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Poisson Cohomology of $SU(2)$-Covariant "Necklace" Poisson Structures on S^2

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Abstract

We compute the Poisson cohomology of the one-parameter family of $SU(2)$-covariant Poisson structures on the homogeneous space $S^2 = \mathbb{C}P^1 = SU(2)/U(1)$, where $SU(2)$ is endowed with its standard Poisson–Lie group structure, thus extending the result of Ginzburg [2] on the Bruhat–Poisson structure which is a member of this family. In particular, we compute several invariants of these structures, such as the modular class and the Liouville class. As a corollary of our computation, we deduce that these structures are nontrivial deformations of each other in the direction of the standard rotation-invariant symplectic structure on S^2; another corollary is that these structures do not admit smooth rescaling.

1 Introduction

The Poisson cohomology of a Poisson manifold (P, π) is the cohomology of the complex $(\mathfrak{X}^{\cdot}(P), d_\pi = [\pi, \cdot])$, where $\mathfrak{X}^k(P)$ is the space of smooth k-vector fields on P, and $[\cdot, \cdot]$ is the Schouten bracket. The Poisson cohomology spaces $H^k_\pi(P)$ are important invariants of (P, π). For instance, $H^0_\pi(P)$ is the space of central (Casimir) functions; $H^1_\pi(P)$ is the space of outer derivations of π; $H^2_\pi(P)$ is the space of non-trivial infinitesimal deformations of π, while $H^3_\pi(P)$ houses obstructions to extending a first-order deformation to a formal deformation. For nondegenerate (symplectic) π, the Poisson cohomology is isomorphic to the de Rham cohomology of P; in general, however, this cohomology is notoriously difficult to compute.

There are two canonical Poisson cohomology classes that merit special attention. The *modular class* $\Delta \in H^1_\pi(P)$ is the obstruction to the existence of an invariant volume form [5]: it vanishes if and only if there exists a measure on P preserved by all Hamiltonian flows. The *Liouville class* is the class of π itself in $H^2_\pi(P)$. This class is the obstruction to smooth rescaling of π: it vanishes if and only if there exists a vector field X such that $L_X\pi = \pi$; the flow of this vector field acts by rescaling π.

The purpose of this note is to compute the Poisson cohomology of all $SU(2)$-covariant Poisson structures on the two-sphere. Here $G = SU(2)$ is endowed with the standard Poisson–Lie group structure and acts on the homogeneous space $P = S^2 = SU(2)/U(1)$ by rotations (recall that a Lie group G is a *Poisson–Lie group* if it is endowed with a *multiplicative* Poisson tensor, i.e. such that the multiplication $G \times G \to G$ is a Poisson map; if a Poisson–Lie group acts on a manifold P, we say that a Poisson structure on P is *G-covariant* if the action $G \times P \to P$ is a Poisson map; see [3] or [1] for details). The $SU(2)$-covariant structures on S^2 form a 1-parameter family π_c, $c \in \mathbb{R}$. For $|c| > 1$ we get nondegenerate (symplectic) Poisson structures; $c = \pm 1$ corresponds to the (isomorphic) *Bruhat-Poisson* structures, so called because their symplectic leaves are the Bruhat cells: a point and an open 2-cell (see [3]); finally, for $|c| < 1$ there are two open symplectic leaves ("caps") of infinite area separated by a circle ("necklace") of zero-dimensional symplectic leaves. All the π_c's are invariant with respect to the residual action of $S^1 = U(1) \subset SU(2)$.

The note is organized as follows. Section 2 is devoted to the explicit description of the Poisson structures π_c, while Section 3 is devoted to the computation of their Poisson cohomology, for $|c| < 1$ (for $|c| > 1$ it is just the deRham cohomology of S^2, whereas the Bruhat case ($c = \pm 1$) was worked out by Viktor Ginzburg [2]). We proceed by first linearizing π_c in a neighborhood of the necklace (in an S^1-equivariant way) and computing its local cohomology, then using the Mayer–Vietoris argument to get the final result, which is

$$
\begin{aligned}
&H^0_{\pi_c}(S^2) = \mathbb{R},\\
&H^1_{\pi_c}(S^2) = \mathbb{R},\\
&H^2_{\pi_c}(S^2) = \mathbb{R}^2
\end{aligned}
$$

independently of the value of c. In fact, this result coincides with that of Ginzburg for $c = 1$. The generator of H^1 is the modular class Δ, whereas H^2 is spanned by the classes of π_c and π, the inverse of the standard $SU(2)$-invariant area form on S^2. This shows that (1) π_c does not admit smooth rescaling, and (2) π_c is not isotopic to $\pi_{c'}$ for $c \neq c'$.

2 Description of the Poisson structures

2.1 The classical r-matrix and the standard Poisson–Lie structure on $SU(2)$

The constructions below can be carried out for any compact semisimple Lie group, but we will only consider $SU(2)$.

Recall that the Lie algebra $\mathfrak{su}(2)$ of 2×2 skew-hermitian traceless matrices has a basis

$$
e_1 = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \qquad e_2 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad e_3 = \frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}
$$

with the commutation relations $[e_\alpha, e_\beta] = \epsilon_{\alpha\beta\gamma} e_\gamma$, where $\epsilon_{\alpha\beta\gamma}$ is the completely skew-symmetric symbol. The span of e_1 is the Cartan subalgebra $\mathfrak{a} \subset \mathfrak{su}(2)$. Recall also that

$$
SU(2) = \left\{ U = \begin{pmatrix} u & -\bar{v} \\ v & \bar{u} \end{pmatrix} \middle| \, u, v \in \mathbb{C}, \quad \det U = u\bar{u} + v\bar{v} = 1 \right\}
$$

identifies $SU(2)$ with the unit sphere in $\mathbb{C}^2$. The *standard r-matrix* $\mathbf{r} = e_2 \wedge e_3 \in \mathfrak{su}(2) \wedge \mathfrak{su}(2)$ defines a multiplicative Poisson structure on $SU(2)$ by

$$
\pi_{SU(2)}(U) = \mathbf{r}U - U\mathbf{r}. \tag{2.1}
$$

In coordinates,

$$\pi\left(\left(\begin{array}{cc} u & -\bar{v} \\ v & \bar{u} \end{array}\right)\right) = \frac{1}{4}\left(\left(\begin{array}{cc} v & \bar{u} \\ -u & \bar{v} \end{array}\right) \wedge \left(\begin{array}{cc} iv & i\bar{u} \\ iu & -i\bar{v} \end{array}\right) - \left(\begin{array}{cc} \bar{v} & u \\ -\bar{u} & v \end{array}\right) \wedge \left(\begin{array}{cc} -i\bar{v} & iu \\ i\bar{u} & iv \end{array}\right)\right)$$
$$= -iv\bar{v}\frac{\partial}{\partial u} \wedge \frac{\partial}{\partial \bar{u}} + \frac{1}{2}\left(iuv\frac{\partial}{\partial u} \wedge \frac{\partial}{\partial v} + \overline{iuv\frac{\partial}{\partial u} \wedge \frac{\partial}{\partial v}}\right)$$
$$+ \frac{1}{2}\left(iu\bar{v}\frac{\partial}{\partial u} \wedge \frac{\partial}{\partial \bar{v}} + \overline{iu\bar{v}\frac{\partial}{\partial u} \wedge \frac{\partial}{\partial \bar{v}}}\right). \tag{2.2}$$

The Poisson brackets are

$$\{u, \bar{u}\} = -iv\bar{v}, \qquad \{u, v\} = \frac{1}{2}iuv, \qquad \{u, \bar{v}\} = \frac{1}{2}iu\bar{v}, \qquad \{v, \bar{v}\} = 0.$$

It is easy to see that these formulas in fact define a smooth real Poisson structure on all of $\mathbb{C}^2$ that restricts to the unit sphere.

2.2 The Bruhat–Poisson structure on $\mathbb{C}P^1$

The r-matrix is invariant under the action of the Cartan subalgebra $\mathfrak{a}$, since

$$[e_1, \mathbf{r}] = [e_1, e_2 \wedge e_3] = [e_1, e_2] \wedge e_3 - e_2 \wedge [e_1, e_3] = e_3 \wedge e_3 + e_2 \wedge e_2 = 0.$$

Hence, the Poisson tensor (2.1) vanishes on the maximal torus (the diagonal subgroup) $A = U(1) \subset SU(2)$. In particular, $U(1)$ is a Poisson subgroup, and hence $\pi_{SU(2)}$ descends to the quotient $SU(2)/U(1) = S^3/S^1 = (\mathbb{C}^2 \setminus 0)/\mathbb{C}^\times = \mathbb{C}P^1 = S^2$. The resulting Poisson structure π_1 on $\mathbb{C}P^1$ is called the *Bruhat–Poisson structure* because its symplectic leaves coincide with the Bruhat cells in $\mathbb{C}P^1$ [3]: the base point where π_1 vanishes, and the complementary open cell where π_1 is invertible. It is $SU(2)$-covariant since $\pi_{SU(2)}$ is multiplicative. It is an easy calculation to deduce from (2.2) that in the inhomogeneous coordinate chart $w = v/u$ covering the base point π_1 is given by

$$\pi_1 = -iw\bar{w}(1 + w\bar{w})\frac{\partial}{\partial w} \wedge \frac{\partial}{\partial \bar{w}}.$$

In particular, it has a quadratic singularity at $w = 0$. The other inhomogeneous chart $z = u/v = 1/w$ gives coordinates on the open symplectic leaf, in which

$$\pi_1 = -i(1 + z\bar{z})\frac{\partial}{\partial z} \wedge \frac{\partial}{\partial \bar{z}}.$$

The corresponding symplectic 2-form is

$$\omega_1 = \frac{idz \wedge d\bar{z}}{1 + z\bar{z}}.$$

Notice that this symplectic leaf has infinite area.

2.3 The other $SU(2)$-covariant Poisson structures on S^2

The difference between any two $SU(2)$-covariant Poisson structures on $\mathbb{C}P^1$ is an $SU(2)$-invariant bivector field which is Poisson because in two dimensions, any bivector field is. Thus, any covariant structure is obtained by adding an invariant structure to the Bruhat structure π_1. To see what these structures look like, it is convenient to embed the Riemann sphere $\mathbb{C}P^1$as the unit sphere $S^2 \subset \mathbb{R}^3$ by the (inverse of) the stereographic projection. The coordinate transformations are given by

$$\begin{aligned}
x_1 &= \frac{2x}{1+x^2+y^2}, & x &= \frac{x_1}{1-x_3},\\
x_2 &= \frac{2y}{1+x^2+y^2}, & y &= \frac{x_2}{1-x_3},\\
x_3 &= \frac{x^2+y^2-1}{1+x^2+y^2}, & x^2+y^2 &= \frac{1+x_3}{1-x_3},
\end{aligned}$$

where $z = x+iy$. We shall identify $\mathbb{R}^3$ with $\mathfrak{su}(2)^*$, with the coadjoint action of $SU(2)$ by rotations. Then the linear Poisson structure on $\mathbb{R}^3 = \mathfrak{su}(2)^*$ is given by

$$-\pi = x_1 \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_3} + x_2 \frac{\partial}{\partial x_3} \wedge \frac{\partial}{\partial x_1} + x_3 \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2}$$

whose restriction to the unit sphere (a coadjoint orbit), also denoted by $-\pi$, is $SU(2)$-invariant and symplectic. Moreover, up to a constant multiple, π is the only rotation-invariant Poisson structure on S^2: any other invariant structure is of the form $\pi' = f\pi$ for some function f, but since both π and π' are invariant, so is f, hence f is a constant. It follows that there is a one-parameter family of $SU(2)$-covariant Poisson structures of the form $\pi' = \pi_1 + \alpha\pi$, $\alpha \in \mathbb{R}$; since $\pi_1 = (1-x_3)\pi$ (straightforward calculation), all $SU(2)$-covariant structures are of the form

$$\pi_c = \pi_1 + (c-1)\pi = (c-x_3)\pi, \qquad c \in \mathbb{R}.$$

It follows that π_c is symplectic for $|c| > 1$, Bruhat for $c = \pm 1$, while for $|c| < 1$ π_c vanishes on the circle $\{x_3 = c\}$ and is nonsingular elsewhere; π_c thus has two open symplectic leaves ("caps") and a "necklace" of zero-dimensional symplectic leaves along the circle. It is these "necklace" structures whose Poisson cohomology we shall compute. Notice that π_c and π_{-c} are isomorphic as Poisson manifolds via $x_3 \mapsto -x_3$.

In the original $\{w, \bar{w}\}$-coordinates we have

$$\pi = -\frac{i}{2}(1+w\bar{w})^2 \frac{\partial}{\partial w} \wedge \frac{\partial}{\partial \bar{w}} = \frac{1}{4}\left(1+x^2+y^2\right)^2 \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}, \tag{2.3}$$

$$\begin{aligned}
\pi_c = \pi_1 + (c-1)\pi &= -\frac{i}{2}(1+w\bar{w})((c+1)w\bar{w} + c - 1)\frac{\partial}{\partial w} \wedge \frac{\partial}{\partial \bar{w}}\\
&= \frac{1}{4}\left(1+x^2+y^2\right)\left((c+1)\left(x^2+y^2\right) + c - 1\right)\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y},
\end{aligned} \tag{2.4}$$

where $w = x+iy$.

2.4 Symplectic areas and modular vector fields

Before we proceed to cohomology computations, we shall compute some invariants of the structures π_c. For $|c| > 1$ π_c is symplectic, and the only invariant is the symplectic area. For the other values of c, the areas of the open symplectic leaves are easily seen to be infinite; instead, we will compute the modular vector field of π_c with respect to the standard rotation-invariant volume form ω on S^2 (the inverse of π). By elementary calculations we obtain the following

Lemma 2.1. *(1) If $|c| > 1$, the symplectic area of (S^2, π_c) is given by*

$$V(c) = 2\pi \ln \frac{c+1}{c-1}.$$

(2) For all values of c the modular vector field with respect to ω is

$$\Delta_\omega = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}.$$

Corollary 2.2. *If $|c|$, $|c'| > 1$, π_c and $\pi_{c'}$ are not isomorphic unless $|c| = |c'|$.*

Corollary 2.3. *If $|c| < 1$, the modular class of π_c is nonzero.*

Proof. The modular vector field Δ_ω rotates the necklace, hence cannot be Hamiltonian. ■

In fact, the modular class of the Bruhat–Poisson structures $\pi_{\pm 1}$ is also nonzero [2].

Unfortunately, the modular vector field does not help us distinguish the different “necklace” structures. The restriction of Δ_ω to the necklace is independent of ω since changing ω changes Δ_ω by a Hamiltonian vector field which necessarily vanishes along the necklace, so the period of Δ_ω restricted to the necklace is an invariant, but it has the same value of 2π for all π_c. When we compute the Poisson cohomology of π_c we will see a different way to distinguish them.

3 Computation of Poisson cohomology

For $|c| > 1$ π_c is symplectic, so its Poisson cohomology is isomorphic to the de Rham cohomology of S^2; the Poisson cohomology of the Bruhat–Poisson structure $\pi_{\pm 1}$ was worked out by Ginzburg [2]. Here we shall compute the cohomology of the necklace structures π_c for $|c| < 1$. Our strategy will be similar to Ginzburg’s: first compute the cohomology of the formal neighborhood of the necklace, show that the result is actually valid in a finite small neighborhood and finally, use a Mayer–Vietoris argument to deduce the global result. The validity of the Mayer–Vietoris argument for Poisson cohomology comes from the simple observation that on any Poisson manifold (P, π) the differential d_π is functorial with respect to restrictions to open subsets (i.e. a morphism of the sheaves of smooth multivector fields on P).

It will be convenient to introduce another change of coordinates:

$$s = \frac{x}{\sqrt{1+x^2+y^2}}, \qquad t = \frac{y}{\sqrt{1+x^2+y^2}}$$

mapping the (x, y)-plane to the open unit disk in the (s, t)-plane. In the new coordinates π_c and π are given by

$$\pi_c = \frac{1}{2}\left(s^2 + t^2 - \frac{1-c}{2}\right)\frac{\partial}{\partial s} \wedge \frac{\partial}{\partial t}, \tag{3.1}$$

$$\pi = \frac{1}{4}\frac{\partial}{\partial s} \wedge \frac{\partial}{\partial t} \tag{3.2}$$

and the necklace is the circle of radius $R = \sqrt{\frac{1-c}{2}}$. Observe that rescaling $s = \alpha s'$, $t = \alpha t'$ $(\alpha > 0)$ takes π_c with necklace radius R to $\pi_{c'}$ with necklace radius $R' = R/\alpha$. But this is only a local isomorphism: it does not extend to all of S^2 since it is not a diffeomorphism of the unit disk. In any case, it shows that all necklace structures are locally isomorphic, so for local computations we may assume that π_c is given in suitable coordinates by

$$\pi_c = \frac{1}{2}\left(s^2 + t^2 - 1\right)\frac{\partial}{\partial s} \wedge \frac{\partial}{\partial t}.$$

3.1 Cohomology of the formal neighborhood of the necklace

Since π_c is rotation-invariant, we can lift the computations in the formal neighborhood of the unit circle in the (s, t)-plane to its universal cover by introducing "action-angle coordinates" (I, θ):

$$s = \sqrt{1+I}\cos\theta, \qquad t = \sqrt{1+I}\sin\theta$$

in which π_c is linear:

$$\pi_c = I\frac{\partial}{\partial I} \wedge \frac{\partial}{\partial \theta}.$$

Of course we will have to restrict attention to multivector fields whose coefficients are periodic in θ. It will be convenient to think of multivector fields as functions on the supermanifold with coordinates (I, θ, ξ, η) where ξ"="∂_I and η"="∂_θ are Grassmann (anticommuting) variables. Then $\pi_c = I\xi\eta$ is a function and

$$d_{\pi_c} = [\pi_c, \cdot] = -I\eta\frac{\partial}{\partial I} + I\xi\frac{\partial}{\partial \theta} - \xi\eta\frac{\partial}{\partial \xi}$$

is a (homological) vector field. Since d_{π_c} commutes with rotations, we can split the complex into Fourier modes

$$\mathfrak{X}^0_n = \left\{f(I)e^{in\theta}\right\}; \qquad \mathfrak{X}^1_n = \left\{(f(I)\xi + g(I)\eta)e^{in\theta}\right\}; \qquad \mathfrak{X}^2_n = \left\{h(I)\xi\eta e^{in\theta}\right\},$$

where $f(I)$, $g(I)$ and $h(I)$ are formal power series in I. It will be convenient to treat the zero and non-zero modes separately; it will turn out that the cohomology is concentrated entirely in the zero mode.

Case 1. **The zero mode ($n = 0$)** consists of multivector fields independent of θ, so d_{π_c} becomes

$$d_{\pi_c}|_{\mathfrak{X}_0} = -I\eta\frac{\partial}{\partial I} + \eta\xi\frac{\partial}{\partial \xi}$$

which preserves the degree in I so the complex $\mathfrak{X}_0$ splits further into a direct product of sub-complexes $\mathfrak{X}_{0,m}$, $m \geq 0$ according to the degree:

$$0 \to \mathfrak{X}^0_{0,m} \to \mathfrak{X}^1_{0,m} \to \mathfrak{X}^2_{0,m} \to 0.$$

These complexes are very small ($\mathfrak{X}^0_{0,m}$ and $\mathfrak{X}^2_{0,m}$ are one-dimensional, while $\mathfrak{X}^2_{0,m}$ is two-dimensional) and their cohomology is easy to compute. For $f = cI^m \in \mathfrak{X}^0_{0,m}$, $d_{\pi_c} f = -cmI^m\eta$, while for $X = aI^m\xi + bI^m\eta \in \mathfrak{X}^1_{0,m}$, $d_{\pi_c} X = a(m-1)I^m\xi\eta$. Therefore, it is clear that for $m > 1$ the complex is acyclic. On the other hand, the cohomology of $\mathfrak{X}_{0,0}$ is generated by $1 \in \mathfrak{X}^0_{0,0}$ and $\eta \in \mathfrak{X}^1_{0,0}$, while the cohomology of $\mathfrak{X}_{0,1}$ is generated by $I\xi \in \mathfrak{X}^1_{0,1}$ and $I\xi\eta \in \mathfrak{X}^2_{0,1}$. Putting these together we obtain

$$\begin{aligned} H^0_0 &= \mathbb{R} = \operatorname{span}\{1\}, \\ H^1_0 &= \mathbb{R}^2 = \operatorname{span}\{\partial_\theta, I\partial_I\}, \\ H^2_0 &= \mathbb{R} = \operatorname{span}\{I\partial_I \wedge \partial_\theta\}. \end{aligned} \tag{3.3}$$

Case 2. **The non-zero modes ($n \neq 0$).** In this case d_{π_c} does not preserve the I-grading so we'll have to consider all power series at once. Let

$$\begin{aligned} f &= \left(\sum_{m=0}^{\infty} f_m I^m\right) e^{in\theta} \in \mathfrak{X}^0_n, \\ X &= \left(\sum_{m=0}^{\infty} a_m I^m\right) e^{in\theta}\xi + \left(\sum_{m=0}^{\infty} b_m I^m\right) e^{in\theta}\eta \in \mathfrak{X}^1_n, \\ B &= \left(\sum_{m=0}^{\infty} c_m I^m\right) e^{in\theta}\xi\eta \in \mathfrak{X}^2_n. \end{aligned}$$

Then

$$\begin{aligned} d_{\pi_c} f &= \left(\sum_{m=1}^{\infty} in f_{m-1} I^m\right) e^{in\theta}\xi + \left(\sum_{m=1}^{\infty} m f_m I^m\right) e^{in\theta}\eta, \\ d_{\pi_c} X &= \left(-a_0 + \sum_{m=1}^{\infty}((m-1)a_m + inb_{m-1})I^m\right) e^{in\theta}\xi\eta \end{aligned}$$

(and, of course, $d_{\pi_c} B = 0$). We see immediately that $d_{\pi_c} f = 0 \Leftrightarrow f = 0$, hence $H^0_n = \{0\}$. Moreover, any B is a coboundary:

$$B = d_{\pi_c}\left(\left(\sum_{m\neq 1}^{\infty} \frac{c_m}{m-1} I^m\right) e^{in\theta}\xi + \frac{c_1}{in} e^{in\theta}\eta\right)$$

so $H^2_n = \{0\}$ as well. Now, X is a cocycle if and only if

$$a_0 = b_0 = 0, \qquad b_m = -\frac{m a_{m+1}}{in}, \quad m \geq 1.$$

Let $f_m = \frac{a_{m+1}}{in}$ for $m \geq 0$, $f = \sum f_m I^m$. Then $X = d_{\pi_c} f$. Hence H^1_n is also trivial. So for $n \neq 0$ $\mathfrak{X}_n$ is acyclic.

It follows that the Poisson cohomology of the formal neighborhood of the necklace is as in (3.3).

3.2 Justification for the smooth case

To see that the cohomology of a finite small neighborhood of the necklace is the same as for the formal neighborhood we apply an argument similar to Ginzburg's [2]. For each Fourier mode consider the following exact sequence of complexes:

$$0 \to \mathfrak{X}^\star_{n,\mathrm{flat}} \to \mathfrak{X}^\star_{n,\mathrm{smooth}} \to \mathfrak{X}^\star_{n,\mathrm{formal}} \to 0,$$

where $\mathfrak{X}^\star_{n,\mathrm{flat}}$ consists of smooth multivector fields whose coefficients vanish along the necklace together with all derivatives. This sequence is exact by a theorem of E Borel. It suffices to show that the flat complex is acyclic. But $\pi_c^\# : \mathfrak{X}^\star_{n,\mathrm{flat}} \to \Omega^\star_{n,\mathrm{flat}}$ is an isomorphism since the coefficient of π_c is a polynomial in I, and every flat form can be divided by a polynomial with a flat result. Furthermore, the flat deRham complex is acyclic by the homotopy invariance of deRham cohomology.

Finally, we observe that a smooth multivector field in a neighborhood of the necklace (given by a *convergent* Fourier series) is a coboundary if and only if each mode is, and the primitives can be chosen so that the resulting series converges, as can be seen from the calculations in the previous subsection (integration can only improve convergence). Therefore, the Poisson cohomology of an annular neighborhood U of the necklace is

$$\begin{aligned} H^0_{\pi_c}(U) &= \mathbb{R} = \mathrm{span}\{1\},\\ H^1_{\pi_c}(U) &= \mathbb{R}^2 = \mathrm{span}\{\partial_\theta, I\partial_I\},\\ H^2_{\pi_c}(U) &= \mathbb{R} = \mathrm{span}\{I\partial_I \wedge \partial_\theta\}. \end{aligned} \tag{3.4}$$

Notice that the generators of $H^1_{\pi_c}(U)$ are the rotation $\partial_\theta = s\partial_t - t\partial_s$ (the modular vector field) and the dilation $I\partial_I = \frac{s^2+t^2-1}{2(s^2+t^2)}(s\partial_s + t\partial_t)$, while the generator of $H^2_{\pi_c}(U)$ is π_c itself, so in particular π_c does not admit rescalings even locally.

3.3 From local to global cohomology

We now have all we need to compute the Poisson cohomology of a necklace Poisson structure π_c on S^2. Cover S^2 by two open sets U and V where U is an annular neighborhood of the necklace as above, and V is the complement of the necklace consisting of two disjoint open caps, on each of which π_c is nonsingular, so that the Poisson cohomology of V and $U \cap V$ is isomorphic to the deRham cohomology. The short exact Mayer–Vietoris sequence associated to this cover

$$0 \to \mathfrak{X}^\star(S^2) \to \mathfrak{X}^\star(U) \oplus \mathfrak{X}^\star(V) \to \mathfrak{X}^\star(U \cap V) \to 0$$

leads to a long exact sequence in cohomology:

$$\begin{array}{cccccccc} 0 & \to & H^0_{\pi_c}(S^2) & \to & H^0_{\pi_c}(U) \oplus H^0_{\pi_c}(V) & \to & H^0_{\pi_c}(U\cap V) & \to \\ & \to & H^1_{\pi_c}(S^2) & \to & H^1_{\pi_c}(U) \oplus H^1_{\pi_c}(V) & \to & H^1_{\pi_c}(U\cap V) & \to \\ & \to & H^2_{\pi_c}(S^2) & \to & H^2_{\pi_c}(U) \oplus H^2_{\pi_c}(V) & \to & H^2_{\pi_c}(U\cap V) & \to & 0. \end{array}$$

Now, the first row is clearly exact since a Casimir function on S^2 must be constant on each of the two open symplectic leaves comprising V, hence constant on all of S^2 by continuity.

On the other hand, $H^1_{\pi_c}(V) = H^2_{\pi_c}(V) = H^2_{\pi_c}(U \cap V) = \{0\}$. Combining this with (3.4), we see that what we have left is

$$\begin{array}{ccccccccc}
 & & & & \mathbb{R}^2 & & \mathbb{R}^2 & & \\
 & & & & \| & & \| & & \\
0 & \to & H^1_{\pi_c}(S^2) & \to & H^1_{\pi_c}(U) \oplus H^1_{\pi_c}(V) & \to & H^1_{\pi_c}(U \cap V) & \to & \\
 & \to & H^2_{\pi_c}(S^2) & \to & H^2_{\pi_c}(U) \oplus H^2_{\pi_c}(V) & \to & 0. & & \\
 & & & & \| & & & & \\
 & & & & \mathbb{R} & & & &
\end{array}$$

Now, on the one hand, we know by Corollary 2.3 that $H^1_{\pi_c}(S^2)$ is at least one-dimensional; on the other hand, the restriction of the dilation vector field $I\partial_I$ to $U \cap V$ is not Hamiltonian: it corresponds under $\pi_c^{\#}$ to the generator of the first de Rham cohomology of the annulus, diagonally embedded into $H^1(U \cap V)$ (a disjoint union of two annuli). It follows that $H^1_{\pi_c}(S^2)$ is exactly one-dimensional, while $H^2_{\pi_c}(S^2)$ is two-dimensional.

It only remains to identify the generators. $H^1_{\pi_c}(S^2)$ is generated by the modular class, while one of the generators of $H^2_{\pi_c}(S^2)$ is π_c itself, since its class was shown to be nontrivial even locally. The other generator is the image of $(I\partial_I, -I\partial_I) \in H^1_{\pi_c}(U \cap V)$ under the connecting homomorphism. This is somewhat unwieldy since it involves a partition of unity subordinate to the cover $\{U, V\}$ which does not yield a clear geometric interpretation of the generator. Instead, we will show directly that the standard rotationally invariant symplectic Poisson structure π on S^2 is nontrivial in $H^2_{\pi_c}(S^2)$ and so can be taken as the second generator.

Lemma 3.1. *The class of the standard $SU(2)$-invariant Poisson structure π on S^2 is nonzero in $H^2_{\pi_c}(S^2)$.*

Proof. We will work in coordinates (s, t) on the unit disk in which π and π_c are given, respectively by (3.2) and(3.1). Locally π is a coboundary whose primitive is given by an Euler vector field $E = \frac{1}{2(c-1)}(s\partial_s + t\partial_t)$: it's easy to check that $[\pi_c, E] = \pi$. But E does not extend to a vector field on S^2 since it does not behave well "at infinity", i.e. on the unit circle in the (s, t)-plane. Therefore, to prove that π is globally nontrivial it suffices to show that there does not exist a Poisson vector field X such that $E + X$ is tangent to the unit circle and the restriction is rotationally invariant. In fact, it suffices to show that there is no Hamiltonian vector field X_f such that $E + X_f$ vanishes on the unit circle (since we can always add a multiple of the modular vector field to cancel the rotation). Assuming that such an f exists, we will have, in the polar coordinates $s = r\cos\phi$, $t = r\sin\phi$:

$$E + X_f = \frac{1}{2(c-1)} r \frac{\partial}{\partial r} + \frac{1}{2r}\left(r^2 - \frac{1-c}{2}\right)\left(\frac{\partial f}{\partial \phi}\frac{\partial}{\partial r} - \frac{\partial f}{\partial r}\frac{\partial}{\partial \phi}\right).$$

Upon restriction to $r = 1$ this becomes

$$(E + X_f)|_{r=1} = \left(\frac{1}{2(c-1)} + \frac{c+1}{4}\left.\frac{\partial f}{\partial \phi}\right|_{r=1}\right)\left.\frac{\partial}{\partial r}\right|_{r=1} + \frac{c+1}{4}\left.\frac{\partial f}{\partial r}\right|_{r=1}\left.\frac{\partial}{\partial \phi}\right|_{r=1}.$$

In order for this to vanish it is necessary, in particular, that $\left.\frac{\partial f}{\partial \phi}\right|_{r=1}$ be a nonzero constant which is impossible since f is periodic in ϕ. ■

We have now arrived at our final result:

Theorem 3.2. *The Poisson cohomology of a necklace Poisson structure* π_c *on* S^2 *is given as follows:*

$$
\begin{aligned}
&H^0_{\pi_c}(S^2) = \mathbb{R} = \mathrm{span}\{1\},\\
&H^1_{\pi_c}(S^2) = \mathbb{R} = \mathrm{span}\{\Delta_\omega\},\\
&H^2_{\pi_c}(S^2) = \mathbb{R}^2 = \mathrm{span}\{\pi_c, \pi\}.
\end{aligned}
$$

Corollary 3.3. π_c *does not admit smooth rescaling.*

Corollary 3.4. *The necklace structures* π_c *and* $\pi_{c'}$ *for* $c \neq c'$ *are nontrivial deformations of each other.*

Proof. $\pi_{c'} - \pi_c$ is a nonzero multiple of π but π is nontrivial in $H^2_{\pi_c}(S^2)$. ■

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