On the Inverse Scattering Approach to the Camassa-Holm Equation

Adrian CONSTANTIN † *and Jonatan LENELLS*

Department of Mathematics, Lund University, P.O. Box 118, SE-221 00 Lund, Sweden
† *Corresponding author. E-mail: adrian.constantin@math.lu.se*

Received November 25, 2002; Revised January 06, 2003; Accepted January 07, 2003

Abstract

A simple algoritm for the inverse scattering approach to the Camassa-Holm equation is presented.

1 Introduction

The nonlinear partial differential equation

$$m_t + 2u_x + um_x + 2mu_x = 0, \quad t > 0,\ x \in \mathbb{R}, \tag{1.1}$$

in dimensionless space-time variables (x, t) models the unidirectional propagation of two-dimensional waves in shallow water over a flat bottom. In (1.1), $u(t, x)$ represents the horizontal component of the fluid velocity, or, equivalently, the water's free surface, and $m = u - u_{xx}$ is the momentum variable cf. [1] (see also [11] for an alternative derivation). The solitary waves of (1.1) are solitons - they regain their shape and speed after interacting nonlinearly with other solitary waves (see [1] [8] [12] [13]). While some initial profiles evolve into waves of permanent form, others yield waves that break in finite time [3] [4]. Another aspect of interest of (1.1) is its bi-Hamiltonian structure [10] and the induced existence of infinitely many conservation laws. This feature is connected to the (formal) integrability of the equation, established in [1] by finding an isospectral problem associated to (1.1).

Physically relevant cases are solutions of (1.1) with a periodic dependence upon the spatial x-variable, as well as solutions which decay at infinity. The objective of this note is to prove the integrability of (1.1) for solutions which decay at infinity (the periodic case is treated in [2] [7] [5]). For a discussion of the scattering problem for (1.1) we refer to [3] and [13]. Recently, in [6], we proposed an algorithm to solve the inverse scattering problem for (1.1). Our aim is to present here a considerable simplification of the approach in [6] and to show the applicability of the new approach by means of an example.

2 The inverse scattering approach

If the initial momentum $m_0 = m(0,\cdot)$ belongs to the Schwartz class of smooth functions $f:\mathbb{R}\to\mathbb{R}$ such that for all $n_1, n_2 \geq 0$, $\sup\limits_{x\in\mathbb{R}} |x^{n_1}\partial_x^{n_2} f(x)| < \infty$, and $m_0 + 1 > 0$, then both properties are preserved by the flow of (1.1) cf. [3] [6]. The isospectral problem in $L^2(\mathbb{R})$ for (1.1) is (see [1])

$$\psi_{xx} = \frac{1}{4}\,\psi + \lambda(m+1)\psi \tag{2.1}$$

with continuous spectrum $(-\infty, -\frac{1}{4}]$ and at most finitely many eigenvalues in the interval $(-\frac{1}{4}, 0)$ cf. [3]. The Liouville transformation

$$\varphi(y) = \Big(m(x)+1\Big)^{\frac{1}{4}}\,\psi(x) \quad \text{where} \quad y = x + \int_{-\infty}^{x}\Big[\sqrt{m(\xi)+1}\,-1\Big]\,d\xi$$

converts (2.1) into

$$-\frac{d^2\varphi}{dy^2} + Q\varphi = \mu\varphi. \tag{2.2}$$

Here

$$Q(y) = \frac{1}{4q(y)} + \frac{q_{yy}(y)}{4\,q(y)} - \frac{3\,q_y^2(y)}{16\,q^2(y)} - \frac{1}{4} \tag{2.3}$$

with $q(y) = m(x)+1$ and spectral parameter $\mu = -\frac{1}{4} - \lambda$. Suitable scattering data for (2.1) happen to be the usual scattering data for the problem (2.2), which evolve linearly at constant speed under the Camassa-Holm flow cf. [6]. Therefore the classical Marchenko approach (see [9]) is applicable and finding $Q(t,y)$ amounts to solving a linear integral equation determined by the scattering data for $Q_0(y)$, data available from the knowledge of $m_0(x)$. The only intricate point of this approach is the recovery of $m(t,x)$ from $Q(t,y)$. This requires us to solve, given Q, the nonlinear second order differential equation (2.3) for q, and then to perform the coordinate transform $y \mapsto x$. Equation (2.3) is a Pinney equation [14] but the solution for q, given Q, obtained in [14] is not convenient for our purposes (this approach was used in [3] and leads to unnecessary complications). A quite intricate (but nevertheless effective) algorithm for the recovery of m was proposed in [6]. Below we present an alternative approach that gives a more direct and less complicated solution. For convenience we drop the time-dependence in the formulation.

Theorem *Let $f(y)$ be the Jost function at $y=\infty$ for the eigenvalue equation*

$$\varphi_{yy} = (Q + \frac{1}{4})\,\varphi, \tag{2.4}$$

i.e. f is the unique solution of (2.4) *with the asymptotic behavior*

$$f(y) \approx e^{-y/2} \quad \text{and} \quad f'(y) \approx -\frac{1}{2}\,e^{-y/2} \quad \text{as} \quad y\to\infty.$$

If $H : \mathbb{R} \to \mathbb{R}$ *is the bijection given by* $H(y) = \displaystyle\int_{-\infty}^{y} \frac{d\xi}{f^2(\xi)}$, *then*

$$m(x) + 1 = e^{2x}\, f^4(H^{-1}(e^x)), \quad x \in \mathbb{R}. \tag{2.5}$$

Proof. Observe that (2.4) is precisely (2.2) with $\mu = -\frac{1}{4}$. Hence $\lambda = 0$, and the Liouville transformation maps (2.2) into

$$\psi_{xx} = \frac{1}{4}\,\psi,$$

where $\varphi(y) = \Big(m(x)+1\Big)^{\frac{1}{4}}\psi(x)$. The above equation has the solution $e^{-x/2}$ so that (2.2) has the corresponding solution $h(y) = \Big(m(x)+1\Big)^{\frac{1}{4}} e^{-x/2}$. But one can easily check that the function h has the asymptotic behavior at $y = \infty$ prescribed for f, so $f = h$, i.e. $f(y) = q^{1/4}(y)\, e^{-x/2}$. This means that $\dfrac{dy}{f^2(y)} = \dfrac{e^x\, dy}{\sqrt{q(y)}} = e^x\, dx$. Therefore $H(y) = \displaystyle\int_{-\infty}^{x} e^s\, ds = e^x$ is clearly invertible and $y = H^{-1}(e^x)$. From the relation $m(x) + 1 = q(y) = f^4(y) e^{2x}$ we now obtain (2.5). $\Box$

Remark The previous result reduces the recovery of $m(x)$ from $Q(y)$ to solving the linear integral equation

$$f(y) = e^{-y/2} + \int_y^{\infty} \Big(e^{(\xi-y)/2} - e^{(y-\xi)/2}\Big)\, Q(\xi)\, f(\xi)\, d\xi, \qquad y \in \mathbb{R},$$

and computing the inverse of the function $H(y) = \displaystyle\int_{-\infty}^{y} \frac{d\xi}{f^2(\xi)}$. $\Box$

Example A solitary wave for (1.1) is a solution $m(t,x) = \Phi(x - ct)$ with a profile Φ that decays at infinity. Solitary waves can exist only for speeds $c > 2$, but no mathematical expression in closed form seems to be available for Φ (see [8]), despite the fact that the corresponding potential $Q(y)$ is given explicitly by

$$Q(y) = -\frac{c-2}{2c\,\cosh^2\Big(\sqrt{\frac{c-2}{4c}}\,(y - y_0)\Big)}, \qquad y \in \mathbb{R},$$

with $y_0 \in \mathbb{R}$ (see [13]). If we choose for simplicity $y_0 = 0$ and $c = 8/3$, then

$$Q(y) + \frac{1}{4} = \frac{2\cosh^2(y/4) - 1}{8\cosh^2(y/4)}, \qquad y \in \mathbb{R}.$$

Note that $g(y) = 4\,\cosh^2(y/4)$ is a solution to (2.4). But then $y \mapsto g(y)\displaystyle\int_y^{\infty} \frac{d\xi}{g^2(\xi)}$ is also a solution to (2.4). Since $\displaystyle\int_y^{\infty} \frac{d\xi}{g^2(\xi)} = \frac{6e^{-y/4} + 2e^{-3y/4}}{3\cosh^3(y/4)}$, we deduce that

$$f(y) = \frac{3e^{-y/4} + e^{-3y/4}}{6\cosh(y/4)}, \qquad y \in \mathbb{R},$$

and an exact formula for the profile of the solitary wave emerges. $\Box$

Acknowledgement The authors thank the referee for valuable remarks.

References

[1] Camassa R and Holm D, An Integrable Shallow Water Equation with Peaked Solitons, *Phys. Rev. Letters* **71** (1993), 1661–1664.

[2] Constantin A, On the Inverse Spectral Problem for the Camassa-Holm Equation, *J. Funct. Anal.* **155** (1998), 352–363.

[3] Constantin A, On the Scattering Problem for the Camassa-Holm Equation, *Proc. Roy. Soc. London A* **457** (2001), 953–970.

[4] Constantin A and Escher J, Wave Breaking for Nonlinear Nonlocal Shallow Water Equations, *Acta Mathematica* **181** (1998), 229–243.

[5] Constantin A and Escher J, Well-Posedness, Global Existence, and Nlowup Phenomena for a Periodic Quasi-Linear Hyperbolic Equation, *Comm. Pure Appl. Math.* **51** (1998), 475–504.

[6] Constantin A and Lenells J, On the Inverse Scattering Approach for an Integrable Shallow Water Equation, *Phys. Lett. A* **308** (2003), 432–436.

[7] Constantin A and McKean HP, A Shallow Water Equation on the Circle, *Comm. Pure Appl. Math.* **52** (1999), 949–982.

[8] Constantin A and Strauss WA, Stability of the Camassa-Holm Solitons, *J. Nonlinear Science* **12**(2002), 415–422.

[9] Drazin P and Johnson RS, Solitons: An Introduction, Cambridge University Press, Cambridge, 1992.

[10] Fuchssteiner B and Fokas A, Symplectic Structures, Their Bäcklund Transformation and Hereditary Symmetries, *Physica D* **4** (1981), 47–66.

[11] Johnson RS, Camassa-Holm, Korteweg-de Vries and Related Models for Water Waves, *J. Fluid. Mech.* **457**(2002), 63–82.

[12] Johnson RS, On Solutions of the Camassa-Holm Equation, *Proc. Roy. Soc. London A* (to appear).

[13] Lenells J, The Scattering Approach for the Camassa-Holm Equation, *J. Nonlinear Math. Phys.* **9** (2002), 389–393.

[14] Pinney E, The Nonlinear Differential Equation $y'' + p(x)y + cy^{-3} = 0$, *Proc. Amer. Math. Soc..* **1** (1950), 681.