On Complete Integrability of the Generalized Weierstrass System

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Abstract

In this paper we study certain aspects of the complete integrability of the Generalized Weierstrass system in the context of the Sinh-Gordon type equation. Using the conditional symmetry approach, we construct the Bäcklund transformation for the Generalized Weierstrass system which is determined by coupled Riccati equations. Next a linear spectral problem is found which is determined by nonsingular 2×2 matrices based on an $sl(2, \mathbb{C})$ representation. We derive the explicit form of the Darboux transformation for the Weierstrass system. New classes of multisoliton solutions of the Generalized Weierstrass system are obtained through the use of the Bäcklund transformation and some physical applications of these results in the area of classical string theory are presented.

1 Introduction

The theory of surfaces has recently had a large resurgence of interest in many applications to very different and diverse areas of theoretical physics. This is in no small measure due to the fact that many equations which are of interest in various areas of mathematical physics also appear, or can be incorporated, in the study of surfaces in three-dimensional space. In particular, the Sine and Sinh-Gordon equations are of a great significance in this respect. Moreover, the theory of constant mean curvature surfaces has been of continued importance in the study of many problems which have physical applications. For example, minimal surfaces and constant mean curvature surfaces in particular have recently found applications in the areas of two-dimensional gravity [1], quantum field theory as well as in the area of string theory [1]. As a string propagates through space-time, it describes a surface in space-time which is called its world sheet. The study of classical strings in three and four-dimensional space is a crucial first step in producing a quantum theory of strings. It can be said that first quantized string theory is the study of conformal field theories on Riemann surfaces.

The stationary two-dimensional sigma model has been shown to be of use in generating two-dimensional surfaces immersed in three-space, and there are links between this model and other models, such as the non-Abelian Chern–Simons theories, which are of great importance in certain areas of condensed matter physics [1]. It has been shown that the Chern–Simons gauged Landau-Ginsburg model plays the role of effective theory for the Fractional Quantum Hall Effect [2]. The Chern–Simons equation of motion can describe time evolving two-dimensional surfaces in such a way that the deformation is not only locally compatible with the Gauss–Codazzi equation, but completely integrable as well [3]. In the static limit, the self-dual version of the model possesses soliton solutions. These correspond to Laughlin's quasiparticles and give a realization of anyon quasiparticles. On the other hand, the self-dual Chern–Simons model can be associated with the stationary two-dimensional continuous classical Heisenberg model, which can be related to the two-dimensional sigma-model.

Another area of recent interest with regard to applications is to the area of liquid crystals and the theory of membranes [4]. Fluid membranes may be idealized as two-dimensional surfaces with each membrane being made up of a double layer of long molecules. Various physical properties of interest such as elastic free energy per unit area can be calculated in terms of quantities which are directly related to the geometry of the surface. In fact the curvature elastic free energy per unit area of the membrane can also be formulated rigorously in terms of two-dimensional differential invariants of the surface. It is of considerable interest with regard to these types of physical applications to obtain shape equations for the membrane. These interrelate the basic parameters and functions which determine the form of a given membrane surface, as in a liquid crystal. In an equilibrium state, the energy of any physical system must be minimized. One usually writes down a shape energy function F in terms of the basic parameters and then minimizes it, and the result is a shape equation. An example of such a shape function is given by

$$F = \frac{1}{2}k_c \int (2H + c_0)^2 dA + \Delta p \int dV + \lambda \int dA,$$

where k_c is the bending rigidity of the membrane, H the mean curvature and the spontaneous curvature c_0 takes account of the asymmetry effect of the membrane or the surrounding environment. The pressure difference between the outside and the inside of the membrane is called Δp, λ the tensile stress acting on the membrane. Mathematically, Δp and λ may be considered as Lagrange multipliers. The shape equation is obtained from the first variation of this F. Specific Delaunay's surfaces of constant mean curvature can be written as $\sin \psi(\rho) = a\rho + d/\rho$. This equation is then substituted into a given shape equation and results in constraint equations between the parameters a and b, which give a characterization of the surface [4, p. 114].

In this paper, we study a connection between the Generalized Weierstrass (GW) system inducing constant mean curvature surfaces immersed in $\mathbb{R}^3$ and a Sinh-Gordon type equation. The objective of this paper is, using this link between these two systems, to derive a representation of a linear spectral problem for which the matrices are nonsingular, and find the corresponding Darboux and Bäcklund transformations for the GW system. These transformations are derived here for the first time. Based on these transformations, we construct solutions of the GW system and investigate minimal surfaces immersed in three-dimensional Euclidean space.

This paper is organized as follows. In Section 2, a short presentation of the conditional symmetry approach for partial differential equations which admit the Painlevé property is given. In Section 3, using the conditional symmetries we derive the linear spectral problem, and we find the Darboux and Bäcklund transformations for the GW system. Section 4 contains new examples of multi-soliton solutions of the GW system and some physical interpretations of these results are given in the area of classical string theory.

2 Conditional symmetries

In this section, we give a brief overview of the conditional symmetry approach for PDEs as developed in [5–8]. In this context, we concentrate on examining certain aspects of integrability of k-th order nonlinear PDEs. The technique which is outlined below is applied only to such classes of PDE which pass the Painlevé test and can be presented in the form of a polynomial in the unknown variable u and its derivatives, possibly after a transformation in the space of independent and dependent variables $X \times U$. The basic terminology and notation used here in the application of Lie groups to differential equations are in conformity with [9]. We are particularly interested in combining singularity structure analysis and Lie point symmetries in order to recover the Auto-Bäcklund transformation (Auto-BT) and Darboux transformation (DT) for PDEs if such exist. In the literature, several attempts at treating this subject can be found, for example [5–8], and references therein. More recently, this subject has been studied for first order systems of PDEs, by one of the authors, leading to the development of a new version of the conditional symmetry method [10, 11]. This approach, like other nonclassical methods (see for a review of the subject [12]) makes possible the explicit determination of certain classes of solutions, invariant under a group of transformations which maps a subset of solutions of the initial equation into other solutions of a different equation, that is, a subsystem composed of the initial PDE and differential constraints (DCs), which are mutually consistent. In this presentation, we postulate in accordance with the method worked out in [11], that these multiple DCs take a specific form for which all derivatives of the unknown function u are expressible in terms of some functions of the independent and dependent variables only. Hence we consider the overdetermined system composed of a nondegenerate k-th order scalar PDE and a first order system of DCs

$$\Delta\left(x, u^{(k)}\right) = 0, \tag{2.1}$$

$$Q_i\left(x, u^{(k)}\right) \equiv \frac{\partial u}{\partial x^i} - \phi_i\left(x, u^{(k)}\right) = 0, \qquad i = 1, \dots, p \tag{2.2}$$

in p independent variables $x = \left(x^1, \dots, x^p\right)$ which form some local coordinates in Euclidean space X. The compatibility conditions for (2.1) and (2.2) are given by

$$\begin{aligned} &(i) \quad \phi_{[i,j]} + \phi_{[j}\phi_{i],u} = 0, \qquad i, j = 1, \dots, p, \\ &(ii) \quad \Delta\left(x, u, \phi^{(k-1)}\right) = 0, \qquad \phi = (\phi_1, \dots, \phi_p). \end{aligned} \tag{2.3}$$

The brackets $[i, j]$ denote the alternation with respect to the indices i and j, that is,

$$\begin{aligned} &\phi_{[i,j]} \equiv 2(\phi_{i,j} - \phi_{j,i}), \\ &\phi_{[i}\phi_{j],u} = 2(\phi_i\phi_{j,u} - \phi_j\phi_{i,u}). \end{aligned}$$

Note the equation (2.2) means that the characteristics Q_i of a set of p-linearly independent vector fields (defined on $X \times U$)

$$Z_i = \partial_{x^i} + \phi_i(x,u)\partial_u, \qquad i = 1, \dots, p, \tag{2.4}$$

are equal to zero.

An Abelian Lie algebra L spanned by the vector fields $Z_1, \dots, Z_p$ is called a conditional symmetry algebra of the k-th order PDE (2.1), if the vector fields $Z_1, \dots, Z_p$ are tangent to the subvariety

$$S = S_\Delta \cap S_Q,$$

where we associate the initial system $\Delta : J^k \to \mathbb{R}$ and a first order system of DCs $Q_i : J^1 \to \mathbb{R}^p$ with the subvarieties of the solution spaces

$$S_\Delta = \left\{ \left(x, u^{(k)}\right) \in J^k : \Delta\left(x, u^{(k)}\right) = 0 \right\},$$
$$S_Q = \left\{ \left(x, u^{(1)}\right) \in J^1 : Q_i\left(x, u^{(1)}\right) = 0, i = 1, \dots, p \right\},$$

respectively. This definition means that the k-th prolongation of the vector fields Z_i belongs to the tangent space to S at $\left(x, u^{(k)}\right)$, that is,

$$\mathrm{pr}^{(k)} Z_i|_S \in T_{\left(x,u^{(k)}\right)} S, \qquad i = 1, \dots, p. \tag{2.5}$$

A solution $u = f(x)$ of the k-th order PDE (2.1) is called a conditionally invariant solution, if its graph $\{(x, f(x))\}$ is invariant under an Abelian distribution of the vector fields $Z_1, \dots, Z_p$ satisfying conditions (2.5).

It has been shown [11] that a nondegenerate k-th order PDE (2.1) admits a p-dimensional conditional symmetry algebra L if and only if there exists a set of p linearly independent vector fields (2.4) for which the C^{k-1} functions ϕ_i satisfy the conditions (2.3). The graph of a solution of the overdetermined system composed of (2.1) and (2.2) is invariant under the vector fields Z_i, $1 \le i \le p$. Hence according to the above definition, this means that there exists a conditionally invariant solution of PDE (2.1).

It has been proved [13] that any PDE (2.1) of the k-th order admits infinitely many compatible first order DCs. However, this statement shows only existence of such constraints, but does not provide a constructive method for finding the explicit form of these DCs. Thus, the construction of conditional symmetries is reduced to the selection of such subsystems composed of initial PDE (2.1) and DCs (2.2) for which conditions (2.3) hold. In general, system (2.3) is a nonlinear one and usually very difficult to solve, except in some particular cases. Nevertheless, there exist many physically interesting systems of PDEs for which particular solutions of (2.3) lead to Bäcklund transformations described by first order differential constraints or to solutions depending on some arbitrary constants [14]. These particular solutions of (2.3) are obtained by expanding each function ϕ_i into a polynomial in the dependent variables u. This polynomial is reduced often to a trinomial. This means that equations (2.2) become a Riccati system of equations which possess the Painlevé property.

In Section 3, we show on a specific example of the generalized Weierstrass system inducing constant mean curvature surfaces in $\mathbb{R}^3$, that the conditions (2.3) for the existence of conditional symmetries can be used to construct a certain class of Bäcklund and Darboux transformations. This construction consists of the following steps.

First, we assume that PDE (2.1) passes the Painlevé test. This means that (2.1) satisfies the necessary conditions for the absence of movable critical singularities in its general solution on arbitrary noncharacteristic surfaces [15, 16]. This test provides us with a tool for assessing the integrability of PDEs. At least we can reject as nonintegrable these PDEs which do not pass the Painlevé test.

We restrict our considerations to the particular case when the singularity structure of a solution of PDE (2.1) consists of only poles. According to Painlevé analysis for PDEs [16], it has been shown that if the choice of the expansion variable χ in terms of the singular manifold variable $\varphi(x) = \varphi_0$, (where χ vanishes as $\varphi - \varphi_0$), then the beginning of the Laurent series for the solution u takes place at the order $-n$ in χ, that is,

$$u \sim \chi^{-n}, \qquad n \in \mathbb{Z} < 0, \tag{2.6}$$

where the power $(-n)$ denotes the multiplicity of a pole.

Secondly, we postulate that the difference of two distinct solutions u and $\hat{u}$ of PDE (2.1) can be represented in a polynomial form in terms of an auxillary variable y up to the degree $(-n)$. The variable y is a mapping of space X into some m-dimensional space B. This demand implies that we perform an embedding transformation of the variable $(u-\hat{u})$ into the space B with coordinates $y = \left(y^1, \dots, y^m\right)$. Thus we assume that it can be realized through a specific Darboux transformation of the form

$$u - \hat{u} = \sum_J c_J y^J, \qquad 1 \le \#J \le (-n), \tag{2.7}$$

where $J = (j_1, \dots, j_m)$ and j_i is a positive integer such that

$$\#J = j_1 + \dots + j_m = -n. \tag{2.8}$$

The coefficients c_J in the expansion (2.7) are assumed to be constants. Eliminating the function u in the initial equation (2.1) through transformation (2.7), we obtain a k-th order PDE for the unknown function y. Denote this PDE by

$$\Delta_1\left(x, \hat{u}^{(k)}, y^{(k)}\right) = 0, \tag{2.9}$$

where $\hat{u}^{(k)}$ is a given function of x. Equation (2.9) is the starting point for our further analysis. In this section, we focus on a close connection between the conditional symmetries and Bäcklund transformations associated with nonlinear systems of PDEs (2.9). This connection is mainly due to the fact that the set of DCs (2.2) in the variable y admits a superposition formula (SF), as is also the case for BTs [17]. Based on the application of Lie's theorem on fundamental sets of solutions [18] to the case of first order PDEs [17], we show this link on a specific example in the next section. We assume that these DCs take the particular form of coupled matrix Riccati equations based on some given representation of the Lie algebra. This means that the DCs (2.2) take a specific form for which the first derivatives of y are decomposable in terms of x and y as follows

$$\frac{\partial y^\alpha}{\partial x^i}(x) = \sum_{l=1}^{r} A_i^l(x) b_l^\alpha(y(x)). \tag{2.10}$$

The set of functions b_l^α is identified with linearly independent vector fields on a space B

$$\hat{b}_l = b_l^\alpha(y)\partial_{y_\alpha}, \qquad l = 1,\dots r, \tag{2.11}$$

which generate a finite-dimensional Lie algebra $\mathcal{G}$

$$[\hat{b}_l, \hat{b}_k] = C_{lk}^a \hat{b}_a, \qquad 1 \le a, k, l \le r, \tag{2.12}$$

where C_{lk}^a are constants of the assumed Lie algebra structure. Note that the problem of construction and classification of all finite-dimensional Lie algebras which can be realized in terms of operators (2.11) remains an open one [18–20]. Nevertheless, this subject has been recently extensively investigated and there exist numerous lists of finite-dimensional Lie algebras which can serve as a source for our selection (see for example [19, 20] and references therein).

We select one of the finite-dimensional Lie algebras $\mathcal{G}$ and its representation in terms of vector fields (2.11) with polynomial coefficients in the variable y. We start with the lowest-dimensional algebra and, if necessary, proceed to consider the higher dimensional ones. For a chosen Lie algebra $\mathcal{G}$ the right hand side of expression (2.10) becomes a polynomial in the dependent variable y. Substituting (2.10) repeatedly $(k-1)$ times into PDE (2.9) and next requiring that the coefficients of successive powers of y in the equation so obtained vanish, we get a system of $(k-1)$ order PDEs for the functions A_i^l. Denote this system by

$$Q^a\left(x, \hat{u}^{(k)}(x), A_i^{l(k-1)}\right) = 0, \qquad a = 1, \dots, r. \tag{2.13}$$

For the assumed Lie algebra $\mathcal{G}$ the compatibility conditions for DCs (2.10) impose zero gauge curvature conditions on the functions A_i^l

$$A^a_{[i,j]} + \frac{1}{2} C_{lk}^a A_i^l A_j^k = 0, \qquad i \ne j = 1, \dots, p. \tag{2.14}$$

As a result, we arrive at an overdetermined system of equations for the functions A_i^l. We denote this system, consisting of equations (2.13) and (2.14), by H. To establish the existence of solutions of this overdetermined system, an analysis of the compatibility conditions of the system is required.

It has been shown [21] on several examples, such as the AKNS, Boussineq, Kadomtsev–Petviashvili, Sawada–Kotera, and Tzitzeica equations, that there exist many physically interesting systems of PDEs for which the general solution of the system H lead to Auto-Bäcklund transformations. This phenomena takes place when the set of solutions of the system H is parametrized by a function $\hat{u}$ satisfying the original PDE (2.1) and by at least one constant parameter λ. In this case, the functions A_i^l can be expressed uniquely in terms of the old solution $\hat{u}$ of (2.1) and some constant λ. Then DCs (2.10) become

$$\frac{\partial y^\alpha}{\partial x^i}(x) = \sum_{l=1}^{r} A_i^l(\hat{u}, \lambda) b_l^\alpha(y(x)). \tag{2.15}$$

From each solution $\hat{u}$ of PDE (2.1) we integrate DCs (2.15) for the functions y^α and find a solution u of original equation (2.1) via Darboux transformation (2.7). Thus, system (2.15) together with (2.7) determines a specific Auto-BT between sets of solutions of the initial PDE (2.1). By eliminating the auxilliary variables y^α from the system (2.15) and (2.7) we can obtain an explicit form of this Auto-BT.

Note that the solution of (2.9) which is obtained from the proposed procedure constitutes a conditionally invariant solution since it is invariant under the p-dimensional conditional symmetry algebra L spanned by the vector fields of the form

$$Z_i = \partial_{x^i} + \sum_{l=1}^{r} A_i^l(\hat{u}, \lambda)\hat{b}_l, \qquad i = 1, \ldots, p, \tag{2.16}$$

where $\hat{b}_l = b_l^\alpha(y)\partial_{y^\alpha}$ generate a finite-dimensional Lie algebra (2.12).

3 The generalized Weierstrass system

We now proceed to apply the conditional symmetry approach to the case of the generalized Weierstrass system (GW) inducing constant mean curvature surfaces embedded in $\mathbb{R}^3$ as derived by B Konopelchenko in [22]. This system is described by a set of Dirac type equations for two complex fields ψ_1 and ψ_2 given by

$$\begin{aligned} &\partial\psi_1 = p\psi_2, \qquad \bar{\partial}\psi_2 = -p\psi_1, \\ &\bar{\partial}\bar{\psi}_1 = p\bar{\psi}_2, \qquad \partial\bar{\psi}_2 = -p\bar{\psi}_1, \end{aligned} \tag{3.1}$$

where $p = |\psi_1|^2 + |\psi_2|^2$, $\partial = \partial/\partial z$ and $\bar{\partial} = \partial/\partial\bar{z}$.

Equations (3.1) possess several conserved quantities

$$\begin{aligned} &\bar{\partial}\left(\bar{\psi}_1\partial\psi_2 - \psi_2\partial\bar{\psi}_1\right) = 0, \qquad \partial\left(\psi_1\bar{\partial}\bar{\psi}_2 - \bar{\psi}_2\bar{\partial}\psi_1\right) = 0, \\ &\partial(\psi_1)^2 + \bar{\partial}(\psi_2)^2 = 0, \qquad \bar{\partial}\left(\bar{\psi}_1\right)^2 + \partial\left(\bar{\psi}_2\right)^2 = 0. \end{aligned} \tag{3.2}$$

It has been shown [22] as a consequence of these conservation laws that there exist three real valued functions $X^i(z, \bar{z})$, $i = 1, 2, 3$ such that

$$\begin{aligned} &X_1 + iX_2 = 2i\int_\Gamma \left(\bar{\psi}_1^2\, dz' - \bar{\psi}_2^2\, d\bar{z}'\right), \qquad X_1 - iX_2 = 2i\int_\Gamma \left(\psi_2^2\, dz' - \psi_1^2\, d\bar{z}'\right), \\ &X_3 = -2\int_\Gamma \left(\bar{\psi}_1\psi_2\, dz' + \psi_1\bar{\psi}_2\, d\bar{z}'\right), \end{aligned} \tag{3.3}$$

Note that by virtue of the conservation laws (3.2), the right hand sides in expression (3.3) do not depend on the choice of contour Γ in the complex plane $\mathbb{C}$, but only on its endpoints. This is due to the fact that the integrals (3.3) have the form

$$\int_\Gamma F(z, \bar{z})\, dz + \bar{F}(z, \bar{z})\, d\bar{z},$$

which satisfy the condition

$$\bar{\partial}F = \partial\bar{F}.$$

Consequently, the differentials of equations (3.3) are exact ones. The functions $X^i(z, \bar{z})$ can be identified as the coordinates of the position vector $\vec{X}$ of a surface embedded in $\mathbb{R}^3$. Hence from equations (3.1) and (3.3), we can determine the induced metric of this surface

$$ds^2 = 4p^2\, dz\, d\bar{z}, \tag{3.4}$$

in isothermic coordinates. The Gaussian curvature and constant mean curvature can be evaluated from $K = -p^{-2}\partial\bar{\partial}(\ln p)$, $H = 1$, respectively, where p is given in (3.1).

The system (3.1) is known to be completely integrable, and a Lax pair for it has been found recently by authors [23, 24]. However, the matrix appearing in the Lax pair is singular and nilpotent, which prevents the construction of solutions of (3.1). The objective of this paper is to demonstrate that it is possible to find such a representation of the Lax pair for which the matrices are nonsingular. This representation is suitable for constructing solutions of GW system (3.1). Next, we construct the Bäcklund transformation and the Darboux transformation by making use of a close connection between these transformations and conditional symmetries. Finally, based on these transformations, we construct several new classes of multisoliton solutions for (3.1) and investigate the corresponding surfaces in $\mathbb{R}^3$.

In our investigation of the integrability of the GW system (3.1) we change the dependent variables ψ_1 and ψ_2 to new dependent variables

$$p = |\psi_1|^2 + |\psi_2|^2,$$

and the current

$$J = \psi_1 \bar{\partial}\bar{\psi}_2 - \bar{\psi}_2 \bar{\partial}\psi_1, \tag{3.5}$$

in order to simplify its structure. In terms of these new variables, we show that GW system (3.1) can be decoupled into a direct sum of the elliptic Sh-Gordon type equation and the conservation of current $\bar{\partial}J = 0$.

In fact, differentiating the function p with respect to z and $\bar{z}$, we obtain,

$$\begin{aligned} &\partial p = \psi_1\left(\partial\bar{\psi}_1\right) + \bar{\psi}_2\left(\partial\psi_2\right), \qquad \bar{\partial}p = \bar{\psi}_1\left(\bar{\partial}\psi_1\right) + \psi_2\left(\bar{\partial}\bar{\psi}_2\right), \\ &\partial\bar{\partial}p = \bar{\partial}\psi_1\partial\bar{\psi}_1 + \bar{\partial}\psi_2\partial\psi_2 - p^3. \end{aligned} \tag{3.6}$$

Making use of conservation laws (3.2) and equations (3.6), GW system (3.1) takes the decoupled form in the variables p and J,

$$\partial\bar{\partial}\ln p = \frac{|J|^2}{p^2} - p^2, \qquad \bar{\partial}J = 0, \qquad \partial\bar{J} = 0. \tag{3.7}$$

If we introduce the new dependent variable

$$p = e^{\varphi/2},$$

into equation (3.7), we then obtain an elliptic sinh-Gordon type equation of the form [25]

$$\partial\bar{\partial}\varphi = -4\sinh\varphi - 2\left(1 - |J|^2\right)e^{-\varphi}, \qquad \bar{\partial}J = 0. \tag{3.8}$$

In particular, if the modulus of the current J is different from zero, $|J| \neq 0$, then we can introduce new independent, dependent variables η, $\bar{\eta}$ and ω

$$d\eta = J^{1/2}\,dz, \qquad d\bar{\eta} = \bar{J}^{1/2}\,d\bar{z}, \qquad \omega = \frac{p^2}{|J|},$$

respectively, such that GW system (2.1) takes the decoupled form

$$(\ln\omega)_{\eta\bar\eta} = 2\left(\frac{1}{\omega} - \omega\right), \qquad \bar\partial J = 0, \qquad \partial\bar J = 0.$$

Hence GW system (2.1) becomes a direct sum of the elliptic Sh-Gordon equation and the conservation of current J.

Equation (3.7) has the Painlevé property. Consequently, we obtain that the general solution p of (3.7) admits double poles with two residues of opposite sign

$$p^{\pm 2} = e^{\pm\varphi} = \pm\chi^{-2}, \tag{3.9}$$

where χ is the expansion variable of the Laurent series. According to the proposed procedure, based on the approach described by Conte and Musette [26] we assume a specific form of the Darboux transformation for the case when equation (3.8) admits opposite residues

$$\varphi - v = 2\ln\frac{\phi_1}{\phi_2}. \tag{3.10}$$

The function v satisfies the initial equation (3.8) and quantities ϕ_1 and ϕ_2 are two entire functions. Note that a similar situation for double poles with opposite residues arises in the study of singularity structure for sine-Gordon and MKDV equations [26]. Introducing a new variable $y = \phi_1/\phi_2$ and changing the variables in (3.10) according to

$$p = e^{\varphi/2}, \qquad q = e^{v/2}, \tag{3.11}$$

we find that the Darboux transformation for equation (3.7) can be realized by the following expression

$$p = q\,y, \tag{3.12}$$

or equivalently,

$$\varphi = v - 2\ln y.$$

A first step on the way to constructing a BT for (3.7) is to look for a conditional symmetry algebra L spanned by two vector fields Z_1, Z_2 which have the characteristic equations of the form (2.10). We have to assume a specific Lie algebra structure for the generators $\{\hat b_l\}$. We start the analysis with the lowest-dimensional case, namely that of the $sl(2,\mathbb{C})$ algebra which admits the one-dimensional representation $(\partial_y, y\partial_y, y^2\partial_y)$ in terms of a coordinate y. This algebra comes up in the study of several completely integrable models eg. [19]. In our case, DCs (2.10) in one complex variable y take the form of coupled scalar Riccati equations with nonconstant coefficients

$$\begin{aligned}\partial y &= A_1^0(z,\bar z) + A_1^1(z,\bar z)y + A_1^2(z,\bar z)y^2,\\ \bar\partial y &= A_2^0(z,\bar z) + A_2^1(z,\bar z)y + A_2^2(z,\bar z)y^2.\end{aligned} \tag{3.13}$$

The zero curvature conditions for (3.13) are given by

$$\begin{aligned}
&\bar\partial A_1^0 - \partial A_2^0 + A_1^1 A_2^0 - A_1^0 A_2^1 = 0,\\
&\bar\partial A_1^1 - \partial A_2^1 + 2\left(A_1^2 A_2^0 - A_2^2 A_1^0\right) = 0,\\
&\bar\partial A_1^2 - \partial A_2^2 - A_1^1 A_2^2 + A_1^2 A_2^1 = 0.
\end{aligned}\tag{3.14}$$

The substitution of the new variables (3.11) and the Ansatz (3.12) into equations (3.7) gives

$$\begin{aligned}
&\frac{1}{qy}(\partial\bar\partial q\, y + \partial q\,\bar\partial y + \bar\partial q\,\partial y + q\bar\partial\partial y) - \frac{1}{q^2 y}(\partial q\, y + q\,\partial y)(\bar\partial q)\\
&\qquad - \frac{1}{qy^2}(\partial q\, y + q\,\partial y)\bar\partial y - \frac{|J|^2}{q^2y^2} + q^2y^2 = 0.
\end{aligned}\tag{3.15}$$

Using equations (3.13) we can eliminate the derivatives of the complex variable y in the expression (3.15). Next, we require that the coefficients of the successive powers of y in the equation so obtained vanish, to give the system

$$\begin{array}{ll}
(1)\ q^2 A_1^0 A_2^0 + |J|^2 = 0, & q^2 A_1^0 A_2^0 + |J|^2 = 0,\\
(2)\ \partial A_2^0 - A_1^1 A_2^0 = 0, & \bar\partial A_1^0 - A_1^0 A_2^1 = 0,\\
(3)\ \partial A_2^1 + A_2^2 A_1^0 - A_1^2 A_2^0 + \partial\bar\partial \ln q = 0, & \bar\partial A_1^1 + A_1^2 A_2^0 - A_1^0 A_2^2 + \bar\partial\partial \ln q = 0,\\
(4)\ \partial A_2^2 + A_2^2 A_1^1 = 0, & \bar\partial A_1^2 + A_1^2 A_2^1 = 0,\\
(5)\ A_2^2 A_1^2 + q^2 = 0, & A_1^2 A_2^2 + q^2 = 0.
\end{array}\tag{3.16}$$

We obtain an overdetermined system (denoted by H in the previous section) composed of (3.14) and (3.16) for the unknown functions A_i^k. In our case, this system is consistent since the compatibility conditions are identically satisfied. This system has a nontrivial, unique solution for the A_i^k. We briefly outline how this solution can be obtained. The second and fourth equations in the first column of (3.16) can be written in the form

$$\partial \ln A_2^0 = A_1^1, \qquad -\partial \ln A_2^2 = A_1^1.$$

Equating these equations, we can integrate to obtain

$$A_2^0 = \frac{\bar g(\bar z)}{A_2^2},\tag{3.17}$$

where $\bar g$ is a complex function of $\bar z$. Similarly, from the second and fourth equations in the second column of (3.16), we obtain

$$\bar\partial \ln A_1^0 = -\bar\partial \ln A_1^2,$$

which can be solved to give,

$$A_1^0 = \frac{h(z)}{A_1^2},\tag{3.18}$$

where h is a complex function of z. Substituting these results into the first equation in (3.16), we find that

$$h(z)\bar{g}(\bar{z}) - |J|^2 = 0.$$

Thus, without loss of generality, one may take $h(z) = J(z)$ and $\bar{g}(\bar{z}) = \bar{J}(\bar{z})$.

From the first equation in (3.16), we can write

$$A_1^0 = -\frac{|J|^2}{q^2 A_2^0}, \tag{3.19}$$

thus A_1^0 is determined in terms of A_2^0. Equating equations (3.18) and (3.19), we eliminate A_1^0 and we get,

$$A_1^2 = -\frac{q^2}{\bar{J}} A_2^0. \tag{3.20}$$

From this, we can substitute A_1^2 into the fifth pair of equations in (3.16), and we obtain A_2^2 in terms of A_2^0, as

$$A_2^2 = \frac{\bar{J}}{A_2^0}. \tag{3.21}$$

Using the first column of (3.16), we substitute the relation $A_1^1 = \partial \ln A_2^0$ from equation (3.16-2) as well as (3.19) through (3.21) into equation (3.16-3). In this way, we can eliminate the coefficients A_1^0, A_1^1, A_1^2 and A_2^2 from differential equation (3.16-3). We obtain a partial differential equation for the function A_2^0 in the form

$$\bar{\partial}\partial \ln A_2^0 - \frac{q^2}{\bar{J}}\left(A_2^0\right)^2 + \frac{|J|^2 \bar{J}}{q^2 \left(A_2^0\right)^2} + \frac{|J|^2}{q^2} - q^2 = 0. \tag{3.22}$$

Moreover, if we introduce a new variable defined by

$$Q = \frac{q A_2^0}{i \bar{J}^{1/2}}, \tag{3.23}$$

into (3.22), it is transformed into the following form

$$\bar{\partial}\partial \ln Q - \frac{|J|^2}{Q^2} + Q^2 = 0. \tag{3.24}$$

This coincides with equation (3.7). Substituting (3.19)–(3.21) into the pair of Riccati equations (3.13), we obtain

$$\begin{aligned} \partial y &= -\frac{|J|^2}{q^2 A_2^0} + \partial \ln A_2^0 \, y - \frac{q^2 A_2^0}{\bar{J}} y^2, \qquad \bar{\partial} J = 0, \\ \bar{\partial} y &= A_2^0 + \bar{\partial} \ln\left(\frac{\bar{J}}{q^2 A_2^0}\right) y + \frac{\bar{J}}{A_2^0} y^2, \qquad \partial \bar{J} = 0. \end{aligned} \tag{3.25}$$

The compatibility condition for (3.25) is satisfied identically whenever (3.7) holds.

Note that Q is some solution to (3.7) which is related to q by (3.23). We could obtain a particular form for A_2^0 by considering the case in which $Q = q$. Then (3.23) implies that

$$A_2^0 = i\sqrt{\bar{J}}.$$

Equation (3.7) is invariant under the transformation

$$J \to \lambda J, \tag{3.26}$$

provided that $|\lambda|^2 = 1$. A Bäcklund parameter λ can be introduced into equations (3.25) by carrying out transformation (3.26). In this case, the pair of Riccati equations (3.25) takes the form

$$\begin{aligned}
&\partial y = i\lambda^{1/2}\left(\frac{J\bar{J}^{1/2}}{q^2} - \frac{q^2}{\bar{J}^{1/2}}y^2\right), \qquad \bar{\partial} J = 0,\\
&\bar{\partial} y = i\bar{\lambda}^{1/2}\bar{J}^{1/2} + \bar{\partial}\ln\left(\frac{\bar{J}^{1/2}}{q^2}\right)y - i\bar{\lambda}^{1/2}\bar{J}^{1/2}y^2, \qquad |\lambda|^2 = 1,
\end{aligned} \tag{3.27}$$

where q satisfies (3.7). Hence the above DCs become an Auto-BT for GW system (3.7), while the Darboux transformation is defined by (3.12). Furthermore, by linearizing the Riccati system (3.27), we obtain the associated linear spectral problem for (3.7) with spectral parameter μ, of the form

$$\begin{aligned}
&\partial\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} = \begin{pmatrix} 0 & i\mu^{1/2}\dfrac{\bar{J}^{1/2}J}{q^2}\\ iq^2\left(\dfrac{\mu}{\bar{J}}\right)^{1/2} & 0\end{pmatrix}\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix},\\
&\bar{\partial}\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} = \begin{pmatrix} \dfrac{1}{2}\bar{\partial}\ln(\dfrac{\bar{J}^{1/2}}{q^2}) & i\bar{\mu}^{1/2}\bar{J}^{1/2}\\ i\bar{\mu}^{1/2}\bar{J}^{1/2} & -\dfrac{1}{2}\bar{\partial}\ln(\dfrac{\bar{J}^{1/2}}{q^2})\end{pmatrix}\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix},
\end{aligned} \tag{3.28}$$

where $y = \phi_1/\phi_2$, $|\mu|^2 = 1$ and $\bar{\partial} J = 0$. The Lax pair (3.28) is based on a nondegenerate $sl(2,\mathbb{C})$ representation. Note that for any holomorphic function J, the compatibility condition for (3.28) reproduces the system (3.7) in the variable q.

4 Multi-soliton solutions

A number of new types of soliton solutions to the GW system (3.1) will now be presented here based on the BT (3.27). Let us define a new dependent variable

$$\rho = \frac{\psi_1}{\bar{\psi}_2}. \tag{4.1}$$

It has been shown in [23], that if functions ψ_1 and ψ_2 are solutions of GW system (3.1), then the function ρ defined by (4.1) is a solution of the two-dimensional Euclidean sigma model equations

$$\partial\bar{\partial}\rho - \frac{2\bar{\rho}}{1+|\rho|^2}\partial\rho\bar{\partial}\rho = 0, \qquad \partial\bar{\partial}\bar{\rho} - \frac{2\rho}{1+|\rho|^2}\partial\bar{\rho}\bar{\partial}\bar{\rho} = 0. \tag{4.2}$$

Conversely, if ρ is a solution of the sigma model (4.2), then the solutions ψ_1 and ψ_2 of GW system (3.1) have the form

$$\psi_1 = \epsilon\rho\frac{(\bar{\partial}\bar{\rho})^{1/2}}{1+|\rho|^2}, \qquad \psi_2 = \epsilon\frac{(\partial\rho)^{1/2}}{1+|\rho|^2}, \qquad \epsilon = \pm 1. \tag{4.3}$$

Thus solutions to GW system (3.1) can be obtained directly by applying the transformation (4.3) when a solution of the sigma model (4.2) is known. Once we have found particular solutions $\hat{\psi}_i$, we can calculate the corresponding value for the function $q = |\hat{\psi}_1|^2 + |\hat{\psi}_2|^2$. Next, we employ succesively the Auto-BT (3.27) in order to find new solutions p and subsequently from (3.1), we construct the multi-soliton solution ψ_i for GW system (3.1).

1. First, we look for simple nonsplitting rational soliton solutions of (4.2) admitting one simple pole given by [27]

$$\rho_j = \frac{z-a_j}{\bar{z}-\bar{a}_j}, \qquad j = 1, \dots, N, \qquad a_j \in \mathbb{C}.$$

Using the Auto-BT (3.27), we get the following algebraic multi-soliton solution of (3.1)

$$\psi_1 = \frac{\epsilon}{2}\left(\sum_{j=1}^{N}\frac{1}{(\bar{z}-\bar{a}_j)}\prod_{k=1}^{N}\frac{z-a_k}{\bar{z}-\bar{a}_k}\right)^{1/2},$$
$$\psi_2 = \frac{\epsilon}{2}\left(\sum_{j=1}^{N}\frac{1}{(z-a_j)}\prod_{k=1}^{N}\frac{z-a_k}{\bar{z}-\bar{a}_k}\right)^{1/2}, \qquad \epsilon = \pm 1.$$

For $N = 1$, the surface is determined by the equation

$$X_3 = \frac{1}{2}\ln\frac{4\left(X_1^2+X_2^2-1\right)}{a_1^2\left(X_1^2+(X_2-1)^2\right)}. \tag{4.4}$$

The corresponding surface of revolution with constant mean curvature $H = 1$ for $a_1 = 2$ is plotted in Fig. 1. Such a surface in $\mathbb{R}^3$ with a similar shape has recently been obtained in a cosmological application to white hole fissioning [28].

2. A large class of hyperbolic nonsplitting solutions, $(\partial\bar{\partial}\rho \neq 0)$, of the sigma model equations (4.2) can be constructed when the function ρ satisfies the algebraic constraint $|\rho|^2 = 1$. Consider a class of nonsplitting hyperbolic solutions of (4.2)

$$\rho = \sum_{i=1}^{N}\exp(\cosh(z-a_i) - \cosh(\bar{z}-\bar{a}_i)).$$

In all of the solutions of (4.2) which are presented below, the a_i will be arbitrary complex constants. The derivatives of ρ with respect to ∂ and $\bar{\partial}$ are given by

$$\partial\rho = \sum_{i=1}^{N}\sinh(z-a_i)\rho, \qquad \bar{\partial}\rho = -\sum_{i=1}^{N}\sinh(\bar{z}-\bar{a}_i)\rho. \tag{4.5}$$

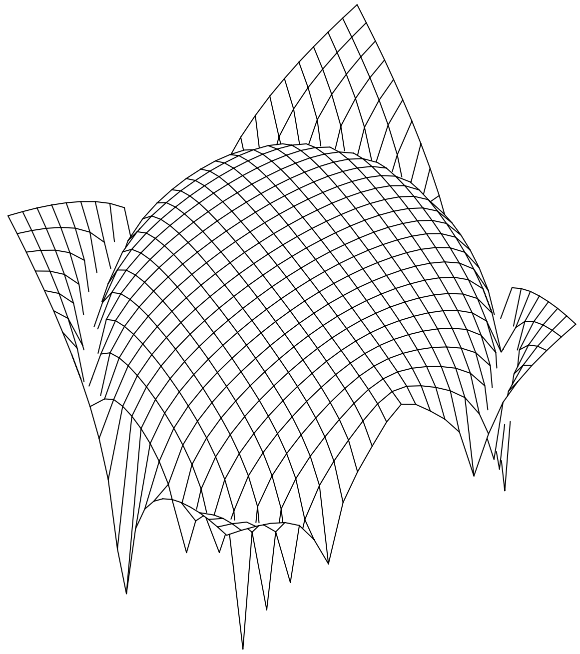

Figure 1. Constant mean curvature surface corresponding to equation (4.4) for $a_1 = 2$.

respectively. Substituting (4.5) into (4.3), we obtain the following solutions to GW system (3.1),

$$\psi_1 = \frac{\epsilon}{2}\left(\bar{\rho}\sum_{i=1}^{N}\sinh(\bar{z}-\bar{a}_i)\right)^{1/2}, \qquad \psi_2 = \frac{\epsilon}{2}\left(\rho\sum_{i=1}^{N}\sinh(z-a_i)\right)^{1/2},$$
$$p = \frac{1}{2}\left|\sum_{i=1}^{N}\sinh(z-a_i)\right|. \tag{4.6}$$

Note that these solutions do not admit any singularities. Now, solution (4.6) for $N = 1$ is substituted into integrals (3.3), we obtain

$$X_1 + iX_2 = i\sinh(-\cosh(z-a)+\cosh(\bar{z}-\bar{a})),$$
$$X_1 - iX_2 = -i\sinh(-\cosh(\bar{z}-\bar{a})+\cosh(z-a)),$$
$$X_3 = -\cosh(\cosh(z-a)-\cosh(\bar{z}-\bar{a})). \tag{4.7}$$

Eliminating the z-dependent factors on the right of (4.7), the following relationship between the X_i variables is obtained

$$X_1^2 + X_2^2 + X_3^2 = 1. \tag{4.8}$$

This represents a sphere of unit radius. Note that similar results hold when sinh is used in place of cosh in expression (4.4).

3. Consider a class of hyperbolic nonsplitting solution of (4.2) which are obtained from the tanh function that satisfies the algebraic condition $|\rho|^2 = 1$. This type of solution

represents kink-type solution, and is generated by

$$\rho = \sum_{i=1}^{N} \exp(\tanh(z - a_i) - \tanh(\bar{z} - \bar{a}_i)). \tag{4.9}$$

The derivatives of ρ in (4.9) are

$$\partial\rho = \sum_{i=1}^{N} \operatorname{sech}^2(z - a_i)\rho, \qquad \bar{\partial}\rho = -\sum_{i=1}^{N} \operatorname{sech}^2(\bar{z} - \bar{a}_i)\rho. \tag{4.10}$$

Substituting (4.10) into (4.3), we obtain the following multi-soliton solutions ψ_i to GW system (3.1),

$$\psi_1 = \frac{\epsilon}{2}\left(\bar{\rho}\sum_{i=1}^{N} \operatorname{sech}^2(\bar{z} - \bar{a}_i)\right)^{1/2}, \qquad \psi_2 = \frac{\epsilon}{2}\left(\rho\sum_{i=1}^{N} \operatorname{sech}^2(z - a_i)\right)^{1/2},$$
$$p = \frac{1}{2}\left|\sum_{i=1}^{N} \operatorname{sech}^2(z - a_i)\right|. \tag{4.11}$$

Note that the functions ψ_i admit only simple poles. For $N = 1$ the surface represents a sphere of radius one.

4. All functions ρ which generate solutions of (4.2) in Examples 1 and 2 can be used to generate larger classes of solution by taking the functions which appear in the sums for ρ in expressions (4.4) and (4.9) and combining them by taking products in different ways. For example, there exists a hyperbolic, nonsplitting solution of (4.2) of the form

$$\rho = \exp\left(\sum_{i=1}^{N}(\cosh(z - a_i) - \cosh(\bar{z} - \bar{a}_i) + \sinh(z - a_i) - \sinh(\bar{z} - \bar{a}_i)\right). \tag{4.12}$$

The corresponding solution of GW system (3.1) has the following form

$$\psi_1 = \frac{\epsilon}{2}\left(\bar{\rho}\sum_{i=1}^{N}(\sinh(\bar{z} - \bar{a}_i) + \cosh(\bar{z} - \bar{a}_i))\right)^{1/2},$$
$$\psi_2 = \frac{\epsilon}{2}\left(\rho\sum_{i=1}^{N}(\sinh(z - a_i) + \cosh(z - a_i))\right)^{1/2},$$
$$p = \frac{1}{2}\left|\sum_{i=1}^{N}(\sinh(z - a_i) + \cosh(z - a_i))\right|. \tag{4.13}$$

This solution does not admit any singularities. For $N = 1$, the surface represents a sphere of radius one exactly of the form (4.8).

5. Let the complex function $g_i(z, \bar{z})$ be a set of N harmonic functions and $f_i(z)$ a set of N arbitrary complex valued functions of one complex variable z, and the complex conjugates of these. Then we can write a general solution of the sigma model as follows

$$\rho = \exp\left[-i\sum_{j=1}^{N} g_j(z, \bar{z})\right]\prod_{j=1}^{N}\frac{f_j(z)}{\bar{f}_j(\bar{z})}.$$

Making use of transformation (4.3), the corresponding general solution of GW system (3.1) has the form

$$\psi_1 = \frac{\epsilon}{2}\left[\rho\left[i\sum_{j=1}^{N}\bar{\partial} g_j + \sum_{j=1}^{N}\frac{\bar{\partial}\bar{f}_j(\bar{z})}{\bar{f}_j(\bar{z})}\right]\right]^{1/2},$$

$$\psi_2 = \frac{\epsilon}{2}\left[\rho\left[-i\sum_{j=1}^{N}\partial g_j + \sum_{j=1}^{N}\frac{\partial f_j(z)}{f_j(z)}\right]\right]^{1/2}, \qquad \epsilon = \pm 1. \tag{4.14}$$

6. As a final example, let us consider a trigonometric solution generated by ρ of the form

$$\rho = \lambda\sin(nz), \qquad n \in \mathbb{Z}, \qquad \lambda \in \mathbb{C}.$$

Using (4.3), the corresponding periodic solution of GW system (3.1) has the form

$$\psi_1 = \epsilon\lambda\sin(nz)\frac{(n\bar{\lambda}\cos(n\bar{z}))^{1/2}}{1+|\lambda|^2\sin(nz)\sin(n\bar{z})}, \qquad \psi_2 = \epsilon\frac{(n\lambda\cos(nz))^{1/2}}{1+|\lambda|^2\sin(nz)\sin(n\bar{z})}.$$

The corresponding constant mean curvature surface for $\lambda = 1$ is given by

$$X_1^2 + X_2^2 = \frac{X_3(X_3-2)^2}{4-X_3},$$

which has the shape of a 'Mexican hat'. It is obtained by rotation of the curve

$$X_1^2(4-X_3) - X_3(X_3-2)^2 = 0,$$

for $X_3 \in [0,4)$, around the X_1 or X_2 axis.

It is useful at this point to mention some physical applications of some of these surfaces. Cylinders and spheres have applications to certain types of cosmological models, and should also be useful in describing event horizons in general relativity [1]. In string theory, the motion of a particle is described by a surface which propagates through space-time. Minimal surfaces can constitute a way of describing particle states.

Certain types of soliton solutions which are localized in time as well as in space are referred to as instantons. Such solutions exist in gauge theories, since the gauge-field equations are relativistic and give rise to topological nontriviality in time as well as in space [29]. The application of the generalized Weierstrass system to strings in three-dimensional Euclidean space will be outlined. This system allows the construction of any surface in $\mathbb{R}^3$, where p is a real-valued function of z and $\bar{z}$. With $u = |\psi_1|^2 + |\psi_2|^2$, the Gaussian curvature is $K = -u^{-2}\partial\bar{\partial}(\log u)$ and the mean curvature is $H = p/u$, so that when the mean curvature is constant, $p = u$. In terms of the variables p, ψ_i the required action for the string has the following form

$$S = 4\mu_0\int\left(|\psi_1|^2 + |\psi_2|^2\right)^2\, dxdy + \frac{4}{\alpha_0}\int p^2\, dxdy.$$

Classical configurations of strings can be described by common solutions of this Nambu–Goto–Polyakov action S and the generalized Weierstrass equations, which provide surfaces in $\mathbb{R}^3$. In generic coordinates, the corresponding Euler–Lagrange equation has the form

$$\Delta H + 2H\left(H^2 - K\right) - 2\alpha_0\mu_0 H = 0, \tag{4.15}$$

where Δ is the Laplace–Beltrami operator, and under the conformal metric

$$\Delta H = u^{-2}\partial\bar{\partial}H.$$

In terms of the variable $\varphi = H^{-1}$, $p = u/\varphi$, the Euler–Lagrange equation takes the form

$$\partial\bar{\partial}\varphi + \left[2p^2 + \partial\bar{\partial}\ln p^2\right]\varphi - 2\alpha_0\mu_0 p^2\varphi^3 = 0.$$

When the mean curvature is constant, $\varphi = \varphi_0$, and the Euler–Lagrange equation reduces to a second order linear equation

$$\partial\bar{\partial}\ln p^2 + 2\left(1 - \alpha_0\mu_0\varphi_0^2\right)p^2 = 0,$$

which can be transformed into the Liouville equation

$$\partial\bar{\partial}\theta + \beta e^{\theta} = 0, \tag{4.16}$$

where $\theta = \ln p^2$. For $\beta \neq 0$, the solution of (4.15) has the form

$$\theta = \ln\partial G + \ln\bar{\partial}\bar{G} - 2\ln\left(|G|^2 + \beta/2\right),$$

where $G(z)$ is an arbitrary analytic function. Solutions of (4.14) which correspond to mean curvature different from zero can be invesigated as well. This task will be undertaken in a future work.

5 Final remarks

In this paper, we demonstrate that the task of finding large classes of solutions of PDEs is related to the group properties of an overdetermined system composed of an initial system of PDEs (2.1) subjected to DCs (2.2). We show that this has a group theoretical interpretation in terms of "conditional symmetries". The main difficulty in this approach is related to finding reasonable ansatzes that yield compatible solutions. The approach adopted here is based on multiple constraints satisfying several specific conditions, such as (2.10), (2.12), (2.13) and (2.14), which enables us to overcome these difficulties. The proposed approach simplifies the task of solving the nonlinear determining equations (2.13) for the initial system (2.9) by providing us with an almost entirely algorithmic procedure. The most important advantage of this method is that it gives us effective tools for constructing certain classes of BT described by first order differential equations in a systematic way. Its effectiveness has been demonstrated by the results, both reconstructed [10, 21, 24, 25] and new in Section 3 and in Section 4 for the GW system (3.1).

In conclusion, we have learned from this example that it is useful to subject system (2.9) with another one, which involves certain auxiliary variables dictated by the possible representations of the "hidden" symmetry algebra (2.12). Selection of appropriate algebras and their representations is still an open problem, requiring further investigation. Finally, the presented example of GW system (3.1) suggests that there may exist a connection between the conditional symmetry method and the prolongation structure approach of Wahlquist and Estabrook [30]. Indeed, the auxiliary function y is nothing else than the pseudopotential introduced in their approach.

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