New classes at Specific Age: Properties and Testing hypotheses

M.A.W. Mahmoud[1], M.E. Moshref[1], A.M. Gadallah[1], and A.I. Shawky[2]

[1] *Department of Mathematics, Faculty of Science, Al-Azhar University, Nasr City (11884), Cairo, Egypt*

[2] *Department of Statistics, Faculty of Sciences, King Abdulaziz University, Saudi Arabia*

mawmahmoud11@yahoo.com, mmoshrefy@yahoo.com, alaadean_mag@yahoo.com, aishawky@yahoo.com

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In this article we introduce new classes of life distributions namely new better (worse) than used in expectation at specific age t_0 *NBUE*-t_0 (*NWUE*-t_0) and harmonic new better (worse) than used in expectation at specific age t_0 *HNBUE*-t_0 (*HNWUE*-t_0). The closure properties under various reliability operations such as convolution, mixture, mixing and the homogeneous Poisson shock model of these classes are studied. Furthermore, nonparametric tests are proposed to test exponentiality versus the *NBUE*-t_0 and *HNBUE*-t_0 classes. The critical values and the powers of this tests are calculated to assess the performance of the tests. It is shown that the proposed tests have high efficiencies for some commonly used distributions in reliability. Sets of real data are used as examples to elucidate the use of the proposed tests for practical problems.

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1. Introduction

Certain classes of life distributions and their variations have been introduced in reliability, the applications of these classes of life distributions can be seen in engineering, social, biological science, maintenance and biometrics. Therefore, statisticians and reliability analysts have shown a growing interest in modeling survival data using classifications of life distributions based on some aspects of aging. For example Barlow and Proschan (1981), Deshpand *et al.* (1986) and Cao and Wang (1991), gave definitions of several classes of distributions, *NBU*, *NBU*(2) and *NBUE*. Rolski (1975) introduced the HNBUE (harmonic new better than used in expectation) class, subsequently studied by Klefsjo (1982). Testing exponentiality versus the classes of life distributions has been a good deal of attention. For testing against *IFR* class we refer to Barlow and Proschan (1981) and Ahmed (1975), several other followed. For testing *IFRA* we refer to Ahmed (1994) and Deshpande (1983), while testing versus NBU are discussed by Hollander and proschan (1972), Koul (1977), Kumazawa (1983) and Ahmed (1994). Mahmoud *et al.* (2003, 2005, 2007) for *NRBU*, *RNBU* and *HNRBUE* classes. Testing versus *NBUL* are discussed by Diab *et al.* (2009) and Diab (2010). Hollander and Proschan (1975), Koul and Susarla (1980), Klefsjoe (1983) and Borges *et al.* (1984) for (*NBUE*) class. Finally testing versus *HNBUE* can be found in the work of Klefsjoe (1983), Basu and Ebrahim (1985), Ahmed (1995) and Hendi *et al.* (1998) among others. Statisticians and reliability analysts studied some aging classes of life distributions at specific age from various points of view. For more details we refer to Hollander *et al.* (1986), Ebrahimi and Habbibullah (1990), Ahmed (1998) and Pandit and Anuradha (2007) for *NBU*-t_0 and Elbatal (2007) for *NBUC*-t_0 and *NBU*(2)-t_0.

In this article we introduce new classes of life distribution at specific age t_0.

Definition 1.1. X is new better than used in expectation at specific age t_0 (denoted by $X \in$ *NBUE*-t_0) if

$$\int_0^\infty \overline{F}(x+t_0)dx \leqslant \mu \overline{F}(t_0) \text{ for all } t_0 \geqslant 0. \tag{1.1}$$

Definition 1.2. X is harmonic new better than used in expectation at specific age t_0 (denoted by $X \in$ *HNBUE*-t_0) if

$$\int_0^\infty \overline{F}(x+t_0)dx \leqslant \mu e^{-t_0/\mu} \text{ for all } t_0 \geqslant 0. \tag{1.2}$$

One can note that

$$NBU \Rightarrow NBU\text{-}t_0 \Rightarrow NBUE\text{-}t_0 \Rightarrow HNBUE\text{-}t_0.$$

In the current investigation, Preservation under convolution, mixture, mixing and the homogeneous Poisson shock model of the new classes are discussed in Section 2. In Section 3 we present a procedure to test X is exponential versus it is *NBUE*-t_0 and not exponential. Another test is constructed to *HNBUE*-t_0 in Section 4. In Section 5, the Pitman asymptotic efficiencies for the two proposed tests are calculated. Monte Carlo null distribution critical points and the power estimates are simulated in Section 6. Finally numerical examples is presented in Section 7.

2. Closure Properties

In this section some properties of our classes are introduced under convolution, mixture, mixing and the shock model in homogeneous case.

Theorem 2.1. *The* NBUE-t_0 *class is preserved under convolution.*

Proof. Suppose that F_1 and F_2 are two independent *NBUE*-t_0 lifetime distributions then their convolution is given by:

$$\overline{F} = \int_0^\infty \overline{F}_1(z-y)dF_2(y).$$

Therefore:

$$\begin{aligned}\int_0^\infty \overline{F}(x+t_0)dx &= \int_0^\infty \int_0^\infty \overline{F}_1(x+t_0-u)dF_2(u)dx \\ &= \int_0^\infty \int_0^\infty \overline{F}_1(x+t_0-u)dxdF_2(u).\end{aligned}$$

Since F_1 is *NBUE-t_0* then

$$\int_0^\infty \overline{F}(x+t_0)dx \leqslant \int_0^\infty \int_0^\infty \overline{F}_1(x-u)\overline{F}_1(t_0)dxdF_2(u)$$
$$= \overline{F}_1(t_0)\int_0^\infty \overline{F}(x)dx,$$

by using $\overline{F}_i(z) \leqslant \overline{F}(z)$ for $i = 1,2$, we get

$$\int_0^\infty \overline{F}(x+t_0)dx \leqslant \overline{F}(t_0)\int_0^\infty \overline{F}(x)dx.$$

Which complete the proof. □

The following example is presented to show that *NWUE-t_0* class is not preserved under convolution.

Example 2.1. The convolution of the exponential distribution $F(x) = 1-e^{-x}$ with itself yields the gamma distribution of order 2: $G(x) = 1-(1+x)e^{-x}$, with strictly increasing failure rate. Thus $G(x)$ is not *NWUE-t_0*.

Theorem 2.2. *The* HNBUE-t_0 *class is preserved under convolution.*

Proof. Suppose that F_1 and F_2 are two independent *HNBUE-t_0* lifetime distributions then their convolution is given by:

$$\overline{F} = \int_0^\infty \overline{F}_1(z-y)dF_2(y).$$

Therefore:

$$\int_0^\infty \overline{F}(x+t_0)dx = \int_0^\infty \int_0^\infty \overline{F}_1(x+t_0-u)dF_2(u)dx$$
$$= \int_0^\infty \int_0^\infty \overline{F}_1(x+t_0-u)dxdF_2(u).$$

Since F_1 is *HNBUE-t_0* then

$$\int_0^\infty \overline{F}(x+t_0)dx \leqslant \int_0^\infty \int_0^\infty e^{-t_0/\mu}\overline{F}_1(x-u)dxdF_2(u)$$
$$= e^{-t_0/\mu}\int_0^\infty \int_0^\infty \overline{F}_1(x-u)dF_2(u)dx$$
$$= e^{-t_0/\mu}\int_0^\infty \overline{F}(x)dx.$$

Which complete the proof. □

The following theorem is presented to show that *HNWUE-t_0* class is preserved under convolution.

Theorem 2.3. *The* HNWUE-t_0 *class is preserved under convolution.*

Proof. The proof is obtained by reversing the inequality in the last proof. □

The following example shows that the *NBUE-t_0* class is not preserved under mixtures.

Example 2.2. Let $\overline{F}_\alpha(x) = e^{-\alpha x}$ and $\overline{G}(x) = \int_0^\infty \overline{F}_\alpha(x)e^{-\alpha}d\alpha = (x+1)^{-1}$. Then the failure rate function is $r_g(x) = (x+1)^{-1}$, which is strictly decreasing thus $\overline{G}(x)$ is not *NBUE-t_0*.

The following theorem is stated and proved to show that the *NWUE-t_0* class is preserved under mixture.

Theorem 2.4. *The* NWUE-t_0 *class is preserved under mixture.*

Proof. Suppose $F(x)$ is the mixture of F_α, where each F_α is *NWUE-t_0*, since

$$\overline{F}(x) = \int_0^\infty \overline{F}_\alpha(x)dG(\alpha),$$

then

$$\begin{aligned}\int_0^\infty \overline{F}(x+t_0)dx &= \int_0^\infty \int_0^\infty \overline{F}_\alpha(x+t_0)dG(\alpha)dx \\ &= \int_0^\infty \int_0^\infty \overline{F}_\alpha(x+t_0)dxdG(\alpha).\end{aligned} \tag{2.1}$$

Since F_α is *NWUE-t_0* then

$$\int_0^\infty \int_0^\infty \overline{F}_\alpha(x+t_0)dxdG(\alpha) \geqslant \int_0^\infty \int_0^\infty \overline{F}_\alpha(t_0)\overline{F}_\alpha(x)dxdG(\alpha). \tag{2.2}$$

Upon using Chebyschev inequality for similarity ordered functions we get

$$\begin{aligned}&\int_0^\infty \int_0^\infty \overline{F}_\alpha(t_0)\overline{F}_\alpha(x)dG(\alpha)dx \geqslant \\ &\int_0^\infty \left\{\int_0^\infty \overline{F}_\alpha(t_0)dG(\alpha)\cdot\int_0^\infty \overline{F}_\alpha(x)dG(\alpha)\right\}dx.\end{aligned} \tag{2.3}$$

Upon using (2.1), (2.2) and (2.3) the proof is completed. □

The following theorem is stated and proved to show that the *HNBUE-t_0* class is preserved under mixture.

Theorem 2.5. *The* HNBUE-t_0 *class is preserved under mixture.*

Proof. Suppose $F(x)$ is the mixture of F_α, where each F_α is *HNBUE-t_0* then,

$$\begin{aligned}\int_0^\infty \overline{F}(x+t_0)dx &= \int_0^\infty \int_0^\infty \overline{F}_\alpha(x+t_0)dG(\alpha)dx \\ &= \int_0^\infty \int_0^\infty \overline{F}_\alpha(x+t_0)dxdG(\alpha).\end{aligned}$$

Since F_α is *HNBUE-t_0* then

$$\begin{aligned}\int_0^\infty \int_0^\infty \overline{F}_\alpha(x+t_0)dxdG(\alpha) &\leqslant \int_0^\infty \int_0^\infty e^{-t_0/\mu}\overline{F}_\alpha(x)dxdG(\alpha) \\ &= e^{-t_0/\mu}\int_0^\infty \int_0^\infty \overline{F}_\alpha(x)dG(\alpha)dx \\ &= e^{-t_0/\mu}\int_0^\infty \overline{F}(x)dx.\end{aligned}$$

□

The following theorem is presented to show that *HNWUE-*t_0 class is preserved under mixture.

Theorem 2.6. *The* HNWUE-t_0 *class is preserved under mixture.*

Proof. The proof is obtained by reversing the inequality in the last proof. □

The following example illustrates that the *NBUE-*t_0 and *HNBUE-*t_0 classes is not preserved under mixing.

Example 2.3. Let $\overline{F}_1 = e^{-\delta x}$ and $\overline{F}_2 = e^{-\gamma x}$. Let $\overline{F} = \frac{1}{2}\overline{F}_1 + \frac{1}{2}\overline{F}_2$. It follows that both $\overline{F}_1$ and $\overline{F}_2$ are *NBUE-*t_0 and *HNBUE-*t_0 but $\overline{F}$ is neither *NBUE-*t_0 nor *HNBUE-*t_0.

2.1. *Homogeneous Poisson Shock Model*

Suppose that a device is subjected to sequence shocks occurring randomly in the time according to a Poisson process with constant intensity λ. Suppose further that the device has probability $\overline{p}_k$ of surviving the first k shocks, where $1 = \overline{p}_0 \geqslant \overline{p}_1 \geqslant \cdots$. Then the survival function of the device is given by

$$\overline{H}(t) = \sum_{k=0}^{\infty} \overline{p}_k \frac{(\lambda t)^k}{k!} e^{-\lambda t}. \tag{2.4}$$

This shock model has been studied by Esary (1969) for *IFR, IFRA, DMRL, NBU*, and *NBUE* classes. Klefsjo (1981) for *HNBUE* and Mahmoud *et al.* (2009) for *NBURFR-*t_0.

Definition 2.1. A discrete distribution p_k, $k = 0, 1, \ldots, \infty$ or its survival probabilities $\overline{p}_k$, $k = 0, 1, \ldots, \infty$ is said to have discrete new better (worse) than used expectation at specified time t_0 (*NBUE-*t_0) (*NWUE-*t_0) if

$$\sum_{r=0}^{\infty} \overline{p}_{r+j} \leqslant (\geqslant) \overline{p}_j \sum_{r=0}^{\infty} \overline{p}_r, \quad j = 0, 1, \ldots. \tag{2.5}$$

Definition 2.2. A discrete distribution p_k, $k = 0, 1, \ldots, \infty$ or its survival probabilities $\overline{p}_k$, $k = 0, 1, \ldots, \infty$ is said to have discrete harmonic new better (worse)than used expectation at specific age t_0 (*HNBUE-*t_0) (*HNWUE-*t_0) if

$$\sum_{r=0}^{\infty} \overline{p}_{r+j} \leqslant (\geqslant) (1 - \frac{1}{m})^j \sum_{r=0}^{\infty} \overline{p}_r, \quad j = 0, 1, \ldots. \tag{2.6}$$

Now, let us introduce the following theorems.

Theorem 2.7. *If* p_k *is discrete* NBUE-t_0*, then* $\overline{H}(t)$ *given by* (2.4) *is* NBUE-t_0.

Proof. It must be shown that

$$\int_0^\infty \overline{H}(x+t_0)dx \leqslant \overline{H}(t_0)\int_0^\infty \overline{H}(x)dx.$$

Upon using (2.4), we get

$$\int_0^\infty \overline{H}(x+t_0)dx = \int_0^\infty \sum_{k=0}^\infty \overline{p}_k \frac{[\lambda(t_0+x)]^k}{k!} e^{-\lambda(x+t_0)}dx$$

$$= e^{-\lambda t_0} \sum_{k=0}^\infty \overline{p}_k \sum_{r=0}^k \tbinom{k}{r} \frac{(\lambda t_0)^{k-r}}{k!} \int_0^\infty (\lambda x)^r e^{-\lambda x}dx.$$

Integrating by parts yields

$$\begin{aligned}
&= \frac{e^{-\lambda t_0}}{\lambda} \sum_{r=0}^\infty \sum_{k=r}^\infty \overline{p}_k \frac{(\lambda t_0)^{(k-r)}}{(k-r)!} \\
&= \frac{e^{-\lambda t_0}}{\lambda} \sum_{r=0}^\infty \sum_{j=0}^\infty \overline{p}_{j+r} \frac{(\lambda t_0)^j}{j!} \\
&\leqslant \frac{e^{-\lambda t_0}}{\lambda} \sum_{j=0}^\infty \sum_{r=0}^\infty \overline{p}_j \overline{p}_r \frac{(\lambda t_0)^j}{j!} \\
&= \overline{H}(t_0) \int_0^\infty \sum_{r=0}^\infty \frac{(\lambda x)^r}{r!} \overline{p}_r e^{-\lambda x}dx \\
&= \overline{H}(t_0) \int_0^\infty \overline{H}(x)dx.
\end{aligned}$$

□

The proof for the *NWUE-t_0* class is obtained by reversing the inequality.

Theorem 2.8. *If p_k is discrete HNBUE-t_0, then $\overline{H}(t)$ given by* (2.4) *is HNBUE-t_0.*

Proof. Upon using (2.4), we have

$$\mu = \int_0^\infty \overline{H}(t)dt = \frac{1}{\lambda}\sum_{k=0}^\infty \overline{p}_k \int_0^\infty \frac{(\lambda t)^k}{k!} d(\lambda t) = \frac{1}{\lambda}\sum_{k=0}^\infty \overline{p}_k = \frac{m}{\lambda},$$

where m is the mean of the discrete distribution $\overline{p}_k$.
It must be shown that

$$\int_0^\infty \overline{H}(x+t_0)dx \leqslant e^{-t_0/\mu_H} \int_0^\infty \overline{H}(x)dx.$$

Upon using (2.4), we get

$$\begin{aligned}
\int_0^\infty \overline{H}(x+t_0)dx &= \int_0^\infty \sum_{k=0}^\infty \overline{p}_k \frac{[\lambda(t_0+x)]^k}{k!} e^{-\lambda(x+t_0)}dx \\
&= e^{-\lambda t_0} \sum_{k=0}^\infty \overline{p}_k \sum_{r=0}^k \tbinom{k}{r} \frac{(\lambda t_0)^{k-r}}{k!} \int_0^\infty (\lambda x)^r e^{-\lambda x}dx.
\end{aligned}$$

Integrating by parts yields

$$\begin{aligned}
&= \frac{e^{-\lambda t_0}}{\lambda}\sum_{r=0}^{\infty}\sum_{k=r}^{\infty}\overline{p}_k\frac{(\lambda t_0)^{(k-r)}}{(k-r)!}\\
&= \frac{e^{-\lambda t_0}}{\lambda}\sum_{r=0}^{\infty}\sum_{j=0}^{\infty}\overline{p}_{j+r}\frac{(\lambda t_0)^{j}}{j!}\\
&\leqslant \frac{e^{-\lambda t_0}}{\lambda}\sum_{j=0}^{\infty}\sum_{r=0}^{\infty}(1-\frac{1}{m})^j\overline{p}_r\frac{(\lambda t_0)^{j}}{j!}\\
&= e^{\frac{-\lambda t_0}{m}}\int_0^{\infty}\sum_{r=0}^{\infty}\frac{(\lambda x)^r}{r!}\overline{p}_r e^{-\lambda x}dx\\
&= e^{-t_0/\mu_H}\int_0^{\infty}\overline{H}(x)dx.
\end{aligned}$$

□

The proof for the *HNWUE-*t_0 class is obtained by reversing the inequality.

3. Testing in the *NBUE-*t_0

Our goal in this section is to propose a test statistic for testing exponentiality against *NBUE-*t_0 class of life distributions. The following is the measure of departure from H_0

$$\triangle_1 = \mu\overline{F}(t_0) - \int_0^{\infty}\overline{F}(x+t_0)dx, \tag{3.1}$$

it is clear that

$$\int_0^{\infty}\overline{F}(x+t_0)dx = \mu - t_0, \tag{3.2}$$

then

$$\triangle_1 = \mu\overline{F}(t_0) - \mu + t_0. \tag{3.3}$$

One can notice that the value of $\triangle_1$ under H_0 equals ζ, where

$$\zeta = \overline{F}(t_0) + t_0 - 1.$$

Consider $\delta_1 = \triangle_1 - \zeta$, δ_1 can be written in the following form

$$\delta_1 = [\overline{F}(t_0) - 1]E[X-1]. \tag{3.4}$$

Note that under $H_0 : \delta_1 = 0$, while under $H_1 : \delta_1 > 0$.

To estimate δ_1, let $X_1, X_2, \dots, X_n$ be a random sample from F, so the empirical form of δ_1 in (3.4) is

$$\hat{\delta}_{1n} = \frac{[\overline{F}(t_0) - 1]}{n} \sum_{i=1}^{n} [X_i - 1], \tag{3.5}$$

To find the limiting distribution of $\hat{\delta}_{1n}$ we resort to the U-statistic theory. Let

$$\phi(X) = [\overline{F}(t_0) - 1][X - 1],$$

and define the symmetric kernel

$$\psi(X) = \sum_R \phi(X_i),$$

where the sum is over all arrangements of X_i, this leads that $\hat{\delta}_{1n}$ in (3.5) is equivalent to U-statistic given by

$$U_n = \frac{1}{n} \sum_R \psi(X_i).$$

The next results summarizes the asymptotic normality of $\hat{\delta}_{1n}$.

Theorem 3.1.

(i) *As $n \to \infty$, $\sqrt{n}(\hat{\delta}_{1n} - \delta_1)$ is asymptotically normal with mean 0 and variance is*

$$\sigma^2 = Var\{[\overline{F}(t_0) - 1](X - 1)\}. \tag{3.6}$$

(ii) *Under H_0, the variance is reduced to*

$$\sigma_0^2 = (e^{-t_0} - 1)^2. \tag{3.7}$$

The proof is emitted because it is clear.

4. Testing in the *HNBUE*-t_0

The main aim of this section is to propose a test statistic for testing H_0: F is exponential versus H_1: F belongs to *HNBUE*-t_0 class and not exponential. We propose the following measure of departure

$$\triangle_2 = \mu e^{-t_0} - \int_0^\infty \overline{F}(x + t_0)dx, \tag{4.1}$$

it is clear that

$$\int_0^\infty \overline{F}(x + t_0)dx = \mu - t_0, \tag{4.2}$$

then

$$\triangle_2 = \mu e^{-t_0} - \mu + t_0. \tag{4.3}$$

One can notice that the value of $\triangle_2$ under H_0 equals ξ, where

$$\xi = e^{-t_0} + t_0 - 1.$$

Consider $\delta_2 = \triangle_2 - \xi$, δ_2 can be written in the following form

$$\delta_2 = [e^{-t_0} - 1]E[X - 1]. \tag{4.4}$$

Based on a random sample $X_1, X_2, X_3, \ldots, X_n$ from a distribution F an unbiased estimate of δ_2 is given by

$$\hat{\delta}_{2n} = \frac{[e^{-t_0} - 1]}{n} \sum_{i=1}^{n} [X_i - 1]. \tag{4.5}$$

To find the limiting distribution of $\hat{\delta}_{2n}$. Set

$$\Phi(X) = [e^{-t_0} - 1][X - 1].$$

Thus, the variance is

$$\sigma^2 = Var[\phi(X)].$$

Under H_0, we get

$$\sigma_0^2 = (e^{-t_0} - 1)^2. \tag{4.6}$$

According to U-statistic theory (cf. Lee, 1990), Theorem 4.1 is immediate.

Theorem 4.1.

(i) *As $n \to \infty$, $\sqrt{n}(\hat{\delta}_{2n} - \delta_2) \sim N(0, \sigma^2)$, where*

$$\sigma^2 = Var([e^{-t_0} - 1][X - 1])$$

(ii) *Under H_0 the variance is reduced to σ_0^2 in* (4.6).

5. The Pitman Asymptotic Efficiency (PAE)

To asses how good this procedures are relative to others in the literature we evaluate its Pittman asymptotic efficiency (PAE) for two alternatives in our classes *NBUE-*t_0 and *HNBUE-*t_0, these are:

1. Linear failure rate family (LFR): $\overline{F}_\theta(x) = \exp(-x - \frac{\theta}{2}x^2)$, $x > 0$, $\theta \geqslant 0$.
2. Makeham family: $\overline{F}_\theta = \exp(-x + \theta(x + e^{-x} - 1))$, $x > 0$, $\theta \geqslant 0$.

The PAE is defined by

$$PAE(\delta) = \frac{1}{\sigma_0} \left| \frac{d\delta}{d\theta} \right|_{\theta \to \theta_0}.$$

In the above cases, we can prove that,

$$PAE(\hat{\delta}_1, F) = \frac{1}{\sigma_0}\left|\overline{F}'_\theta(t_0) + \overline{F}_\theta(t_0)\int_0^\infty \overline{F}'_\theta(x)dx - \int_{t_0}^\infty \overline{F}'_\theta(x)dx\right|,$$

$$PAE(\hat{\delta}_2, F) = \frac{1}{\sigma_0}\left|e^{-t_0}\int_0^\infty \overline{F}'_\theta(x)dx - \int_{t_0}^\infty \overline{F}'_\theta(x)dx\right|,$$

$$PAE(\hat{\delta}_1, LFR) = \frac{1}{\sigma_0}\left|t_0 e^{-t_0}\right|,$$

$$PAE(\hat{\delta}_1, Makeham) = \frac{1}{\sigma_0}\left|\frac{1}{2}e^{-t_0}(1-e^{-t_0})\right|,$$

$$PAE(\hat{\delta}_2, LFR) = \frac{1}{\sigma_0}\left|e^{-t_0}(\frac{t_0^2}{2}+t_0)\right|,$$

and

$$PAE(\hat{\delta}_2, Makeham) = \frac{1}{\sigma_0}\left|\frac{1}{2}e^{-t_0}(2t_0+e^{-t_0}-1)\right|.$$

We notice that $t_0 = 0.1$ is the value that maximizes the above values.

We compare the above PAE's at $t_0 = 0.1$ with that of Hollander and Proschan (1975) and Ahmad *et al.* (1999), and the results are shown in Table 1.

Table 1 shows that our tests outperforms the others tests for the two alternatives.

6. Monte Carlo Null Distribution Critical Points

Many practitioners, such as applied statisticians, and reliability analysts are interested in simulated percentiles. Table 2 gives these percentile points of the statistics $\hat{\delta}_{1n}$ and $\hat{\delta}_{2n}$ given in (3.5) and (4.5) at $t_0 = 0.1$ and the calculations are based on 1000 simulated samples of sizes $n = 2(1)50$.

6.1. *The Power Estimates*

Tables 3 and 4 show the power estimate of the test statistic $\hat{\delta}_{1n}$ and $\hat{\delta}_{2n}$ given in (3.5) and (4.5) respectively at the significant level 0.05 using LFR and Makeham distributions. The estimates are based on 1000 simulated samples for sizes $n = 10,\ 20$ and 30.

From Tables 3 and 4 we can show that our tests have perfect power.

7. Numerical Examples

Example 7.1. The following data represent 39 liver cancers patients taken from Elminia cancer center Ministry of Health – Egypt, which entered in (1999). The ordered life times (in years) are:

0.027	0.038	0.038	0.038	0.038	0.038	0.041	0.047	0.049	0.055
0.055	0.055	0.055	0.055	0.063	0.063	0.066	0.071	0.082	0.082
0.085	0.110	0.314	0.140	0.143	0.164	0.167	0.184	0.195	0.203
0.206	0.238	0.263	0.288	0.293	0.293	0.293	0.318	0.411	

It was found that

- $\hat{\delta}_{1n} = 0.509$ and this value greater than the tabulated critical value in Table 1. Then we accept H_1 which states that the data has *NBUE-t_0* property.

- $\hat{\hat{\delta}}_{2n} = 0.098$ which is greater than the tabulated critical value in Table 1. Then we accept H_1 which states that the data set has *HNBUE-t_0* property.

Example 7.2. Consider the well-known Darwin data (Fisher, 1966) that represent the differences in heights between cross- and self-fertilized plants of the same pair grown together in one pot

4.9	−6.7	0.8	1.6	2.3	2.8	4.1	1.4
2.9	0.6	5.6	2.4	7.5	6.0	−4.8	

It was found that

- $\hat{\hat{\delta}}_{1n} = 0.9$ which is greater than the tabulated critical value in Table 1. Then we conclude that this data set has *NBUE-t_0* property.
- $\hat{\hat{\delta}}_{2n} = -465.079$ and this value less than the tabulated critical value in Table 1. There is enough evidence to accept H_0 which states that the data set has exponential property.

8. Appendix

Table 1. The PAE's for LFR and Makeham families

Test	LFR	Makeham
Hollander-Proschan	0.8660	0.2886
Ahmad *et al.*	0.7490	0.2800
δ_1	0.951	0.4524
δ_2	0.998	0.4984

Table 2. Critical values of statistic $\hat{\hat{\delta}}_{1n}$ and $\hat{\hat{\delta}}_{2n}$ at $t_0 = 0.1$

n	0.01	0.05	0.10	0.90	0.95	0.99
2	−0.214	−0.137	−0.088	0.071	0.080	0.090
3	−0.178	−0.103	−0.075	0.061	0.070	0.084
4	−0.144	−0.087	−0.065	0.053	0.062	0.076
5	−0.126	−0.079	−0.060	0.048	0.057	0.069
6	−0.115	−0.074	−0.051	0.045	0.054	0.068
7	−0.102	−0.065	−0.047	0.043	0.051	0.066
8	−0.089	−0.060	−0.046	0.041	0.048	0.060
9	−0.086	−0.053	−0.042	0.038	0.045	0.057
10	−0.080	−0.053	−0.039	0.037	0.044	0.054
11	−0.084	−0.053	−0.037	0.036	0.043	0.052
12	−0.079	−0.046	−0.035	0.036	0.042	0.054
13	−0.075	−0.044	−0.033	0.034	0.041	0.053
14	−0.072	−0.046	−0.033	0.032	0.039	0.049
15	−0.063	−0.044	−0.031	0.030	0.036	0.049
16	−0.065	−0.042	−0.033	0.031	0.037	0.048
17	−0.063	−0.041	−0.030	0.029	0.036	0.047
18	−0.060	−0.039	−0.028	0.030	0.036	0.047
19	−0.061	−0.038	−0.029	0.028	0.034	0.044
20	−0.056	−0.037	−0.026	0.029	0.035	0.045
21	−0.059	−0.036	−0.027	0.027	0.032	0.041
22	−0.054	−0.034	−0.025	0.027	0.034	0.043
23	−0.054	−0.034	−0.025	0.025	0.032	0.043
24	−0.054	−0.033	−0.025	0.026	0.032	0.041
25	−0.050	−0.036	−0.026	0.026	0.031	0.041
26	−0.050	−0.031	−0.024	0.024	0.029	0.039
27	−0.050	−0.032	−0.025	0.024	0.030	0.040
28	−0.049	−0.030	−0.024	0.023	0.028	0.038
29	−0.047	−0.031	−0.024	0.024	0.030	0.039
30	−0.044	−0.033	−0.021	0.021	0.027	0.037
32	−0.042	−0.028	−0.022	0.022	0.027	0.037
33	−0.043	−0.029	−0.022	0.021	0.027	0.035
34	−0.042	−0.028	−0.022	0.021	0.027	0.036
35	−0.042	−0.029	−0.022	0.021	0.026	0.033
36	−0.041	−0.027	−0.020	0.021	0.026	0.033
37	−0.042	−0.027	−0.020	0.020	0.025	0.033
38	−0.040	−0.027	−0.020	0.020	0.026	0.035
39	−0.040	−0.026	−0.019	0.020	0.025	0.031
40	−0.041	−0.027	−0.020	0.019	0.024	0.033
41	−0.038	−0.026	−0.020	0.019	0.023	0.031
42	−0.037	−0.026	−0.019	0.018	0.023	0.033
43	−0.039	−0.025	−0.019	0.019	0.023	0.032
44	−0.041	−0.025	−0.018	0.018	0.023	0.032
45	−0.037	−0.026	−0.019	0.019	0.024	0.030
46	−0.037	−0.026	−0.017	0.019	0.023	0.030
47	−0.038	−0.026	−0.019	0.018	0.023	0.030
48	−0.039	−0.024	−0.019	0.018	0.022	0.029
49	−0.037	−0.024	−0.018	0.018	0.023	0.029
50	−0.035	−0.023	−0.018	0.017	0.022	0.029

Table 3. Power estimates using $\alpha = 0.05$ at $t_0 = 0.1$

	n	$\theta = 1$	$\theta = 2$	$\theta = 3$	$\theta = 4$
LFR	10	1.000	1.000	1.000	1.000
	20	1.000	1.000	1.000	1.000
	30	1.000	1.000	1.000	1.000
Makeham	10	1.000	1.000	1.000	1.000
	20	1.000	1.000	1.000	1.000
	30	1.000	1.000	1.000	1.000

Table 4. Power estimates using $\alpha = 0.05$ at $t_0 = 0.1$

	n	$\theta = 1$	$\theta = 2$	$\theta = 3$	$\theta = 4$
LFR	10	1.000	1.000	1.000	1.000
	20	1.000	1.000	1.000	1.000
	30	1.000	1.000	1.000	1.000
Makeham	10	1.000	1.000	1.000	1.000
	20	1.000	1.000	1.000	1.000
	30	1.000	1.000	1.000	1.000

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