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A Universal Solution

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Abstract

The phenomenon of an implicit function which solves a large set of second order partial differential equations obtainable from a variational principle is explicated by the introduction of a class of universal solutions to the equations derivable from an arbitrary Lagrangian which is homogeneous of weight one in the field derivatives. This result is extended to many fields. The imposition of Lorentz invariance makes such Lagrangians unique, and equivalent to the Companion Lagrangians introduced in [1].

1 Introduction

The simplest example of a universal solution is that of a linear function, i.e.

$$f(x_i) = a_1 x_1 + a_2 x_2 + \cdots + a_n x_n$$

satisfies all partial differential equations in n variables each of whose terms contains at least one factor which is a derivative of second order or higher. The purpose of this article is to display a large class of functions of which the linear function is a particular case which are solutions of a set, infinite in general, of partial differential equations derived by variation of a Lagrangian. The result is given by the following theorem, and is then extended to the case of several unknowns. It is shown that the functions also provide a solution to the iterated variations of this class of Lagrangians, which terminates in the so-called Universal Field Equation. Finally, the first order formulation of the equations of motion provides further understanding of the mechanism behind the result claimed.

Theorem. *Suppose $\phi(x_i)$, $i = 1, \ldots, n$ is a differentiable function of n variables x_i. Let ϕ_j denote $\frac{\partial \phi}{\partial x_j}$, $j = 1, \ldots, n$ and let $F^j(\phi)$ be any arbitrary differentiable functions of the single argument ϕ subject to the single constraint*

$$\sum_{j=1}^{j=n} x_j F^j(\phi) = \text{constant}. \tag{1.1}$$

Then an implicit solution of this constraint for ϕ is a solution to any equation of motion for ϕ derivable from a Lagrangian of the form

$$\mathcal{L} = \mathcal{L}(\phi, \phi_j),$$

where $\mathcal{L}$ is homogeneous of weight one in the first derivatives of ϕ, i.e.

$$\sum_{j=1}^{j=n} \phi_j \frac{\partial \mathcal{L}}{\partial \phi_j} = \mathcal{L}. \tag{1.2}$$

The proof of this result is relatively easy. First of all from (1.1) it follows by differentiation that

$$\begin{aligned} \frac{\partial \phi}{\partial x_j} &= -\frac{F^j}{\sum x_i (F^i)'}, \\ \frac{\partial^2 \phi}{\partial x_j \partial x_k} &= \frac{F^j (F^k)' + F^k (F^j)'}{\left(\sum x_i (F^i)'\right)^2} + \frac{F^j F^k \left(\sum x_r (F^r)''\right)}{\left(\sum x_i (F^i)'\right)^3} \\ &= -\frac{\phi_j (F^k)' + \phi_k (F^j)'}{\sum x_i (F^i)'} + \frac{\phi_j \phi_k \left(\sum x_r (F^r)''\right)}{\sum x_i (F^i)'}. \end{aligned} \tag{1.3}$$

Here a prime denotes differentiation once with respect to the argument ϕ, i.e. $(F^j)' = \frac{dF^j}{d\phi}$. The equation of motion for $\mathcal{L}$ is

$$\begin{aligned} &\frac{\partial \mathcal{L}}{\partial \phi} - \frac{\partial}{\partial x_j} \frac{\partial \mathcal{L}}{\partial \phi_j} \\ &\quad = \frac{\partial \mathcal{L}}{\partial \phi} - \frac{\partial \phi}{\partial x_j} \frac{\partial^2 \mathcal{L}}{\partial \phi_j \partial \phi} - \frac{\partial^2 \phi}{\partial x_j \partial x_k} \frac{\partial^2 \mathcal{L}}{\partial \phi_j \partial \phi_k} = 0. \end{aligned} \tag{1.4}$$

The first two terms cancel because $\frac{\partial \mathcal{L}}{\partial \phi}$ is also homogeneous of degree one in ϕ_j leaving

$$\frac{\partial^2 \phi}{\partial x_j \partial x_k} \frac{\partial^2 \mathcal{L}}{\partial \phi_j \partial \phi_k} = 0 \tag{1.5}$$

as equation of motion. But from differentiating (1.2)

$$\sum_j \frac{\partial \phi}{\partial x_j} \frac{\partial^2 \mathcal{L}}{\partial \phi_j \partial \phi_k} = 0.$$

Using this result, together with (1.3) which expresses the second derivatives of ϕ in terms of the first, the theorem is established. Note that a characteristic feature of this equation of motion is that it is covariant, i.e. if ϕ is a solution so is any function of ϕ and this feature is manifest in part of the arbitrariness of the universal solution.

2 Multifield extension

The theorem extends to the the case where the Lagrangian depends upon the first derivatives of several fields ϕ^α, and satisfies the following orthogonality relations for the gradients of each field

$$\sum_j \frac{\partial \phi^\alpha}{\partial x_j} \frac{\partial \mathcal{L}}{\partial \frac{\partial \phi^\beta}{\partial x_j}} = \delta_{\alpha\beta} \mathcal{L}. \tag{2.1}$$

(These are somewhat stronger conditions than to demand homogeneity in the first derivatives of each ϕ^α.) In this situation the equations of motion are

$$\sum_\beta \frac{\partial^2\phi^\beta}{\partial x_j \partial x_k} \frac{\partial^2\mathcal{L}}{\partial\phi_j^\alpha \partial\phi_k^\beta} = 0 \tag{2.2}$$

and the equations which determine the universal solution take the form

$$\sum_i x_i F_i^\alpha(\phi^\beta) = c^\alpha. \tag{2.3}$$

Here the arbitrary functions F_i^α may be regarded as matrix valued functions of all the fields ϕ^β and the constants c^α depend upon α. These equations, linear in x_i provide an implicit solution for the unknowns ϕ^β The proof is along similar lines to the previous case. Differentiating (2.3) twice with respect to $(x_j,\ x_k)$ gives

$$\left(\sum_i \frac{\partial^2 F_i^\alpha}{\partial\phi^\sigma \partial\phi^\tau} x_i\right) \phi_j^\sigma \phi_k^\tau + \frac{\partial F_j^\alpha}{\partial\phi^\sigma}\phi_k^\sigma + \frac{\partial F_k^\alpha}{\partial\phi^\sigma}\phi_j^\sigma = -\left(\sum_i \frac{\partial F_i^\alpha}{\partial\phi^\sigma} x_i\right)\phi_{jk}^\sigma.$$

This implies that ϕ_{jk}^β has the structure

$$\phi_{jk}^\beta = \phi_j^\beta G_k + \phi_k^\beta G_j, \tag{2.4}$$

where G_j are functions of ϕ^σ and their derivatives whose precise form is unnecessary for the proof. Differentiating (2.1) yields

$$\sum_j \phi_j^\alpha \frac{\partial^2\mathcal{L}}{\partial\phi_j^\beta \partial\phi_k^\gamma} + \delta^{\alpha\gamma}\frac{\partial\mathcal{L}}{\partial\phi_k^\beta} = \delta^{\alpha\beta}\frac{\partial\mathcal{L}}{\partial\phi_k^\gamma}. \tag{2.5}$$

When the result for the form of ϕ_{jk}^β is substituted into the equations of motion (2.2) the consequences of homogeneity (2.5) then imply that the equations of motion are satisfied identically.

3 Iterated Lagrangians

In this section it is demonstrated that the universal solution (1.1) is not only a solution to the equation of motion derived from any Lagrangian of weight one, but also to that arising from iterations of this Lagrangian. If $\mathcal{E}$ denotes the Euler operator

$$\mathcal{E} = -\frac{\partial}{\partial\phi} + \partial_i \frac{\partial}{\partial\phi_i} - \partial_i\partial_j \frac{\partial}{\partial\phi_{ij}} + \cdots\,. \tag{3.1}$$

(In principle the expansion continues indefinitely but it is sufficient for our purposes to terminate at the stage of second derivatives ϕ_{ij}, since it turns out that the iterations do not introduce any derivatives higher than the second thanks to the weight one requirement on $\mathcal{L}$.)

Then the r fold iteration, defined by

$$\mathcal{L}^r = \mathcal{L}\mathcal{E}\mathcal{L}\mathcal{E}\mathcal{L}\cdots\mathcal{L}\mathcal{E}\mathcal{L}, \tag{3.2}$$

where $\mathcal{E}$ acts on everything to the right gives rise to the generic equation of motion

$$\epsilon_{i_1 i_2 \ldots i_{r+1}}\epsilon_{j_1 j_2 \ldots j_{r+1}} \frac{\partial^2 \mathcal{L}}{\partial\phi_{i_1}\partial\phi_{j_1}} \frac{\partial^2 \mathcal{L}}{\partial\phi_{i_2}\partial\phi_{j_2}} \cdots \frac{\partial^2 \mathcal{L}}{\partial\phi_{i_{r+1}}\partial\phi_{j_{r+1}}} \det\left|\frac{\partial^2\phi}{\partial x_{i_\alpha}\partial x_{j_\beta}}\right| = 0, \tag{3.3}$$

where a summation over all choices of $r+1$ out of the n variables $x_{i_1}, x_{i_2}, \ldots x_{i_n}$, and similarly $r+1$ out of the n variables $x_{j_1}, x_{j_2}, \ldots x_{j_n}$ is implied, and the determinant is that of the corresponding $(r+1)\times(r+1)$ matrix. Then using the structure of ϕ_{ij} (1.3) implied by the universal solution, together with the homogeneity property it is straightforward to prove that the universal solution also solves each member of (3.3). In fact after summation of each term in the determinantal expansion of (3.3) over all permutations of the indices, it is readily seen that every such sum vanishes. The culminating equation, corresponding to the $(n-1)$st equation of motion is the Universal Field Equation (so-called because it is independent of the choice of initial Lagrangian) which was introduced in [2]. In this paper it was already noted that the universal solution provides a class of solutions of this equation. Moreover, in the iteration (3.2) each successive factor $\mathcal{L}$ may be replaced by a different function of weight one in ϕ_j, viz

$$\mathcal{L}^r = \mathcal{L}_{r+1}\mathcal{E}\mathcal{L}_r\mathcal{E}\mathcal{L}_{r-1}\cdots\mathcal{L}_2\mathcal{E}\mathcal{L}_1 \tag{3.4}$$

with the same result: the universal solution is a solution to the resulting equation of motion.

4 Implications of Lorentz invariance

If there is any application of the results of this study, it is necessary to restrict the class of Lagrangians further. If one asks for the obvious requirement that the initial Lagrangian should be Lorentz invariant in addition to being homogeneous of weight one in field derivatives, then the answer is unique, up to field redefinitions and is just

$$\mathcal{L} = \sqrt{\sum \eta^{\mu\nu}\phi_\mu\phi_\nu},$$

or

$$\mathcal{L} = \sqrt{\sum J^{\mu_1\mu_2\ldots\mu_m} J_{\mu_1\mu_2\ldots\mu_m}}$$

in the case of multiple fields. $J_{\mu_1\mu_2\ldots\mu_m}$ is a typical Jacobian of the fields with respect to the base co-ordinates, and the sum is over all combinations of the indices $\mu_1, \mu_2, \ldots, \mu_m$. These Lagrangians are just the Companion Lagrangians proposed in [1, 3] as covariant analogues of the Klein Gordon Lagrangian and its extension to several fields.

5 The secret revealed!

In order to obtain more insight into the nature of the universal solution, it is illuminating to transform to a first order formulation. Suppose one sets $u_j = \frac{\phi_j}{\phi_n}$, $j \neq n$; then in the ϕ independent case the Lagrangian $\mathcal{L}(\phi_k)$ can be expressed as

$$\mathcal{L}(\phi_k) = \phi_n \mathcal{K}(u_j),$$

where the u_j satisfy the constraints

$$u_j \frac{\partial u_k}{\partial x_n} - u_k \frac{\partial u_j}{\partial x_n} = \frac{\partial u_k}{\partial x_j} - \frac{\partial u_k}{\partial x_j} \qquad \forall\, j,k = 1,\dots,n-1. \tag{5.1}$$

The equation of motion can be written as

$$\begin{aligned}
&\frac{\partial}{\partial x_n}\left(\phi_n \frac{\partial \mathcal{K}}{\partial \phi_n} + \mathcal{K}\right) + \sum_{j=1}^{n-1} \frac{\partial}{\partial x_j}\left(\phi_n \frac{\partial \mathcal{K}}{\partial \phi_j}\right) \\
&\quad = \sum_{j=1}^{n-1}\left(\frac{\partial \mathcal{K}}{\partial u_j}\frac{\partial u_j}{\partial x_n} - \frac{\partial}{\partial x_n}\left(\frac{\partial \mathcal{K}}{\partial u_j} u_j\right)\right) + \sum_{j=1}^{n-1} \frac{\partial}{\partial x_j}\left(\frac{\partial \mathcal{K}}{\partial u_j}\right) \\
&\quad = \sum_{j=1}^{n-1}\sum_{k=1}^{n-1} \frac{\partial^2 \mathcal{K}}{\partial u_j \partial u_k}\left(u_j \frac{\partial u_k}{\partial x_n} - \frac{\partial u_k}{\partial x_j}\right) = 0.
\end{aligned}$$

If there is to be a universal solution, then since $\mathcal{K}(u_j)$ is completely arbitrary, we must have

$$u_j \frac{\partial u_k}{\partial x_n} + u_k \frac{\partial u_j}{\partial x_n} - \frac{\partial u_k}{\partial x_j} - \frac{\partial u_j}{\partial x_k} = 0 \qquad \forall\, j,k = 1,\dots,n-1, \tag{5.2}$$

or, in terms of ϕ,

$$\phi_{nn}\phi_j\phi_k - \phi_{nj}\phi_n\phi_k - \phi_{nk}\phi_n\phi_j + \phi_{jk}\phi_n^2 = 0.$$

This partial differential equation is satisfied for all viable choices of the indices (j,k) in virtue of (1.3) for an implicit solution of (1.1). Combining (5.1) with (5.2) there results the simpler set of equations

$$u_j \frac{\partial u_k}{\partial x_n} - \frac{\partial u_k}{\partial x_j} = 0 \qquad \forall\, j,k = 1,\dots,n-1, \tag{5.3}$$

which may be made the basis for the deduction of the solution given.

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