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A Symmetric Component Alpha Normal Slash Distribution: Properties and Inferences

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In this paper, we introduce a new class of symmetric bimodal distribution. We define the distribution by means of a stochastic representation as the mixture of a symmetric component alpha normal random variable with respect to the power of a uniform random variable. The proposed distribution is more flexible in terms of its kurtosis than the slashed normal distribution. Properties involving moments and moment generating function are studied. The proposed distribution is illustrated with a real application by maximum likelihood procedure.

Keywords: Alpha Normal distribution, Kurtosis, Skewness, Slash distribution, Symmetric Component.

AMS (2000) *Subject classification*: 62E10, 62F10

1. Introduction

The slash distribution is defined as the ratio of two independent random variables, which is described as follows: a random variable S has a standard slash distribution $SL(q)$ if S can be represented as

$$S = \frac{Z}{U^{\frac{1}{q}}}, \tag{1.1}$$

where $Z \sim N(0,1)$ and $U \sim U(0,1)$ are independent random variables. $q > 0$ is the shape parameter. The distribution was named by William H. Rogers and John Tukey in a paper published in 1972 [21].

The slash distribution is more flexible than the normal distribution which plays a kew role in statistical analysis but the inference based on that is sensitive to statistical errors that have a heavier-tailed distribution [3]. Slash distribution has been very popular in robust statistical analysis [11, 14, 17, 21].

The $SL(q)$ density function is given by

$$f(x) = q\int_0^1 u^q \phi(xu)du, \quad -\infty < x < \infty, \tag{1.2}$$

where $\phi(t) = \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}$ is the standard normal density function. The mean and variance of the slash distribution are given by $\mathbb{E}(S) = 0$ for $q > 1$ and $\mathrm{Var}(S) = \frac{q}{q-2}$ for $q > 2$.

For the limit case $q \to \infty$, it yields the standard normal distribution. The standard slash density has heavier tails than those of the normal distribution and has larger kurtosis. For $q = 1$, we obtain the canonical slash distribution having the following density [12]

$$f(x) = \begin{cases} \frac{\phi(0)-\phi(x)}{x^2} & x \neq 0, \\ \frac{\phi(0)}{2} & x = 0. \end{cases} \tag{1.3}$$

Rogers and Tukey [21] and Mosteller and Tukey [18] studied the general properties of this canonical slash distribution. Kafadar [13] found the maximum likelihood estimators of the location and scale parameters for the standard slash distribution. Wang and Genton [22] discussed the multivariate skew version of the distribution and studied its properties and inferences and used it to fit some skew data sets. They considered a skew normal distribution introduced by Azzalini [4] for the standard normal random variable Z to define a skew extension of the slash distribution.

Arslan [1] developed an alternative asymmetric extension of the distribution by using the variance-mean mixture of the multivariate normal distribution. Arslan [2] introduced an EM-based maximum likelihood estimation procedure to estimate the parameters of this distribution. Arslan and Genç [3] proposed a generalization of the multivariate symmetric slash distribution. Gómez *et al.* [7] defined some symmetric extensions of the distribution based on elliptical distributions and studied its general properties of the resulting families, including their moments.

We consider a symmetric component random variable which generalizes the normal distribution Z and define an extension of the slash distribution and study its properties.

The paper is organized as follows: in Section 2, we present the new class and study the properties, including the stochastic representation etc. Section 3 discusses the inference, maximum likelihood estimation and bimodality test for the parameters. A simulation is conducted in Section 4 to investigate the behavior of maximum likelihood estimate (MLE). Section 5, a real dataset analyzed by other distributions is reported and illustrated to fit the model. We conclude our work in Section 6.

2. Symmetric Component Alpha Normal Slash Distribution and its properties

2.1. *Stochastic representation*

Definition 2.1. A random variable X has a symmetric component normal distribution, denoted by $X \sim SCN(\alpha)$, if X has density function given by

$$f_X(x) = \frac{1+\alpha x^2}{1+\alpha}\phi(x), \quad -\infty < x < \infty, \alpha \geqslant 0. \tag{2.1}$$

Remark 2.1. The density function (2.1) is a mixture between a normal density and a bimodal-normal density, as shown below

$$f_X(x) = \frac{1}{1+\alpha}\phi(x) + \frac{\alpha}{1+\alpha}x^2\phi(x). \tag{2.2}$$

If we take $\alpha = 0$, we obtain the standard normal distribution $Z \sim N(0,1)$. If $\alpha \to \infty$, it is a bimodal-normal random variable. An algebraic computation shows that it is bimodal if $\alpha > 0.5$ and it is unimodal if $0 \leqslant \alpha \leqslant 0.5$. Elal-Olivero *et al.* [5] studied the equivalent form of this density when introducing an alternative form of generate asymmetry in the normal distribution that allows to fit unimodal and bimodal data sets.

Proposition 2.1. *Let $X \sim SCN(\alpha)$, then the k^{th} non-central moments are given by*

$$\mathbb{E}X^k = \begin{cases} \frac{1+(2m+1)\alpha}{1+\alpha} \prod_{j=1}^{m} (2j-1) & k = 2m \\ 0 & k = 2m-1 \end{cases}, \quad for \quad m = 1,2,3,\ldots \tag{2.3}$$

Proof. Using (2.2), if $k = 2m-1$, we get $\mathbb{E}X^k = 0$; If $k = 2m$,

$$\mathbb{E}X^k = \frac{1}{1+\alpha}\mathbb{E}Z^k + \frac{\alpha}{1+\alpha}\mathbb{E}Z^{k+2} = \frac{1+(2m+1)\alpha}{1+\alpha}\prod_{j=1}^{m}(2j-1).$$

□

Proposition 2.2. *Let $X \sim SCN(\alpha)$, then the moment generating function of X is given by*

$$M_X(t) = (1 + \frac{\alpha t^2}{1+\alpha})e^{\frac{t^2}{2}}. \tag{2.4}$$

It can be directly obtained by integration and the proof is omitted.

Now we focus on introducing a new distribution using the idea of the slash construction as the ratio of two independent random variables.

Definition 2.2. A random variable Y has a symmetric component alpha normal slash distribution with parameters $\alpha \geqslant 0$ and $q > 0$ if it can be represented as the ratio

$$Y = \frac{X}{U^{\frac{1}{q}}}, \tag{2.5}$$

where $U \sim U(0,1)$ and $X \sim SCN(\alpha)$ are independent. We denote it as $Y \sim SCNS(\alpha, q)$.

Proposition 2.3. *Let $Y \sim SCNS(\alpha, q)$. Then, the density function of Y is given by*

$$f_Y(y) = \begin{cases} \frac{q}{y^{q+1}} \int_0^y \frac{1+\alpha t^2}{1+\alpha}\phi(t)t^q dt & y \neq 0 \\ \frac{q}{q+1}\frac{\phi(0)}{1+\alpha} & y = 0 \end{cases}, \tag{2.6}$$

Or

$$f_Y(y) = q\int_0^1 \frac{1+\alpha y^2 t^2}{1+\alpha}\phi(yt)t^q dt. \tag{2.7}$$

Proof. The joint probability density function of X and U is given by $g(x,u) = \frac{1+\alpha x^2}{1+\alpha}\phi(x)$, for $-\infty < x < \infty, 0 < u < 1$. Using the following transformation: $y = \frac{x}{u^{1/q}}, u = u$, the joint probability density function of Y and U is given by

$$h(y,u) = \frac{1+\alpha(yu^{\frac{1}{q}})^2}{1+\alpha}\phi(yu^{\frac{1}{q}})u^{\frac{1}{q}}, \tag{2.8}$$

where $-\infty < y < \infty, 0 < u < 1$. The density function of Y is given by

$$f_Y(y) = \int_0^1 \frac{1+\alpha(yu^{\frac{1}{q}})^2}{1+\alpha}\phi(yu^{\frac{1}{q}})u^{\frac{1}{q}}du. \tag{2.9}$$

If $y = 0$, $f_Y(y) = \frac{q}{q+1}\frac{\phi(0)}{1+\alpha}$. If $y \neq 0$, the result follows by using $t = yu^{\frac{1}{q}}$. On the other hand, From (2.9), let $t = u^{\frac{1}{q}}$, the density function can also be written as (2.7). □

Remark 2.2. If $\alpha = 0$, the density function in (2.7) reduces to (1.2). $Y \sim SCNS(\alpha = 0, q)$ is the slashed normal distribution obtained by William H. Rogers and John Tukey [21]. If $\alpha = 0$ and $q = 1$, the density function in (2.6) reduces to (1.3). $Y \sim SCNS(\alpha = 0, q = 1)$ is the canonic slashed normal distribution. As $q \to \infty$, the limit case of the symmetric component alpha normal slash distribution $Y \sim SCNS(\alpha, q = \infty)$ is the symmetric component alpha normal distribution $SCN(\alpha)$.

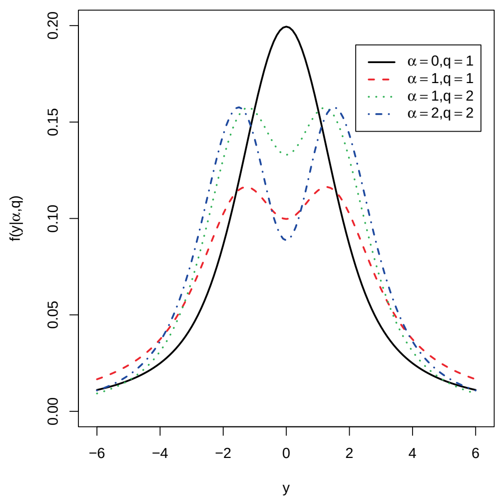

Fig. 1. The density function of $SCNS(\alpha,q)$ for selected parameters

Figure 1 depicts some plots of the density function of the symmetric component alpha normal slash distribution for selected parameters.

The cumulative distribution function (CDF) of the symmetric component alpha normal slash distribution $Y \sim SCNS(\alpha,q)$ is given as follows.

$$F_Y(y) = \int_{-\infty}^{y} f_Y(u)du = q\int_0^1 t^{q-1}\left[\Phi(yt) - \frac{\alpha}{1+\alpha}yt\phi(yt)\right]dt. \tag{2.10}$$

2.2. *Location-scale form of the distribution*

Now we apply the well-known location-scale transformation and consider the general form of the distribution,

$$Y = \sigma \frac{X}{U^{\frac{1}{q}}} + \mu, \tag{2.11}$$

where $U \sim U(0,1)$ and $X \sim SCN(\alpha)$ are independent, $q > 0$, $-\infty < \mu < \infty$ and $\sigma > 0$. μ is called the location parameter and σ the scale parameter. We denote it as $Y \sim SCNS(\alpha, q; \mu, \sigma)$. If $\mu = 0$ and $\sigma = 1$, it is the standard case we discussed above. In this case, we use $Y \sim SCNS(\alpha, q)$.

The density function of the general form is given by

$$f_Y(y|\theta) = \frac{q}{\sigma} \int_0^1 \frac{1 + \alpha \left(\frac{y-\mu}{\sigma}\right)^2 t^2}{1+\alpha} \phi \left[\left(\frac{y-\mu}{\sigma} \right) t \right] t^q dt, \tag{2.12}$$

where $\theta = (\alpha, q, \mu, \sigma)$. The results about the moments, CDF etc. are similar to the standard case and then are ommitted.

Proposition 2.4. *Let $Y|U = u \sim SCN(\alpha; \mu = 0, \sigma = u^{-\frac{1}{q}})$ and $U \sim U(0,1)$, then $Y \sim SCNS(\alpha, q; \mu = 0, \sigma = 1)$.*

Proof. $f_Y(y) = \int_0^1 f_{Y|U}(y|u) f_U(u) du = \int_0^1 \frac{1}{u^{-\frac{1}{q}}} \frac{1+\alpha\left(\frac{y}{u^{-\frac{1}{q}}}\right)^2}{1+\alpha} \phi\left(\frac{y}{u^{-\frac{1}{q}}}\right) du$, let $t = u^{\frac{1}{q}}$, the result follows. □

Remark 2.3. Proposition 2.4 shows that the symmetric component alpha normal slash distribution can be represented as a scale mixture of the symmetric component alpha normal distribution and uniform $U(0,1)$ distribution. The result provides another way to generate random numbers from the the symmetric component alpha normal slash distribution.

2.3. *Moments and Moment generating function*

Proposition 2.5. *Let $Y \sim SCNS(\alpha, q)$. For $k = 1, 2, \ldots$ and $q > k$, the k^{th} non-central moment of Y is given by*

$$\mu_k = \mathbb{E}Y^k = \begin{cases} \frac{q}{q-2m} \frac{1+(2m+1)\alpha}{1+\alpha} \prod_{j=1}^{m} (2j-1) & k = 2m \\ 0 & k = 2m-1 \end{cases}. \tag{2.13}$$

Proof. From the stochastic representation defined in (2.5) and the results in (2.3), we have

$$\mu_k = \mathbb{E}Y^k = \mathbb{E}(XU^{-\frac{1}{q}})^k = \mathbb{E}(X^k) \times \mathbb{E}(U^{-\frac{k}{q}}) = \frac{q}{q-k} \mathbb{E}(X^k).$$

□

Corollary 2.1. *Let $Y \sim SCNS(\alpha, q)$. The mean and variance of Y are given by*

$$\mathbb{E}Y = 0 \quad and \quad Var(Y) = \frac{q}{q-2} \frac{1+3\alpha}{1+\alpha}, \quad for \quad q > 2. \tag{2.14}$$

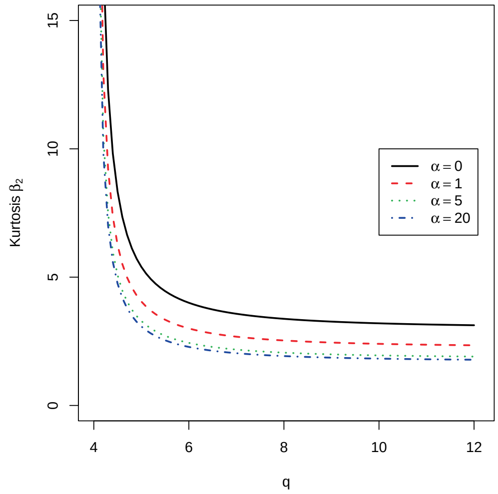

Fig. 2. The plot for the kurtosis coefficient β_2 for selected parameters

Corollary 2.2. *Let $Y \sim SCNS(\alpha, q)$. The skewness and kurtosis coefficients of Y are*

$$\sqrt{\beta_1} = 0, \quad for \quad q > 3, \quad and \quad \beta_2 = \frac{3(q-2)^2(1+\alpha)(1+5\alpha)}{q(q-4)(1+3\alpha)^2}, \quad for \quad q > 4. \tag{2.15}$$

As $q \to \infty$, $\beta_2 \to \frac{3(1+\alpha)(1+5\alpha)}{(1+3\alpha)^2}$, which is the kurtosis coefficient for the symmetric component alpha normal distribution. Kurtosis range is $(\frac{5}{3}, \infty)$. Figure 2 show the kurtosis coefficient for selected parameters.

Proposition 2.6. *Let $Y \sim SCNS(\alpha, q)$, the moment generating function of Y is given by*

$$M_Y(t) = q \int_1^\infty \left(1 + \frac{\alpha w^2 t^2}{1+\alpha}\right) e^{\frac{w^2 t^2}{2}} w^{-(1+q)} dw. \tag{2.16}$$

Proof.

$$\begin{aligned} M_Y(t) = \mathbb{E}(\mathbb{E}(e^{tY}|U)) &= \mathbb{E}\left[\left(1 + \frac{\alpha}{1+\alpha}(U^{-\frac{1}{q}}t)^2\right) e^{\frac{(U^{-\frac{1}{q}}t)^2}{2}}\right] \\ &= \int_0^1 \left(1 + \frac{\alpha}{1+\alpha}(u^{-\frac{1}{q}}t)^2\right) e^{\frac{(u^{-\frac{1}{q}}t)^2}{2}} du \\ &= q \int_1^\infty \left(1 + \frac{\alpha w^2 t^2}{1+\alpha}\right) e^{\frac{w^2 t^2}{2}} w^{-(1+q)} dw \end{aligned}$$

□

3. Inference

In this section, we consider the maximum likelihood estimation about the parameter $\theta = (\alpha, q, \mu, \sigma)$ of the location-scale form defined in (2.12).

Suppose $Y_1, Y_2, \ldots, Y_n$ is a random sample of size n from the distribution $Y \sim SCNS(\theta)$. The log-likelihood function is expressed as

$$l(\theta) = n\log q - n\log\sigma + \sum_{i=1}^{n}\log\int_0^1 \frac{1+\alpha\left(\frac{y_i-\mu}{\sigma}\right)^2 t^2}{1+\alpha}\phi\left[\left(\frac{y_i-\mu}{\sigma}\right)t\right]t^q dt. \tag{3.1}$$

Taking the partial derivatives of the log-likelihood function with respect to α, q, μ, σ respectively, we have the following maximum likelihood estimating equations

$$l_\alpha = \sum_{i=1}^{n} \frac{\int_0^1 \frac{\left(\frac{y_i-\mu}{\sigma}\right)^2 t^2 - 1}{(1+\alpha)^2}\phi\left[\left(\frac{y_i-\mu}{\sigma}\right)t\right]t^q dt}{\int_0^1 \frac{1+\alpha\left(\frac{y_i-\mu}{\sigma}\right)^2 t^2}{1+\alpha}\phi\left[\left(\frac{y_i-\mu}{\sigma}\right)t\right]t^q dt} = 0, \tag{3.2}$$

$$l_q = \frac{n}{q} + \sum_{i=1}^{n} \frac{\int_0^1 \frac{1+\alpha\left(\frac{y_i-\mu}{\sigma}\right)^2 t^2}{1+\alpha}\phi\left[\left(\frac{y_i-\mu}{\sigma}\right)t\right]t^q \log t dt}{\int_0^1 \frac{1+\alpha\left(\frac{y_i-\mu}{\sigma}\right)^2 t^2}{1+\alpha}\phi\left[\left(\frac{y_i-\mu}{\sigma}\right)t\right]t^q dt} = 0, \tag{3.3}$$

$$l_\mu = \sum_{i=1}^{n} \frac{\int_0^1 \frac{t^{q+1}}{(1+\alpha)\sigma}\left\{(1-2\alpha)\left(\frac{y_i-\mu}{\sigma}\right)t + \alpha\left(\frac{y_i-\mu}{\sigma}\right)^3 t^3\right\}\phi\left[\left(\frac{y_i-\mu}{\sigma}\right)t\right]dt}{\int_0^1 \frac{1+\alpha\left(\frac{y_i-\mu}{\sigma}\right)^2 t^2}{1+\alpha}\phi\left[\left(\frac{y_i-\mu}{\sigma}\right)t\right]t^q dt} = 0, \tag{3.4}$$

$$l_\sigma = -\frac{n}{\sigma} + \sum_{i=1}^{n} \frac{\int_0^1 \frac{(y_i-\mu)t^{q+1}}{(1+\alpha)\sigma^2}\left\{(1-2\alpha)\left(\frac{y_i-\mu}{\sigma}\right)t + \alpha\left(\frac{y_i-\mu}{\sigma}\right)^3 t^3\right\}\phi\left[\left(\frac{y_i-\mu}{\sigma}\right)t\right]dt}{\int_0^1 \frac{1+\alpha\left(\frac{y_i-\mu}{\sigma}\right)^2 t^2}{1+\alpha}\phi\left[\left(\frac{y_i-\mu}{\sigma}\right)t\right]t^q dt} = 0. \tag{3.5}$$

The maximum likelihood estimating equations above are not in a simple form. In general, there are no implicit expression for the estimates. The estimates can be obtained through some numerical procedures such as Newton-Raphson method. Many programs provide routines to solve such maximum likelihood estimating equations.

In this paper, all the computations are performed using software *R*. The MLEs are computed by the *optim* function which uses L-BFGS-B method. In the following section, a simulation is conducted to illustrate the behavior of the MLE.

For asymptotic inference of $\theta = (\alpha, q, \mu, \sigma)$, the Fisher information matrix $I(\theta)$ plays a key role. It is known that its inverse is the asymptotic variance matrix of the maximum likelihood estimators. For the case of a single observation ($n = 1$), we take the second order derivatives of the log-likelihood function in (3.1) and the Fisher information matrix of the MLEs is defined as, for $i = 1, \ldots, 4$ and $j = 1, \ldots, 4$,

$$(I(\theta))_{i,j} = -\mathbb{E}\left[\frac{\partial^2 \log f_Y(y)}{\partial\theta_i \partial\theta_j}\right]. \tag{3.6}$$

Proposition 3.1. *Let $Y_1, Y_2, \ldots, Y_n$ be a random sample of size n from the distribution $SCNS(\alpha, q, \mu, \sigma)$, and let $\theta = (\alpha, q, \mu, \sigma)$ and $\hat{\theta}$ is the MLE of θ, we have*

$$\sqrt{n}(\hat{\theta} - \theta) \xrightarrow{d} N_4(0, I(\theta)^{-1}). \tag{3.7}$$

Proof. It follows directly by the large sample theory for the MLEs and the Fisher information matrix given above. □

For an individual parameter θ_j $(j = 1,2,3,4)$, as n goes to infinity, $\sqrt{n}(\hat{\theta}_j - \theta_j) \xrightarrow{d} N(0,(I(\theta)^{-1})_{jj})$. If we want to test a hypothesis about θ_j, $H_0 : \theta_j \leqslant \theta_j^0$ v.s. $H_1 : \theta_j > \theta_j^0$, then we would use

$$z = \sqrt{n}\frac{\hat{\theta}_j - \theta_j^0}{(I(\theta)^{-1})_{jj}} \xrightarrow{d} N(0,1). \tag{3.8}$$

The test is simply the z-score test in statistics. Specially, if we want to test the bimodality within the proposed model $SCNS(\theta)$ class, $\alpha(= \theta_1)$ is the modal parameter and the model is bimodal if $\alpha > 0.5$ and otherwise unimodal. Therefore, we need perform an asymptotical inference about $\alpha(= \theta_1)$: $H_0 : \alpha \leqslant 0.5$ v.s. $H_1 : \alpha > 0.5$.

4. Simulation Study

4.1. *Data Generation*

In this section, we present how to generate the random numbers from the symmetric component alpha normal slash distribution.

Proposition 4.1. *Let T, V be two independent random variables, where $T \sim \chi^2_{(3)}$ and V is such that $P(V = 1) = P(V = -1) = \frac{1}{2}$, then $W = \sqrt{T}V \sim w^2\phi(w)$, with $w \in \mathbb{R}$.*

Proof. See Elal-Olivero *et al.* [5]. □

Proposition 4.2. *Let $Z \sim N(0,1)$ and $W \sim w^2\phi(w)$, $w \in \mathbb{R}$, be two independent random variables, if $X = \sqrt{\frac{1}{1+\alpha}}Z + \sqrt{\frac{\alpha}{1+\alpha}}W$, then $X \sim SCN(\alpha)$, where $\alpha \geqslant 0$.*

Proof. Let $a = \sqrt{\frac{1}{1+\alpha}}$ and $b = \sqrt{\frac{\alpha}{1+\alpha}}$, the moment generating function of X

$$M(t) = \mathbb{E}e^{taZ+tbW} = e^{\frac{t^2a^2}{2}} \int_{-\infty}^{\infty} e^{tbw}w^2\phi(w)dw = (1+t^2b^2)e^{\frac{t^2(a^2+b^2)}{2}} = (1+\frac{\alpha t^2}{1+\alpha})e^{\frac{t^2}{2}},$$

which is the moment generating function of $SCN(\alpha)$ in Proposition 2.2. □

A random number from $SCNS(\alpha,q,\mu,\sigma)$ can be generated as follows: generate $Z \sim N(0,1)$, $T \sim \chi^2_{(3)}$, $S \sim U(0,1)$ and $U \sim U(0,1)$; let $V = 1$ if $S < \frac{1}{2}$. Otherwise $V = -1$; set $W = \sqrt{T}V$; set $X = \sqrt{\frac{1}{1+\alpha}}Z + \sqrt{\frac{\alpha}{1+\alpha}}W$; set $Y = \sigma\frac{X}{U^{\frac{1}{q}}} + \mu$.

4.2. *Behavior of MLE*

In this section, a simulation is conducted to investigate bias properties of the MLEs. It is known that the distribution of the MLE tends to the normal distribution, as the sample size n goes to infinity, with mean (α,q,μ,σ) and covariance matrix equal to the inverse of the Fisher matrix. However, it is not generally possible to derive the explicit form of the Fisher information matrix since the expression of the log-likelihood function given in (3.1) is complex. The theoretical properties (asymptotically normal, unbiased etc) of the MLEs are hardly derived and obtained. We study the properties of the estimators numerically.

We first generate 100 samples of size $n = 100$ and $n = 200$ from the symmetric component normal slash distribution $SCNS(\alpha, q, \mu, \sigma)$ for fixed parameters. The MLEs are calculated by the *optim* function in software R. The empirical means and standard deviations of the estimators are reported in Table 1. From Table 1, the parameters are well estimated and the estimates are asymptotically unbiased as expected.

5. Data Analysis

In this section, we consider the height data set which consists of the heights in inches of 126 students from University of Pennsylvania. This data set has been previously analyzed and studied in the literature. See [16] and [10] etc.

Table 2 shows the basic descriptive statistics for the data set, where $\sqrt{\beta_1}$ which is almost equal to 0 is the sample asymmetry coefficient. Figure 3 displays that the height distribution is roughly symmetric bimodal.

We fit the height data set with the $N(\mu, \sigma)$, $SCN(\alpha, \mu, \sigma)$ and $SCNS(\alpha, q, \mu, \sigma)$ distributions and examine the performances of the distributions. The MLEs of the parameters are obtained and the results are listed in Table 3. The Akaike information criterion (AIC) is used to measure of the goodness of fit of the models. $AIC = 2k - 2\log L$, where k is the number of parameters in the model and L is the maximized value of the likelihood function for the estimated model. Given a set of candidate models for the data, the preferred model is the one with the minimum AIC value.

From Table 3, for the height data set, AIC shows that SCNS is a best fit. It has smallest AIC and highest likelihood values. It can effectively depict the features (symmetry, bimodality, etc) of the data (see Figure 3). The bimodality test for the hypothesis $H_0 : \alpha \leqslant 0.5$(unimodal) v.s. $H_1 : \alpha > 0.5$(bimodal) is $z = \sqrt{126}\frac{1.453-0.5}{0.808} = 13.24$, $p - value < 0.05$, reject H_0, the distribution is bimodal.

For SCN model, even if it contains unimodal and bimodal cases. But we apply the MLE method and obtain $\hat{\alpha} = 0.189 < 0.5$, the fitted is a unimodal model. It can not capture the bimodality for the height data set.

6. Concluding Remarks

In this paper, we propose a new two parameter distribution called symmetric component alpha normal slash $SCNS(\alpha, q)$. This distribution includes some distributions as special cases such as the slashed normal and symmetric component alpha normal distributions.

Properties involving moments and moment generating function are studied. A simulation is conducted to investigate the behavior of MLE. We apply the distrubution to the height dataset where model fitting is implemented by maximum likelihood procedure. The data analysis shows the proposed model is very useful in real applications and can even present better fit than other symmetric models. It can successfully capture the bimodality.

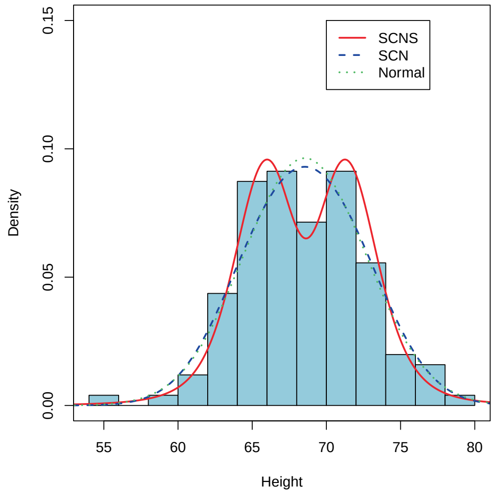

Fig. 3. Histogram and models fitted for the height data set

7. Appendix

Table 1. Empirical means and SD for the MLE estimators of $\theta = (\alpha, q, \mu, \sigma)$

				$n = 100$				$n = 200$			
α	q	μ	σ	$\hat{\alpha}(SD)$	$\hat{q}(SD)$	$\hat{\mu}(SD)$	$\hat{\sigma}(SD)$	$\hat{\alpha}(SD)$	$\hat{q}(SD)$	$\hat{\mu}(SD)$	$\hat{\sigma}(SD)$
2	2	1	1	2.4854(0.4215)	2.1661(0.5446)	1.0084(0.2339)	1.0410(0.2326)	2.2035(0.1105)	2.0787(0.3089)	0.9701(0.1422)	1.0342(0.1474)
2	4	1	1	2.4785(0.9615)	4.2676(1.4846)	1.0054(0.1553)	1.0255(0.1555)	2.2873(0.9358)	4.1451(1.2606)	1.0187(0.1174)	1.0152(0.1136)
4	2	1	1	4.2819(0.9775)	2.1366(0.4513)	1.0157(0.2045)	1.0333(0.2261)	4.1837(0.9736)	2.0422(0.3302)	0.9812(0.1194)	0.9999(0.1006)
4	4	1	1	4.3020(0.8584)	4.8404(1.7120)	1.0027(0.1353)	1.0536(0.1324)	4.2840(0.8168)	4.2319(1.1900)	1.0049(0.0983)	1.0038(0.0809)
1	1	2	4	1.2618(0.9311)	1.1068(0.1965)	1.8848(1.2618)	4.5323(1.6464)	0.9824(0.9153)	1.0591(0.1362)	1.9044(0.8858)	4.2814(1.3545)
1	3	2	4	1.3677(0.7224)	3.1705(1.1256)	2.0823(0.8830)	4.1437(1.0187)	1.1571(0.4813)	3.1323(1.0944)	2.0818(0.5101)	4.0404(0.8078)
3	1	2	4	3.4432(0.9352)	1.0586(0.1439)	2.1555(0.1533)	4.3883(1.2193)	3.3425(0.8501)	1.0355(0.1116)	2.0050(0.0728)	4.2019(0.7226)
3	3	2	4	3.3271(0.9243)	3.4467(1.1800)	1.9716(0.6564)	4.1796(0.6416)	3.1533(0.5078)	3.1704(0.6545)	2.0060(0.4905)	4.0358(0.3794)

Table 2. Descriptive statistics for the height data set

n	Mean	Median	Standard Deviation	Range	Skewness $\sqrt{\beta_1}$
126	68.54	68.44	4.16	23.5	-0.04

Table 3. Maximum likelihood parameter estimates (with standard deviations (SD)) of the SCNS, SCN and Normal models for the height data set

Model	$\hat{\alpha}$	$\hat{q}$	$\hat{\mu}$	$\hat{\sigma}$	loglik	AIC
Normal	–	–	68.55	4.14	−357.83	719.66
			(0.369)	(0.261)		
SCN	0.189	–	68.55	3.608	−357.82	721.65
	(0.569)		(0.371)	(1.117)		
SCNS	1.453	3.7	68.628	1.968	−355.58	719.16
	(0.808)	(1.231)	(0.291)	(0.319)		

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