^2} \tag{4.2}$$

where $g_\pm$ is holomorphic functions of $\rho_\pm := x_0 \pm ir$. In particular, the solutions are shown to have finite instanton number, provided that $g_\pm$ is the meromorphic functions with finite zeroes and poles. We can find that the $\ell^2[I_q]$ solutions (3.22) are also enjoys (4.1) with meromorphic functions of infinite zeroes and poles, *i.e.*,

$$g_- = \prod_{n=0}^{\infty} \frac{\frac{2q^{-n}}{1-q} + \rho_-}{\frac{2q^{-n}}{1-q} - \rho_-}, \tag{4.3}$$

(and similar expression for g_+.) This means the solutions (3.22) are axisymmetric instantons with infinite instanton number.

As pointed out in section 3, we can generalize the Δ operator in our q-analog formulations into $k > 1$ sector. In this case the SD conditions inherit to finite dimensional matrices $T_i(z;q)$ in (3.4) enjoying the following equations,

$$D_q T_i(z) = \epsilon_{ijk}(T_j(qz)T_k(z) - T_k(qz)T_j(z)), \tag{4.4}$$

similarly to the Nahm construction. As mentioned in the introduction, there is another difference analog of the Nahm equations, the discrete Nahm equations [11], which yield a monopole on hyperbolic 3-space [12]. In contrast to the discrete Nahm equations, the q-analog formulation considered in this paper gives SD solutions in flat $\mathbb{R}^4$ with infinite instanton number. About the relationship between these difference equations, and for the integrability of the q-Nahm equations, we will publish elsewhere.

References

[1] Mason L J and Woodhouse N M J, Integrability, selfduality, and twistor theory, Oxford University Press - Oxford, 1996

[2] Mason L J and Sparling G A J, *Phys. Lett.* **A 137** (1989), 29

[3] Schiff J, Integrability of Chern-Simons-Higgs vortex equations and a reduction of the self-dual Yang-Mills equations to three dimensions, in Painlevé Transcendents, Editors Levi D and Winternitz P, Plenum Press - New York, 1992

[4] Bogomol'nyi E B, *Sov. J. Nucl. Phys.* **24** (1976), 449

[5] Atiyah M F, Hitchin N J, Drinfeld V G, and Manin Yu I *Phys. Lett.* **A65** (1978), 185

[6] Drinfeld V G and Manin Yu I *Commun. Math. Phys.* **63** (1978), 177

[7] 'tHooft G, *Phys. Rev. Lett.* **37** (1976) 8

[8] Kamata M and Nakamula A, *Phys. Lett.* **B463** (1999), 257; preprint "A q-analog of ADHMN construction and axisymmetric multi-instanton"

[9] Nahm W *Phys. Lett.* **B90** (1980), 413

[10] Nahm W, The construction of all self-dual multimonopoles by the ADHM method, in Monopoles in Quantum Field Theory, Proceedings of the Monopole Meeting (Trieste Italy), Editors Craigie N S, Goddard P, and Nahm W, World Scientific - Singapore, (1982), 87

[11] Braam P J and Austin D M, *Nonlinearity* **3** (1990), 809

[12] Atiyah M F, Magnetic monopoles in hyperbolic spaces, in *Proc. Bombay Colloq. on Vector Bundles on Algebraic Varieties*, Oxford University Press - Oxford, 1984, 1

[13] Prasad M K and Sommerfield C M, *Phys. Rev. Lett.* **35** (1975), 760

[14] Gasper G and Rahman M, Basic Hypergeometric Series, Cambridge University Press - Cambridge, 1990

[15] Witten E, *Phys. Rev. Lett.* **38**, (1977), 121

[16] Belavin A A, Polyakov A M, Schwartz A, and Tyupkin Y, *Phys. Lett.* **B59** (1975), 85

[17] Bowman M C, Corrigan E, Goddard P, Puaca A, and Soper A, *Phys. Rev.* **D28** (1983), 3100

[18] Hitchin N J, *Commun. Math. Phys.* **89** (1983), 145

[19] Rossi P, *Nucl. Phys.* **B149** (1979), 170

[20] Kraan T C and van Baal P, *Nucl. Phys.* **B533** (1998), 627

[21] Harrington B J and Shepard H K, *Phys. Rev.* **D**17 2122, **D18** (1978), 2990; *Nucl. Phys.* **B124** (1977), 409

[22] Frenkel I B and Reshetikhin N Yu, *Commun. Math. Phys.* **146** (1992), 1