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Nonlinear Schrödinger, Infinite Dimensional Tori and Neighboring Tori

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Abstract

In this work, we explain in what sense the generic level set of the constants of motion for the periodic nonlinear Schrödinger equation is an infinite dimensional torus on which each generalized nonlinear Schrödinger flow is reduced to straight line almost periodic motion, and describe how neighboring generic infinite dimensional tori are connected.

1 Introduction

We consider the Hamiltonian equation

$$iu_t + u_{xx} - |u|^2 u = 0 \tag{1}$$

of the periodic nonlinear Schrödinger equation, where $u(x,t)$ is a complex valued function in the class of smooth period one functions. In this work, we explain in what sense the generic level set of the constants of motion for the periodic nonlinear Schrödinger equation is an infinite dimensional torus, why the solution of the Hamiltonian equation is almost periodic in time, and describe how neighboring generic infinite dimensional tori are connected. Bourgain [1] has solved the initial value problem for the periodic nonlinear Schrödinger equation. Ma and Ablowitz [2] have reduced the periodic nonlinear Schrödinger equation to an inverse spectral problem for periodic potentials. They provide explicit formulas for the special class of N-soliton solutions of the periodic nonlinear Schrödinger equation and found an infinite sequence of functionals that are in involution and constant along solutions of (1). For the nonlinear Schrödinger equation, Batig et al [3] and Schmidt [4] used the method of inverse spectral theory and integrated the equation in the class of analytic [4] and smooth periodic functions [3]. They identified the generic invariant set of the constants of motion with an infinite dimensional tori. Their study [2, 3] did not describe how neighboring tori are connected.

The nonlinear Schrödinger equation is an example of the Hamiltonian equation

$$\frac{\partial u}{\partial t} = K(u), \tag{2}$$

where $K(u)$ is a nonlinear operator and u a complex valued function in the class of smooth periodic functions. Let $F_m(u)$ denote functionals that are in involution and constant along solutions of (2). In [5], we give a proof of an infinite dimensional version of Liouville's theorem and explain in what sense the generic level set of the functionals $F_m(u)$ is an infinite dimensional torus on which the solution of (2) reduces to straight line motion that is almost periodic in time. Furthermore, we explain in what sense neighboring generic tori and solutions of (2) are connected. The approach in [5] is related to Lax's [6] study of finite-dimensional level sets of completely integrable partial differential equations and is independent of the method of inverse spectral theory and the viewpoint of algebraic curves. An application of the theorem in [5] to the nonlinear Schrödinger equation yields a different proof of the result of Batig [3] and Schmidt [4]. In addition, the present work describes how neighboring generic tori and the solutions of (2) are connected.

In the classical case

$$\frac{dv}{dt} = K(v), \qquad v \in \mathbb{R}^{2N}$$

a theorem of Liouville [7] states that the system is completely integrable. If the involutive constant functions $F_m(v)$, $m = 1, 2, \ldots, N$ are independent in the sense that their gradients are linearly independent and if the N dimensional level set satisfying $F_m(v) = F_m(v_0)$, $m = 1, 2, \ldots, N$ is compact; in fact,

(a) the level set is an N dimensional torus on which the flow is quasiperiodic and

(b) neighboring Louville tori are diffeomorphic to one another.

The proof of the classical Liouville theorem is based on the inverse function theorem. It verifies that the composition of the commuting flows associated with $F_m(v)$, $m = 1, \ldots, N$ identifies a neighborhood of $\mathbb{R}^N$ with a neighborhood of the level set. The basic periods of this map are used to identify a connected component of the level set with an N dimensional torus on which the Hamiltonian flow associated with each $F_m(v)$ is reduced to straight line quasiperiodic motion.

In [5], we gave a proof of an infinite dimensional version of Liouville's theorem. We were unable to use the inverse function theorem. We introduced instead a local open mapping theorem for certain types of nondifferentiable maps and established that the composition of the commuting flows associated with $F_m(u)$, $m \geq 1$ defines a continuous open map from the Hilbert space l_2 of square summable sequences onto a connected component of a generic compact level set. This map is not locally diffeomorphic because l_2 in not locally compact. The periods of this mapping are contained in any neighborhood of l_2. A complete set of basic periods was used to identify a connected component of the level set with an infinite dimensional torus on which the Hamiltonian flow associated with each $F_m(u)$ is reduced to straight line almost periodic motion. Furthermore we established that the complete set of basic periods that characterized a generic level set may be continuously extended to a complete set of basic periods that describe a neighboring generic level set. We established a sense in which neighboring generic level sets are homeomorphic to the standard infinite dimensional torus, and determined a sense in which these neighboring level sets are connected. This present study of the periodic nonlinear Schrödinger equation is an illustration of the result [5].

2 Theorem

Let W_n $(n \geq 0)$ denote the usual Sobolev space of functions on $[0, 1]$, of period one, having derivatives of all orders up to n with norm

$$\|w\|_n^2 = \sum_{j \leq n} \int_0^1 |D^j w(x)|^2 dx.$$

The norm in the space L_2 is denoted by $\|w\|$. For $w \in W_n$ and integers j, k, and p with $p \geq 2$, it is known that

$$\sqrt[p]{\int_0^1 |D^j w(x)|^p dx} \leq 2^{p-2/2p} \|D^k w\|^a \|w\|^{1-a},$$

where $a = \left(j + \frac{1}{2} - \frac{1}{p}\right)/k$ and $1 \leq j < k \leq n$. We denote by C_1^n the space of functions of period one having continuous derivatives of order less than or equal n. The value of n does not enter into the proof of the result [5]. The value of n specifies the class of solutions for the nonlinear Schrödinger equation or generalized equations. The subscript of W_n is generally suppressed.

The Hamiltonian formulation of (2) is due to Gardiner [8] and Lax [6]. Let $F(u)$ denote a functional whose argument is a smooth function of period one and let $(\cdot, \cdot)$ denote the scalar product in L_2. Then

$$\lim_{\epsilon \to 0} \epsilon^{-1}(F(u + \epsilon v) - F(u)) = (G_F(u), v)$$

for appropriate u and v defines $G_F(u)$, the gradient of F at u. Define the Poisson bracket of $F(u)$ with $H(u)$ by

$$\{F(u), H(u)\} = (G_F(u), JG_H(u)),$$

where J is an antisymmetric operator independent of u. If $\mathrm{K}(u) = JG_F(u)$, then the equation (2) is said to be Hamiltonian. We denote by $S_F(t)u$ the nonlinear operator determining the solution of (2) on the basis of its initial values at $t = 0$: $u(t) = S_F(t)u_0$. If $\{F, H\} = 0$ for all u, then the solutions of (2) and of $u_t = JG_H(u)$ commute: $S_H(t)S_F(t') = S_F(t')S_H(t)$ for all t and t'.

As for the nonlinear Schrödinger equation, Ma and Ablowitz have constructed explicitly an infinite sequence of functionals $I_m(u)$ that are constant along the flow (1). The first three are

$$\int_0^1 u\overline{u}\, dx, \qquad \frac{i}{2} \int_0^1 (\overline{u}u_x - u\overline{u}_x)\, dx, \qquad \int_0^1 \left(|u_x|^2 + \|u|^4\right).$$

Let $G_{I_m}(u)$ denote the gradient of I_m with respect to $\overline{u}$ at u, and let

$$\langle u, v \rangle = \int_0^1 (u\overline{v} + v\overline{u})\, dx$$

denote the product in L_2. Then the Poisson bracket

$$\{I_m(u), I_n(u)\} = \langle G_{I_m}(u), JG_{I_n}(u) \rangle$$

where the symplectic structure is introduced through $J = i$. It is known that

$$\{I_m(u), I_n(u)\} = 0$$

for all m and n and smooth periodic functions u. Therefore $I_m(u)$ are constant along solutions of

$$\frac{\partial u}{\partial t} = K_{I_m}(u) = JG_{I_m}(u), \qquad m \geq 1$$

the generalized nonlinear Schrödinger equation, where $m = 3$ is equation (1). The nonlinear Schrödinger equation (1) is of the form

$$u_t = [L, A],$$

where the operator L and A depend on u. The nonlinear Schrödinger flow preserves the spectrum of L determined by

$$Lf = \begin{pmatrix} -D & u \\ -\overline{u} & D \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = i\lambda \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \tag{3}$$

in the class of functions $f[0,1) \to C^2$ with $f(x+1) = mf(x)$ for $0 \leq x < 1$. The periodic and antiperiodic spectra $[m = \pm 1]$ will be of special interest. Use of (3) and a direct calculation shows that the spectrum of L is real. Ma and Ablowitz determined that the periodic and antiperiodic spectrum of L is comprised of simple and double eigenvalues λ_m with eigenfunctions $f = (f_{1,m}, f_{2,m})^T$, and that the functionals $\lambda_m(u)$ are in involution. This study concerns the general situation in which the spectrum of L is simple. The exceptional case of mixed simple and double spectra offers no additional technical difficulties. Let M denote the portion of the space W of smooth periodic functions for which the spectrum of L is simple. For $u_0 \in M$, we consider the level set

$$M_{\lambda(u_0)} = \{u \,|\, \lambda_m(u) = \lambda_m(u_0),\ m \geq 1\}$$

in W_n. We prove that M_λ is generated by the sequence of the generalized nonlinear Schrödinger flows and that the generic level set is identified with an infinite dimensional torus on which each generalized nonlinear Schrödinger flow is reduced to straight line motion that is almost periodic in time. Furthermore, we make precise the sense in which neighboring generic level sets are connected.

To identify $M_{\lambda u_0}$ with the standard infinite-dimensional torus $T^\infty = [0,1)^\infty$ we first state the result in [5]. Consider the Hamiltonian equation (2), the sequence $F_m(u)$ that are in involution and constant along solutions of (2), and the level set

$$M_{u_0} = \{u \,|\, F_m(u) = F_m(u_0),\ m \geq 1\}.$$

Let u be an element of M_{u_0} and view the latter as a subset of L_2. Define $G_{F_m}(u)$ to be the gradient of $F_m(u)$ at u. $G_{F_m}(u)$ is a vector that is normal to M_{u_0} at u. Let N_u be the closure in L_2 of the span of $G_{F_m}(u)$ and assume that $G_{F_m}(u)$ is a basis of N_u; by which we mean a) each element G_u in N_u is uniquely expressible as $G_u = tG(u) = \sum_{m=1}^{\infty} t_m G_{F_m}(u)$ for t in the Hilbert space l_2 and b) G_u admits the estimate

$$c_1(u)|t|_{l_2} \leq \|tG(u)\| \leq c_2(u)|t|_{l_2},$$

where c_1 and c_2 depend continuously on u in M. N_u is the normal space of M_{u_0} at u and, by our assumptions, no single gradient $G_{F_m}(u)$ lies in the closure in L_2 of the other gradients $G_{F_n}(u)$. The Poisson bracket of $F_m(u)$ and $F_n(u)$ vanishes for all m and n and for u in the class of smooth period one functions. The functionals $F_m(u)$ generate commuting flows

$$\frac{\partial u}{\partial t} = K_{F_m}(u) = JG_{F_m}(u), \qquad m \geq 1 \tag{4}$$

on M_{u_0} and $F_1(u), \ldots, F_m(u), \ldots$ are constants of these motions. $K_{F_m}(u)$ is tangent to M_{u_0} at u. Denote by T_u the closure in L_2 of the span of $K_{F_m}(u)$. Suppose that $K_{F_m}(u)$ is a basis of T_u; each element K_u in T_u is uniquely expressible as $K_u = \sum_{m=1}^{\infty} t_m K_{F_m}(u) = tK(u)$ for t in l_2, and

$$c_1(u)|t|_{l_2} \leq \|tK(u)\| \leq c_2(u)|t|_{l_2}, \tag{5}$$

where c_1 and c_2 depend continuously on u in M. Assume that T_u equals the orthogonal complement of N_u. T_u represents the tangent space and every direction of L_2 has been accounted for. Let $S_{F_m}(t_m)u_0$ denote the nonlinear operator uniquely determining the solution of (4) on the basis of its initial values at $t = 0$: $u(t) = S_{F_m}(t_m)u_0$. For t in l_2 we show that

$$S(t)u = \lim_{N \to \infty} \prod_{m=1}^{N} S_{F_m}(t_m)u_0$$

in W_n where $S(t+t')u = S(t)S(t')u$ for t, t' in l_2, and for $t \in l_2$, $S(t)u \in W_m$ is continuous in t uniformly in u on M_{u_0}. Denote by $dG_F(u)$ the second derivative of F defined by

$$\lim_{\epsilon \to 0} \epsilon^{-1}(G_F(u + \epsilon v) - G_F(u)) = dG_F(u)v.$$

Let $v(\tau)$, $\tau \geq 0$ be a curve in M_{u_0} that satisfies $\frac{dv(\tau)}{d\tau} = K_{v(\tau)}$ with $v(0) = v_1$ and let $dG_{v(\tau)}K_{v(\tau)} = \frac{dG_{v(\tau)}}{d\tau}$ admit the estimate

$$\frac{(dG_u K_v, K'_v)}{\|K_v\|\|K'_v\|} \leq c\|G_v\|, \tag{6}$$

where $K'_v \in T_v$, c is independent of $v(0) \in M_{u_0}$, and $v = v(\tau)$ for small τ. Then $S(t)u_0$ is an open map of l_2 onto M_{u_0} in W_n. Let L_{u_0} denote the set of t in l_2 for which $S(t)u = u$ for all u in M_{u_0}. S is a homeomorphism of l_2/L_{u_0} onto M_{u_0} in W_n. l_2/L_{u_0} is compact and may be identified as in [5] with the standard infinite-dimensional torus T^∞: in more detail, there exist ω_m, $m \geq 1$ from L_{u_0} so that each t of l_2/L_{u_0} is uniquely represented by $t = \sum_{m=1}^{\infty} \tau_m \omega_m$, where $0 \leq \tau_m < 1$ for all m. M_{u_0} is an infinite-dimensional torus and the solution $S_{F_m}(t_m)u$ is almost periodic on l_2/L_{u_0}, uniformly with respect to initial values $u \in M_{u_0}$. The motion $S_{F_m}(t_m)u$ of each Hamiltonian equation related to F_m is identified with straight line motion on l_2/L: in detail, for e_m in the m-th coordinate direction in

l_2/L and $\hat{\omega}_m \in l_2$ with $\omega_n \cdot \hat{\omega}_m = \delta_{n,m}$, then $S_{F_m}(t_m)u$ is identified with straight line motion in the direction $\sum_{m=1}^{\infty}(e_m \cdot \hat{\omega}_m)\omega_m$ on l_2/L.

For the generalized nonlinear Schrödinger flow, $K_{I_m}(u)$ is an element of the closure in L_2 of the span of $K_{\lambda_m}(u)$ and the flow of each generalized nonlinear Schrödinger equation is identified with straight line motion that is almost periodic in time on l_2/L.

We next identify as in [5], neighboring generic tori and then state a sense in which they are related. Let $F_m(u)$ denote a sequence of analytic functions of $u \in W$ and suppose $\sum_{m=1}^{\infty} F_m(u)^2$ is bounded uniformly in u on bounded sets in W. Let $F(u) = (F_1, \ldots, F_m, \ldots)$ and $l = F(M)$. Let $v_0 \in M$ and $f_0 = F(v_0) \in l$. Write u_{f_0} for v_0. For f in a small neighborhood of f_0, there exists u_f in M and $f = F(u_f)$ and

$$|f - f_0|_{l_2} \geq c\|u_f - u_{f_0}\|,$$

where c is locally independent of $u \in M$ and $f \in l$. The curve that joins u_{f_0} to u_f depends uniquely on u_{f_0}. The curve is relatively short in the sense that the length of the curve in W joining u_{f_0} with u_f is bounded by a fixed multiple of $|f - f_0|_{l_2}$. The torus $M_f = F^{-1}(f)$ is homeomorphic to the standard torus T^∞. $M_{u_{f_0}}$ is characterized by basic generators $\omega_m(f_0)$ that are the periods of $S(t)u_{f_0}$, and for f in a small enough neighborhood of f_0 in l, the $\omega_m(f_0)$ may be continuously extended to the basic generators $\omega_m(u_f)$ that describe M_{u_f}. $M_{u_f} = F^{-1}(f)$ is identified with the set T_f of convergent sums $\sum_{m=1}^{\infty} \tau_m\omega_m(f)$, $0 \leq \tau_m < 1$, which converge in l_2 uniformly in τ_m and f. Furthermore, there exists a curve that is continuous in l_2 that connects $\sum_{m=1}^{\infty} \tau_m\omega_m(f_0)$ with $\sum_{m=1}^{\infty} \tau_m\omega_m(f)$ and is relatively short in the sense that the length of the curve in l_2 is less than $\frac{1}{2}\left|\sum_{m=1}^{\infty} \tau_m\omega_m(f_0)\right|_{l_2}$. T_f is homeomorphic to T_{f_0} and T_f is uniformly close to T_{f_0}. M_{u_f} is homeomorphic to $M_{u_{f_0}}$ and to the standard infinite-dimensional torus T^∞. Furthermore, M_{u_f} and $M_{u_{f_0}}$ are connected by a relatively short continuous curve in W that is contained in M except for a countable number of elements. This leads to the result of this work on the nonlinear Schrödinger equation.

Theorem. *Let $u \in M$ and $F_m(u) = \lambda_m(u) - \lambda_m(0)$. $F_m(u)$ is a sequence of analytic functions of $u \in W$ that are in involution and the level set M_u is bounded. The sequence $G_m(u)$ and $K_m(u)$ is a basis for N_u and T_u respectively with $N_u \oplus T_u = L_2$ and dG admits the estimate $(dG_uK_u, K_u)/\|K_u\|\|K_u\| \leq c\|G_u\|$, where c is independent of u. For $u_0 \in M$, $S(t)u_0$ is a homeomorphism of l_2/L_{u_0} onto M_{u_0} in W. l_2/L_{u_0} is compact and identified with an infinite dimensional torus: there exists a sequence ω_m from L_{u_0} for which each element of $l_2/L_{u_{f_0}}$ is uniquely represented by $\sum_{m=1}^{\infty} \tau_m\omega_m$, $0 \leq \tau_m < 1$ for all m. The flow of each generalized nonlinear Schrödinger equation is identified with straight motion that is almost periodic in time on $l_2/L_{u_{f_0}}$. For u in M, $F_m(u)$ is square summable uniformly in u on bounded sets in W. For directions v transverse to M_u at u, $dG_m(u)$ admits the estimate $\|dG_m(u)v\| \leq c_m\|v\|$, where c_m is square summable independently of u and v. For f in a small neighborhood of $f_0 = F(u_{f_0})$ in l there exists $u_f \in M$ satisfying $F(u_f) = f$ that admits the estimate $|f - f_0|_{l_2} \geq c\|u_f - u_{f_0}\|$, where c is locally independent of u in M*

and $f \in l$. M_{u_f} is homeomorphic to M_{f_0} and to the standard infinite-dimensional torus. Furthermore M_{u_f} and $M_{u_{f_0}}$ are connected by a relatively short continuous curve in W that is contained in M except for a countable number of elements.

This completes the statement of the Theorem.

A modification of the proof of Lemma 1 in [5] establishes that M_{u_f} and $M_{u_{f_0}}$ are connected by a relatively short smooth curve in W that is contained in M except for a countable number of elements. This verifies that M_{u_f} is diffeomorphic to M_{f_0} and to the standard infinite-dimensional torus. The exceptional case of mixed simple and double spectra can be solved by a slight modification of the approach taken in [5]. The application of [5] to the Hamiltonian flow

$$iu_t + u_{xx} + |u|^2 u = 0$$

offers no additional technical problem. In this case the torus is of lower dimension.

3 Proof

In this section we prove the Theorem. We establish first the properties of M_{u_0} and obtain an apriori estimate of dG. For u in M, we verify that $G_m(u)$ is a basis for N_u and $K_m(u)$ is a basis for T_u and that $N_u \oplus T_u = L_2$.

<u>*Item 1.*</u> The functionals $\lambda_m(u)$ are in involution and M_{u_0} is bounded in W. For $u_0 \in M$, consider

$$M_{u_0} = \{u \,|\, F_m(u) = \lambda_m(u) - \lambda(0) = F_m(u_0),\ m \geq 1\}$$

and use the result in [2] or the periodic version of the result of Zakharov and Shabat [9] to show that $I_m(u)$ are directly related to λ_m and that $I_m(u)$ is constant on M_{u_0}. The functional

$$I_1 = \int_0^1 |u|^2 dx$$

gives

$$\|u\| \leq c.$$

Rearrange the functional

$$I_3 = \int_0^1 \left(|u_x|^2 + |u|^4\right) dx$$

and estimate to find that

$$\int_0^1 |u_x|^2 dx \leq c + c\|u_x\| \|u\|^3 \leq c + c\|u_x\| \leq c + \frac{1}{2}\|u_x\|^2,$$

where we have applied the general inequality $|u|_\infty \leq \sqrt{2}\|u_x\|^{1/2}\|u\|^{1/2}$ and previous bounds. This leads to the estimate $\|u_x\| \leq c$. The integral

$$I_5 = \int_0^1 |u_{xx}\|^2 + \frac{1}{2}|u|^6 - \frac{1}{2}\left(\frac{d}{dx}|u|^2\right)^2 - 3|u_x|^2|u|^2 dx$$

is estimated as follows:

$$\|u_{xx}\|^2 \leq c\|u_x\|^{1/3}\|u\|^{2/3} + c|u|_\infty^2\|u_x\|^2 \leq c,$$

where we have used the general estimate $|u|_6 \leq 2^{1/3}\|u_x\|^{1/3}\|u\|^{2/3}$ and previous bounds to find

$$\|u_{xx}\| \leq c.$$

For $n \geq 4$, the functionals I_n have weight $2n$ where the weight is a sum of the weights of its factors and the weight of $D^r u$ is $1+r$. Use previous estimates to obtain

$$\|u\|_n \leq c$$

for any n. M_{u_0} is bounded in W_n.

Item 2. For simple eigenvalue λ, the gradient of λ_m with respect to $\overline{u}$ equals

$$\frac{d\lambda_m}{d\bar{u}} = if_1\overline{f}_2 = if_1^2.$$

Begin with the equation (3) for f_1 and f_2. Let

$$(u,v) = \int_0^1 uv\,dx.$$

Let $u^\epsilon = \overline{u} + \epsilon v$ and compute the derivatives

$$\frac{d}{d\epsilon}\lambda \qquad \text{and} \qquad \overline{\frac{d}{d\epsilon}}\lambda.$$

Begin with the equation for f_2 in (3) and compute the derivative with respect to ϵ. Multiply the resulting equation by $\overline{f}_2$ and integrate with respect to x from zero to one. Integrate by parts and use the equations once again and substitute $(\dot{f}_{2x}, \overline{f}_2)$ and $(\dot{\overline{u}}_{2x}, f_2)$ into the previous expression, and find

$$\begin{aligned}
i\dot{\overline{\lambda}}(\overline{f}_2, f_2) &= (\dot{\overline{f}}_2, \overline{u}f_1 + i\lambda f_2) + (\dot{u}, f_2\overline{f}_1) + (u, \dot{\overline{f}}_1 f_2) - i\overline{\lambda}(\dot{\overline{f}}_2, f_2),\\
i\dot{\lambda}(f_2, \overline{f}_2) &= -(\dot{f}_2, u\overline{f}_1 - i\lambda\overline{f}_2) - (\dot{\overline{u}}, f_1\overline{f}_2) - (\overline{u}, \overline{f}_2\dot{f}_1) - i\lambda(\dot{f}_2, \overline{f}_2).
\end{aligned}$$

The properties $\overline{\lambda} = \lambda$, $f_1 = \overline{f}_2$, and $f_2 = \overline{f}_1$ follow directly from (3). Substitute these identities into the previous equations with $\|f_1\| = \|f_2\| = 1$ and obtain

$$\dot{\overline{\lambda}} + \dot{\lambda} = (\dot{\overline{u}}, if_1\overline{f}_2) + (\dot{u}, \overline{i\overline{f}_2 f_1}).$$

Use the inner product $\langle u, v\rangle$ and find that

$$\frac{d\lambda}{d\overline{u}} = if_1\overline{f}_2 = if_1^2.$$

Item 3. For u in M and v transverse to M_u, let $u(\tau) = \tau v + (1-\tau)u$. Then

$$|f_1^2(u)|_\infty \leq c$$

and

$$\left| \frac{d}{d\tau} f_1^2 \right|_\infty \leq \frac{c}{|\lambda_m|} \|u - v\|$$

at $\tau = 0$, where c is independent of u on bounded sets in M. Use that $F_m(u)$ is an analytic functional of u in W to establish that the curve $u(\tau)$, $0 \leq \tau \leq 1$, remains in M except for a countable number of values of τ. Begin with

$$f_{1,x} - u f_2 = -i\lambda f_1$$

and

$$f_{2,x} - \overline{u} f_1 = i\lambda f_2.$$

Multiply the equation for f_1 by f_1 and rewrite as

$$\frac{1}{2}\partial_x f_1^2 = u f_1 f_2 - i\lambda f_1^2.$$

A similar calculation gives

$$\frac{1}{2}\partial_x f_2^2 - \overline{u} f_1 f_2 = i\lambda f_2^2.$$

Next multiply the equation for f_1 by f_2 and combine with the equation for f_2 multiplied by f_1 and find an expression for $\partial_x(f_1 f_2)$. The function

$$f_1 f_2 = \int_0^x \left(u f_2^2 + \overline{u} f_1^2\right) dx$$

satisfies the differential expression for $f_1 f_2$. Substitute the above identity for $f_1 f_2$ into the preceding equation for f_1^2 and f_2^2 and find

$$-\frac{1}{2}\partial_x f_1^2 + u \int_0^x \left(u f_2^2 + \overline{u} f_1^2\right) dx = i\lambda f_1^2 \tag{7}$$

and

$$\frac{1}{2}\partial_x f_2^2 - \overline{u} \int_0^x \left(u f_2^2 + \overline{u} f_1^2\right) dx = i\lambda f_2^2.$$

Multiply (7) by $\exp(2i\lambda x)$ and rewrite the first equation as

$$\partial\left(e^{2i\lambda x} f_1^2\right) = u e^{2i\lambda x} \int^x \left(u f_2^2 + \overline{u} f_1^2\right) dx.$$

Take the absolute value of this expression, use the inequality

$$\left|\partial\left(e^{2i\lambda x} f_1^2(x)\right)\right| \geq \partial\left|e^{2i\lambda x} f_1^2(x)\right|$$

and then integrate in x and find that

$$\left|f_1^2(x)\right| \leq \int^x \left| u \int^y \left(u f_2^2 + \overline{u} f_1^2\right) dy \right| dx \leq |u|_\infty^2 \int^x \left(\left|f_1^2\right| + \left|f_2^2\right|\right) dx.$$

A similar calculation gives

$$\left|f_2^2(x)\right| \le |u|_\infty^2 \int^x \left(\left|f_1^2\right| + \left|f_2^2\right|\right) dx.$$

Combine and form $\left|f_1^2(x)\right| + \left|f_2^2(x)\right|$, use Gronwall and find that

$$\left|f_1^2(x)\right| \le c,$$

where c is independent of u on bounded sets in W.

For $u(\tau) = \tau v + (1-\tau)u$, differentiate (7) with respect to τ at $\tau = 0$ and find that

$$\begin{aligned}\dot{f}_1^2 + \frac{1}{2i\lambda}\partial \dot{f}_1^2 &= \frac{\dot{\lambda}}{\lambda} f_1^2 - \frac{\dot{u}}{i\lambda}\int^x \left(u f_2^2 + \overline{u} f_1^2\right) dy \\ &\quad - \frac{u}{i\lambda}\int^x \left(\dot{u} f_2^2 + \dot{\overline{u}} f_1^2\right) dy - \frac{u}{i\lambda}\int^x \left(u \dot{f}_2^2 + \overline{u}\dot{f}_1^2\right) dy\end{aligned}$$

and a similar expression for $\dot{f}_2^2$. Multiply the previous identity by $\exp(x2i\lambda)$ and write the left hand side of the resulting expression as

$$\partial\left(e^{x2i\lambda}\dot{f}_1^2\right)/2i\lambda.$$

Next take the absolute value of the expression, use the inequality

$$c\partial\left|\dot{f}_1^2(x)\right| \le \partial\left|e^{x2i\lambda}\dot{f}_1^2(x)/2i\lambda\right| \le \left|\partial\left(e^{x2i\lambda}\dot{f}_1^2(x)\right)/2i\lambda\right|,$$

where we have used that λ is an analytic functional of u in W and c is an absolute constant. Integrate the resulting expression in x and use the estimates

$$\begin{aligned}&\left|\int^x e^{y/2i\lambda}\frac{\dot{u}}{i\lambda}\int^y \left(u f_2^2 + \overline{u} f_1^2\right)\right| \le \int^x \frac{|\dot{u}|}{|\lambda|}\int^y \left|u f_2^2 + \overline{u} f_1^2\right| \\ &\qquad \le \frac{\|u-v\|}{|\lambda|}\|u\|\left(|f_2|_\infty^2 + |f_1|_\infty^2\right) \le \frac{c}{|\lambda|}\|u-v\|, \\ &\left|\int^x e^{y/2i\lambda}\frac{u}{i\lambda}\int^y \left(u \dot{f}_2^2 + \overline{u}\dot{f}_1^2\right)\right| \le \int^x \frac{|u|}{|\lambda|}\int^y \left|u\dot{f}_2^2 + \overline{u}\dot{f}_1^2\right| \\ &\qquad \le \frac{|u|_\infty^2}{|\lambda|}\int^x \left(\left|\dot{f}_2^2\right| + \left|\dot{f}_1^2\right|\right) dy \le \frac{c}{|\lambda|}\int^x \left(\left|\dot{f}_2^2\right| + \left|\dot{f}_1^2\right|\right) dy, \\ &\left|\frac{1}{\lambda}\int^x \left(i f_1^2, u - v\right) f_1^2\right| \le \frac{\|u-v\||f_1|_\infty^2}{|\lambda|} \le \frac{c\|u-v\|}{|\lambda|},\end{aligned}$$

where we have used previous bounds and find that

$$\left|\dot{f}_1^2(x)\right| \le \frac{c\|u-v\|}{|\lambda|} + \frac{c}{|\lambda|}\int^x \left(\left|\dot{f}_2^2\right| + \left|\dot{f}_1^2\right|\right) dy.$$

A similar calculation gives

$$\left|\dot{f}_2^2(x)\right| \le \frac{c\|u-v\|}{|\lambda|} + \frac{c}{|\lambda|}\int^x \left(\left|\dot{f}_2^2\right| + \left|\dot{f}_1^2\right|\right) dy.$$

Combine and form $\left|\dot{f}_1^2(x)\right| + \left|\dot{f}_2^2(x)\right|$ and use Gronwall to obtain

$$\left|\dot{f}_1^2(x)\right| + \left|\dot{f}_2^2(x)\right| \le \frac{c\|u-v\|}{|\lambda|},$$

where c is independent of u on bounded sets of M.

Item 4. For u in M, the normal vectors $G_{\lambda_m}(u) = if_{1,m}^2$ and the tangent vectors $K_{\lambda_m}(u) = -f_{2,m}^2$ is a basis for N_u and T_u respectively and $N_u \oplus T_u = L_2$. Each G_u in N_u is uniquely $G_u = \sum_{m=1}^{\infty} t_m G_{\lambda_m}(u)$ and

$$c_1|t|_{l_2} \le \left\| \sum_{m=1}^{\infty} t_m G_{\lambda_m}(u) \right\| \le c_2|t|_{l_2},$$

where c_1, c_2 are independent of u on bounded sets in M. Each element K_u in the tangent space is uniquely represented as $K_u = \sum_{m=1}^{\infty} t_m K_{\lambda_m}(u)$ and

$$c_1|t|_{l_2} \le \left\| \sum_{m=1}^{\infty} t_m K_{\lambda_m}(u) \right| \le c_2|t|_{l_2},$$

where c_1, c_2 are independent of u on bounded sets in M.

We modify an idea of Borg [10] and establish this result by comparing the sequence G_{λ_m} and K_{λ_m} at u with the sequence at $u = 0$. The work of McKean and Trubowitz [11] used a similar comparison in their study of the basis properties of the normal and tangent space for the isospectral set of the periodic Korteweg-de Vries equation. For $u = 0$, we begin with the periodic and antiperiodic spectrum $\lambda_\pm = n\pi$ and eigenfunctions $f_{2m} = e^{ix\lambda_m}$. Then $G_{\lambda_m} = ie^{-i2x\lambda_m}$ and $K_{\lambda_m} = -e^{-i2x\lambda_m}$. For $u = 0$, the closure in L_2 of the linear span of G_{λ_m} and K_{λ_m} equals the closure in L_2 of the linear span of

$$(a_n \sin(2n\pi x) + b_n \cos(2n\pi x)) + i(a_n \cos(2n\pi x) + b_n \sin((2n\pi x)),$$

a basis for the space of complex valued functions that are square integrable.

For u=0, $G_{1,m}(0) = g_{1,m}^2$ is orthogonal and admits the estimate

$$c_1|c|_{l_2} \le \left\| \sum_{m=1}^{\infty} c_m g_{1,m}^2 \right\| \le c_2|c|_{l_2},$$

where c_1 and c_2 are absolute constants. We next establish an apriori estimate of $f_{1,m}^2(u)$.

For u in M, define T by

$$T\left(\sum_m c_m g_{1,m}^2\right) = \sum_m c_m \left(f_{1,m}^2(u) - g_{1,m}^2\right).$$

Use of the estimate in item 3 confirms that

$$\sum_m \left\|f_{1,m}^2(u) - g_{1,m}^2\right\|^2 < \infty$$

and establishes that T is a bound linear operator and is Hilbert Schmidt. If $f_{1,m}^2(u)$ is minimal then $(I+T)$ is invertible and

$$\left|(I+T)^{-1}\sum_m c_m f_{1,m}^2(u_1)\right| = \sum_m |c_m|^2 \left\|g_{1,m}^2\right\|^2 \le c'|c|_{l_2},$$

where we have used that $\left(g_{1,m}^2, g_{1,n}^2\right) = 0$, $m \ne n$, the estimate of item 3, and c' is independent of u on bounded sets from M. It follows that

$$\left\|\sum_m c_m f_{1,m}^2(u)\right\| \ge c_1|c|_{l_2}, \tag{8}$$

where c_1 is independent of u on bounded sets in M.

We use the apriori estimate and establish that the basis $g_{1,m}^2$ may be continuously extended to a basis $f_{1,m}^2$. Let $u(\tau) = \tau u$ and use that $F_m(u)$ is an analytic functional of u in W to establish that the curve $u(\tau)$, $0 \le \tau \le 1$ remains in M except for a countable number of τ and write

$$f_{1,m}^2(\tau u) - g_{1,m}^2 = \int_0^\tau \frac{d}{d\tau} f_{1,m}^2(su)\,ds.$$

Use the estimate of item 3 and find

$$\|f_{1,m}^2(\tau u) - g_{1,m}^2\| \le \tau \left\|\frac{d}{ds} f_{1,m}^2(su)\right\| \le \frac{\tau}{\|\lambda_m\|}\, c\|u\|,$$

where c is independent of u on bounded sets of M. Use (7), the previous bound, and select $\tau = \tau_1$ independently of u on bounded sets of M and find $u_1 = \tau_1 u$ in M, where

$$\left\|\sum_m c_m f_{1,m}^2(u_1)\right\| \ge \frac{c_1}{2}|c|_{l_2}.$$

This estimate confirms that $f_{1,m}^2(u_1)$ is a minimal sequence and that the closure in L_2 of the linear span of $f_{1,m}^2(u_1)$ equals the closure in L_2 of the linear span of $g_{1,m}^2$. Use of the apriori bound (8) gives the estimate

$$\left\|\sum_m c_m f_{1,m}^2(u_1)\right\| \ge c_1|c|_{l_2}.$$

Iteration of this construction gives the result.

Item 5. For u in M, dG_u admits the estimate $(dG_u K_u, K_u)/\|K_u\|\|K_u\| \le c\|G_u\|$, where c is independent of u on bounded sets in M. For the direction v transverse to M_u at u, $dG_m(u)$ satisfies the estimate $\|dG_m(u)v\| \le c_m\|v\|$, where c_m is square summable uniformly in u and v on bounded sets in M.

Use of (3) and a direct calculation shows that $\lambda_m/m \to 1$ uniformly in u on bounded sets in M as $m \to \infty$. Combine this result and the estimates of item 3 to obtain the estimates of this section.

This completes the proof of the Theorem.

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