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On Vortex Solutions and Links between the Weierstrass System and the Complex Sine-Gordon Equations

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Abstract

The connection between the complex Sine and Sinh-Gordon equations associated with a Weierstrass type system and the possibility of construction of several classes of multivortex solutions is discussed in detail. We perform the Painlevé Test and analyse the possibility of deriving the Bäcklund transformation from the singularity analysis of the complex sine-Gordon equation. We make use of the analysis using the known relations for the Painlevé equations to construct explicit formulæ in terms of the Umemura polynomials which are τ-functions for rational solutions of the third Painlevé equation. New classes of multivortex solutions of a Weierstrass system are obtained through the use of this proposed procedure. Some physical applications are mentioned in the area of the vortex Higgs model when the complex sine-Gordon equation is reduced to coupled Riccati equations.

1 Introduction

The complex Sine and Sinh-Gordon equations have been of considerable interest recently in many areas of mathematical physics. They originally appeared in the reduction of the $O(4)$ nonlinear sigma model [1, 2]. They have also appeared in a number of other physical contexts, for example, in the study of a massless fermion model with chiral symmetry, and also in the study of the motion of a vortex filament in an inviscid incompressible fluid [3] and in a model of relativistic strings [4]. The equations have been found to be completely integrable, and some work on the construction of multi-soliton solutions has been carried out [1–5]. It has also been shown that the complex sine-Gordon theory may be reformulated in terms of the Wess–Zumino–Witten action and interpreted as the integrably deformed $SU(2)/U(1)$-coset model, (as in [6] and references therein). These studies are based on the complex Sine and Sinh-Gordon theory in $(1+1)$-dimensional Minkowski space-time. These models have energy functionals which are related to Ginsburg–Landau type models and can be written as [7]

$$E = \int \left[\frac{\vec{\nabla}\psi|^2}{1 \pm |\psi|^2} + 1 \pm |\psi|^2 \right] d^2x. \tag{1.1}$$

The complex Sine and Sinh-Gordon equations which are studied here are obtained from this type of Hamiltonian. In the limit $|\psi|^2 \leq 1$, equation (1.1) becomes an energy functional for a Ginzburg–Landau type model, which would have applications to superconductivity. From the physical point of view, this expression constitutes the basis of the phenomenological theory of superfluidity [7, 8] and has appeared in particle physics as well [1]. The Ginsburg–Landau functional is minimized by the Gross–Pitaevski vortices [7]. These are topological solitons of the form

$$\psi(x,y) = \Phi_n(r)e^{in\theta}, \qquad \lim_{r\to\infty} \Phi_n(r) = 1.$$

One of the purposes of our work is to study vortex solutions for the equations of interest discussed in this paper.

Another important physical application is to the area of superconductivity [3], where vortex solutions play an essential role. The Lagrangian for the superconducting system usually takes the form

$$L = -\frac{1}{4}F_{\mu\nu}^2 + |\nabla_\mu \psi|^2 - V(\psi). \tag{1.2}$$

Here ψ plays the role of the Higgs field, $F_{\mu\nu}$ pertains to the electrodynamic term and $V(\psi)$ is the scalar or Higgs potential function which is responsible for mass generation in the system. This is essentially the Lagrangian of the abelian Higgs model and, although the equations of motion differ from those we study here, there are known vortex solutions to them.

In particular the classical sine-Gordon equation has relevance to models which are of interest to particle theory. The complete integrability of the Gross–Neveu model, in the large N approximation, is quite analogous to the complete integrability of the classical sine-Gordon equation, where $1/N$ plays the role of the coupling constant [1, 4]. For $N=1$, a large coupling case, the model reduces to the massless Thirring model, which is scale invariant with anomalous dimensions. For $N>1$ the theory exhibits nontrivial renormalization group behavior and mass generation through dimensional transmutation.

Another area of recent importance with regard to applications is the area of liquid crystals and membranes [9]. Fluid membranes may be idealized as two-dimensional surfaces in solution with each membrane being made up of a double layer of long molecules. The curved fluid membrane may be treated as a bending liquid crystal cell with uniaxial molecular order. Various physical properties of interest can be calculated in terms of quantities which are directly related to the geometry of the surface. For example in one model for uniaxial liquid crystals, with normal $\vec{n}$ to the membrane of the liquid crystal, it has been shown [9] that the elastic energy of curvature per unit area of the membrane is

$$F = \int g\,dA, \qquad g = \frac{1}{2}k(c_1 + c_2 - c_0)^2 + \bar{k}c_1c_2. \tag{1.3}$$

Here c_1 and c_2 are the two principal curvatures of the surface of the membrane, and c_0 is called the spontaneous curvature of the surface. The quantity F, is referred to as the total bending energy of the membrane. The constant c_0 is related to the asymmetry of the layers. The positive constant k is the bending rigidity and $\bar{k}$, which could have either sign, is the elastic modulus of the Gaussian curvature. The curvature elastic free energy per unit area of the membrane can also be formulated rigorously in terms of two-dimensional differential invariants of the surface [9]. Many of these free energies have reductions to sine-Gordon equations and, hence, their solutions can be connected with Weierstrass data for surfaces.

We begin this paper with establishing the connection between a generalized Weierstrass system and the complex Sine and Sinh-Gordon equations. Then we analyze various properties of the complex sine-Gordon equation and show how they transform to corresponding properties of the Weierstrass system. The paper is organized as follows. A generalization of a Weierstrass system for inducing two-dimensional surfaces in $\mathbb{R}^4$ is presented in Section 2. Later in that section it is shown how this system is related to the complex Sine and Sinh-Gordon equations. Several properties, namely the Lax pair, the Painlevé Property and the Bäcklund transformation following from the singularity analysis, are investigated for the complex sine-Gordon equation and then extended to the Weierstrass system in Sections 3 and 4. Multivortex solutions are constructed and the Painlevé structure of the associated radial equations is studied in Section 5. The τ functions for the rational class of solutions of the Painlevé equation, P5, are written in terms of Umemura polynomials and explicit forms of such solutions are given in that section. We also present a particular class of solutions of the Weierstrass system via the solution of the complex sine-Gordon equation. Section 6 contains examples of solving the Weierstrass system by means of solutions of the complex sine-Gordon equation.

2 The generalized Weierstrass system and associated complex Sine and Sinh-Gordon equations

The Gauss–Codazzi equations describing a two-dimensional surface immersed in a three-dimensional sphere which is itself again immersed into a four-dimensional Euclidean space have been studied by Darboux [10]. He investigated the nonlinear Dirac-type system for four complex-valued functions ψ_i and φ_i, $i = 1, 2$ satisfying the following system of equations

$$\begin{aligned}
&\partial\psi_1 = Q_1\left(\psi_1 + \frac{\varphi_1}{2\psi_2\bar{\varphi}_2}\right), \qquad \bar{\partial}\psi_2 = Q_1\psi_2,\\
&\bar{\partial}\varphi_1 = Q_2\left(\varphi_1 - \frac{\psi_1}{2\varphi_2\bar{\psi}_2}\right), \qquad \partial\varphi_2 = Q_2\varphi_2,\\
&Q_1 = |\psi_2|^2 \pm |\psi_1|^2, \qquad Q_2 = |\varphi_2|^2 \pm |\varphi_1|^2,
\end{aligned} \tag{2.1}$$

and its respective complex conjugate equations. The partial derivatives are denoted $\partial = \partial/\partial z$ and $\bar{\partial} = \partial/\partial\bar{z}$ and the bar denotes complex conjugation The above system can be considered as a variant of the Weierstrass representation [11] for surfaces immersed in $\mathbb{R}^4$ and we refer to it as such. System (2.1) is a nonlinear first order system of eight equations, for which eight of sixteen first order derivatives with respect to z or $\bar{z}$ are known in terms of functions ψ_i and φ_i. System (2.1) admits several conservation laws such as

$$\partial(\ln\bar{\psi}_2) = \bar{\partial}(\ln\psi_2), \qquad \partial(\ln\varphi_2) = \bar{\partial}(\ln\bar{\varphi}_2), \qquad \partial\left(\frac{\psi_1\bar{\varphi}_1}{\bar{\psi}_2\varphi_2}\right) = \bar{\partial}\left(\frac{\bar{\psi}_1\varphi_1}{\psi_2\bar{\varphi}_2}\right). \tag{2.2}$$

As a consequence of the conservation laws, (2.2), there exist four real-valued functions $X^i(z,\bar{z})$ which are defined by

$$\begin{aligned} X^1 &= \int_\Gamma \ln\psi_2 dz + \ln\bar{\psi}_2 d\bar{z}, \qquad X^2 = \int_\Gamma \ln\varphi_2 dz + \ln\bar{\varphi}_2 d\bar{z}, \\ X^3 &= \int_\Gamma \ln\psi_2\varphi_2 dz + \ln\bar{\psi}_2\bar{\varphi}_2 d\bar{z}, \qquad X^4 = \int_\Gamma \frac{\bar{\psi}_1\varphi_1}{\psi_2\bar{\varphi}_2}dz + \frac{\psi_1\bar{\varphi}_1}{\bar{\psi}_2\varphi_2}d\bar{z} \end{aligned} \tag{2.3}$$

for any contour Γ in the complex plane which begins at a fixed z_0 and ends at the variable point z. The right hand side of (2.3) does not depend on the choice of the curve Γ since the differentials of equations (2.3) are exact. Thus equations (2.1) and (2.2) allow us to identify the real-valued functions, $X^i(z,\bar{z})$, $i = 1,\ldots,4$, as the coordinates of a surface immersed in four-dimensional Euclidean space.

At this point we want to underline that, for the Weierstrass system (2.1), few explicit solutions have been found up to now, and the link obtained below with the complex Sine and Sinh-Gordon equations allows us to construct new classes of solutions explicitly. To our knowledge the connection between these two systems is observed here for the first time.

We subject system (2.1) to several transformations in order to simplify its structure. We start by defining two new complex valued functions

$$u = \frac{\psi_1}{\bar{\psi}_2}, \qquad v = \frac{\varphi_1}{\bar{\varphi}_2}. \tag{2.4}$$

It is easy to show that, if the complex functions ψ_i and φ_i, $i = 1,2$, are solutions of the first order system (2.1), then the complex-valued functions u and v defined by (2.4) are solutions of the first order system of two equations

$$\partial u = \frac{1}{2}\left(1 \pm |u|^2\right)v, \qquad \bar{\partial}v = -\frac{1}{2}\left(1 \pm |v|^2\right)u, \tag{2.5}$$

and their respective complex conjugate equations. The elimination of one of the functions u or v in system (2.5) leads to the complex Sinh-Gordon (CShG) equation when the sign is positive in (2.5), and sine-Gordon (CSG) equation when the sign is negative in (2.5). Thus we get for both cases

$$\partial\bar{\partial}u \mp \frac{\bar{u}}{1 \pm |u|^2}\partial u\bar{\partial}u + \frac{1}{4}u\left(1 \pm |u|^2\right) = 0. \tag{2.6}$$

If u is assumed real then the substitutions $u = \sinh(\Phi/2)$ for the CShG or $u = \sin(\Phi/2)$ for the CSG yield

$$\partial\bar{\partial}\Phi + (1/4)\sinh\Phi = 0, \qquad \text{or} \qquad \partial\bar{\partial}\Phi + (1/4)\sin\Phi = 0,$$

which are the well-known sinh-Gordon and sine-Gordon equations respectively.

As was shown in [1], equation (2.6) was derived in the context of the reduction of the $O(4)$ nonlinear sigma model and, as well, the reduction of the self-dual Yang–Mills equations and relativistic equations [2, 12, 13].

Note that, if v tends to one in the CSG equations (2.5), then Q_2 vanishes and the system (2.1) takes the form

$$\partial\psi_1 = Q_1\left(\psi_1 + \frac{1}{2\psi_2}\right), \qquad \bar\partial\psi_2 = Q_1\psi_2. \tag{2.7}$$

Conversely, if u tends to one in CSG equations (2.5), then Q_1 vanishes and system (2.1) becomes

$$\bar\partial\varphi_1 = Q_2\left(\varphi_1 - \frac{1}{2\varphi_2}\right), \qquad \partial\varphi_2 = Q_2\varphi_2. \tag{2.8}$$

These limits characterize the properties of the solutions of system (2.1).

Let us now state the following Proposition.

Proposition 1. *If the set of complex-valued functions, ψ_i, φ_i, $i = 1, 2$, is a solution of the system (2.1), and u and v are defined in terms of them by (2.4), then the pair (u, v) is a solution of equations (2.5).*

Proof. Differentiation of equations (2.4) with respect to z and $\bar z$ gives

$$\partial u = \frac{\partial\psi_1}{\bar\psi_2} - \frac{\psi_1}{\bar\psi_2^2}\partial\bar\psi_2, \qquad \bar\partial v = \frac{\bar\partial\varphi_1}{\bar\varphi_2} - \frac{\varphi_1}{\bar\varphi_2^2}\bar\partial\bar\varphi_2,$$

respectively. Making use of system (2.1), we get

$$\partial u = \left(|\psi_2|^2 \pm |\psi_1|^2\right)\frac{\varphi_1}{2|\psi_2|^2\bar\varphi_2}, \qquad \bar\partial v = -\left(|\varphi_2|^2 \pm |\varphi_1|^2\right)\frac{\psi_1}{2|\varphi_2|^2\bar\psi_2},$$

and, by virtue of (2.4), we obtain (2.5), which ends the proof. ■

Now we discuss a set of conditions which allow the system (2.1) to become a decoupled system of equations.

Proposition 2. *Let the complex functions u and v be solutions of system (2.5). Let the functions ψ_i and φ_i be defined in terms of u and v by*

$$\begin{aligned} &\psi_1 = \epsilon u\left(1 \pm |u|^2\right)^{-1/2}, \qquad \varphi_1 = \epsilon v\left(1 \pm |v|^2\right)^{-1/2}, \\ &\psi_2 = \epsilon\left(1 \pm |u|^2\right)^{-1/2}, \qquad \varphi_2 = \epsilon\left(1 \pm |v|^2\right)^{-1/2}, \qquad \epsilon = \pm 1. \end{aligned} \tag{2.9}$$

Then the general integrals of system (2.1) are given by

$$\begin{aligned} &\psi_1 = u\bar A(\bar z)e^{z}, \qquad \varphi_1 = vB(z)e^{\bar z}, \\ &\psi_2 = A(z)e^{\bar z}, \qquad \varphi_2 = \bar B(\bar z)e^{z}, \end{aligned} \tag{2.10}$$

where the complex functions A and B satisfy the following conditions,

$$|A|^2 = e^{-(z+\bar z)}\left(1 \pm |u|^2\right)^{-1}, \qquad |B|^2 = e^{-(z+\bar z)}\left(1 \pm |v|^2\right)^{-1}. \tag{2.11}$$

Proof. Substituting (2.4) into system (2.1) we obtain an overdetermined system for the functions ψ_2 and φ_2 of the following form

$$\partial(u\bar{\psi}_2) = \left(1 \pm |u|^2\right)|\psi_2|^2\left(u\bar{\psi}_2 + \frac{v}{2\psi_2}\right), \qquad \bar{\partial}\psi_2 = \left(1 \pm |u|^2\right)|\psi_2|^2\psi_2,$$
$$\bar{\partial}(v\varphi_2) = \left(1 \pm |v|^2\right)|\varphi_2|^2\left(v\bar{\varphi}_2 - \frac{u}{2\varphi_2}\right), \qquad \partial\varphi_2 = \left(1 \pm |v|^2\right)|\varphi_2|^2\varphi_2. \tag{2.12}$$

Consider the first equation in the first line of (2.12). By expansion of the derivative term $\partial(u\bar{\psi}_2)$ and the use of (2.5) this equation reduces to the form

$$\partial\bar{\psi}_2 = \left(1 \pm |u|^2\right)|\psi_2|^2\bar{\psi}_2.$$

Using (2.9) to eliminate $|\psi_2|^2$ in this result, this equation reduces to $\partial\bar{\psi}_2 = \psi_2$. In a similar way, (2.9) can be used to treat the remaining three equations in (2.12). Thus the initial system (2.1) becomes a linear system of the form

$$\bar{\partial}\psi_2 = \psi_2, \qquad \partial\varphi_2 = \varphi_2. \tag{2.13}$$

These two equations can be easily integrated to give ψ_2 and φ_2 as given in (2.10). Then (2.4) can be used to obtain ψ_1 and φ_1. The results in (2.10) must be consistent with those in (2.9). If we calculate the modulus of ψ_2 and φ_2 from (2.10) and equate to the modulus calculated from (2.9), the conditions (2.11) are obtained. In fact, equating ψ_i, φ_i in (2.10) to their corresponding forms in (2.9), we must also have that

$$\bar{A}(\bar{z})e^{z} = \epsilon\left(1 \pm |u|^2\right)^{-1/2} = A(z)e^{\bar{z}}, \qquad B(z)e^{\bar{z}} = \epsilon\left(1 \pm |v|^2\right)^{-1/2} = \bar{B}(\bar{z})e^{z}.$$

A set of differential constraints which must be satisfied can be obtained by the substitution of (2.9) into (2.13) and we find that

$$\left(\bar{\partial}u\bar{u} + \frac{1}{2}uv\left(1 \pm |u|^2\right)\right) = \mp 2\left(1 \pm |u|^2\right),$$
$$\left(\partial v\bar{v} - \frac{1}{2}\bar{u}v\left(1 \pm |v|^2\right)\right) = \mp 2\left(1 \pm |v|^2\right). \qquad \blacksquare$$

From the computational point of view, it is more convenient to deal with the CShG or CSG equations (2.5) than with the original system (2.1). From every solution of CShG or CSG equations (2.5), we can integrate a linear system (2.13) and, consequently, a very large class of solutions of system (2.1) can be found explicitly by making use of formulae (2.10) and (2.11).

Using the connection between the CSG equation (2.6) and Weierstrass system (2.1), we discuss in the next section in detail the Painlevé analysis of the CSG equation which allows us to extend this analysis to the Weierstrass system.

3 Painlevé analysis of the complex sine-Gordon equation

Integrability of the CSG equation (2.6) is confirmed by tests for the Painlevé Property. We perform the classical test of [14] extended to partial differential equations in [15], assuming a solution in the form of a Laurent series about an arbitrary singularity manifold $F(z,\bar{z}) = 0$ and checking compatibility of the resulting recurrence formulae. Detailed discussion of the meaning and validity of this test may be found in [16, 17]. The test is carried out for the CSG equation (2.6) and for the system (2.5). Both versions of the CSG pass the test. However, the possibility of obtaining the Bäcklund transformation through truncation of the Laurent series [15] is restricted to special cases.

For the purpose of the Painlevé test z and $\bar{z}$ should be treated as two independent variables and extended to two separate complex planes. Similarly the functions u and $\bar{u}$ need not be complex conjugates of each other when their independent variables are separately extended; the same holds for v and $\bar{v}$. To avoid the misleading complex conjugate symbol, we denote $\bar{z}$ by t, $\bar{u}$ by w and $\bar{v}$ by s, while symbols of the derivatives ∂ and $\bar{\partial}$ will be replaced by alphabetic subscripts z and t, respectively.

In principle the Painlevé test could be performed either for the system (2.5) or (2.6). However, the latter has a singularity at $|u| = 1$. It is not encompassed by the usual test, which assumes $|u| \to \infty$. Therefore we start from equation (2.5).

This system has an apparent symmetry, $u \leftrightarrow v$, $z \leftrightarrow \bar{z}$. For the purpose of the "Painlevé Test" equations (2.5) constitute a 4×4 system. The CSG version (lower sign in (2.5)) in our notation is given by

$$\begin{aligned} &u_z - (1-wu)v/2 = 0, \qquad w_t - (1-wu)s/2 = 0,\\ &v_t + (1-sv)u/2 = 0, \qquad s_z + (1-sv)w/2 = 0. \end{aligned} \tag{3.1}$$

The initial exponent is -1 for u and w and zero for v and s or the other way round. Both choices are equivalent due to the aforementioned symmetry. The second one implies $uw \to 1$ when $F \to 0$. Thus our test encompasses the extra singularity $|u| \to 1$ of equations (2.6).

At this point we mention the following fact. Usually, the classical "Painlevé Test" is not possible when the leading order term of an expanded function is of order zero in F. Such a term lacks the property (necessary for the algorithm of [14] and [15]) that differentiation decreases its order of magnitude by one. With the first choice of the exponents this problem could emerge in the last two equations of (3.1), containing the derivatives v_t and s_t. However, in our case the leading terms in these equations are the nonlinear ones, which are proportional to F^{-1} (their balance is achieved by $s_0 v_0 = 1$), while the troublesome derivatives are of order zero in F. Hence the test may be performed in the usual way.

The initial coefficients for the first choice of the exponents are

$$\begin{aligned} &u_0 = 2(F_z F_t)^{1/2}/Q_0, \qquad w_0 = 2(F_z F_t)^{1/2} Q_0,\\ &v_0 = (F_z/F_t)^{1/2}/Q_0, \qquad s_0 = (F_t/F_z)^{1/2} Q_0, \end{aligned} \tag{3.2}$$

where Q_0 is an arbitrary function of z and t. The remaining terms are derived from the linear system of recurrence formulae

$$\begin{aligned} &nF_z u_n + \left(F_z/Q_0^2\right) w_n + 2F_z F_t v_n\\ &\qquad = -(u_{n-1})_z + \frac{1}{2} v_{n-2} - \frac{1}{2} u_0 \sum_{k=1}^{n-1} w_k v_{n-k} - \frac{1}{2} \sum_{k=0}^{n-1} \sum_{l=1}^{n-k-1} w_k u_l v_{n-k-l}, \end{aligned} \tag{3.3a}$$

$$\left(F_t Q_0^2\right) u_n + nF_t w_n + 2F_z F_t s_n$$
$$= -(w_{n-1})_t + \frac{1}{2}s_{n-2} - \frac{1}{2}w_0 \sum_{k=1}^{n-1} u_k s_{n-k} - \frac{1}{2}\sum_{k=0}^{n-1}\sum_{l=1}^{n-k-1} u_k w_l s_{n-k-l}, \tag{3.3b}$$

$$(n-1)F_t v_n - \left(F_z/Q_0^2\right) s_n$$
$$= -(v_{n-1})_t + \frac{1}{2}u_0 \sum_{k=1}^{n-1} s_k v_{n-k} + \frac{1}{2}\sum_{k=0}^{n-1}\sum_{l=1}^{n-k-1} s_k u_l v_{n-k-l}, \tag{3.3c}$$

$$-\left(F_t Q_0^2\right) v_n + (n-1)F_z s_n$$
$$= -(s_{n-1})_z + \frac{1}{2}w_0 \sum_{k=1}^{n-1} v_k s_{n-k} + \frac{1}{2}\sum_{k=0}^{n-1}\sum_{l=1}^{n-k-1} v_k w_l s_{n-k-l}, \tag{3.3d}$$

where v and s with negative subscripts are both set equal to zero, but have been included for reasons of notation.

The determinant of the system (3.3) is

$$(F_z F_t)^2 (n+1)n(n-1)(n-2). \tag{3.4}$$

Hence the indices at which it becomes zero, are -1, 0, 1 and 2. Tedious but straightforward calculations show that all the compatibility conditions are satisfied, whence we conclude that equation (2.5) has the Painlevé Property.

This Painlevé integrability obviously extends to the Weierstrass system (2.1) as the Painlevé Property is invariant under the homographic transformation (2.4) which converts the equations (2.5) to (2.1).

The method of the Laurent expansion may often be extended to deriving the Bäcklund transformation and further an explicit integration scheme [15]. The usual approach relies on truncation of the Laurent series, usually at the term of order F^0, which is expected to be the transformed function, satisfying the original equation. The truncation requires appropriate choice of the arbitrary functions (first integrals). Some extra assumptions on the coefficients and expansion variable may also be necessary. A systematic approach to that problem may be found in [16–19].

However, the usual method contains assumptions which are too restrictive for application to equations (3.1), namely the Laurent series of v and s, which begin with the F^0 terms, reduce to a single term each. This means that v and s would not be transformed at all. Moreover, the truncation at F^0 implies vanishing of terms proportional to F^1. This imposes further constraints on these variables: from their recurrence equations (3.3) at $n = 1$

$$-(F_z/Q_0^2)s_1 = -(v_0)_t = 0, \tag{3.5a}$$
$$-(F_t Q_0^2)v_1 = -(s_0)_z = 0. \tag{3.5b}$$

It follows that v_0 should be independent of t, while s_0 should be independent of z. As these coefficients are reciprocals of each other (see 3.2), neither of them may depend on z or t. This, together with the truncation of the series, reduces u and v to constants. If we denote

$$s = k, \qquad v = 1/k, \qquad k = \text{const}, \tag{3.6}$$

then the original equations (3.1) reduce to a system of coupled Riccati equations

$$u_z - (1 - wu)/(2k) = 0, \qquad w_t - (1 - wu)k/2 = 0, \tag{3.7}$$

which may immediately be linearized by substitution

$$u = (2/k)(\ln \Psi)_t, \qquad w = 2k(\ln \Psi)_z, \tag{3.8}$$

to the Helmholtz equation

$$\Psi_{zt} = (1/4)\Psi. \tag{3.9}$$

Obviously this linearization also yields an (almost trivial) transformation of u and v, a superposition principle and other properties.

We have also tried a more general truncation scheme of [20, 21]. For better comparison with the usual SG equation we start from a trigonometric representation of (2.6) as in [21]. Let

$$u = \sin(\Phi/2)\exp(i\alpha). \tag{3.10}$$

The polar coordinates Φ and α satisfy a system of equations similar to that given by Lund [4]

$$\partial\bar{\partial}\Phi - 2\frac{\sin(\Phi/2)}{\cos^3(\Phi/2)}\partial\alpha\bar{\partial}\alpha + \frac{1}{4}\sin\Phi = 0, \tag{3.11}$$

$$\partial\left(\tan^2(\Phi/2)\bar{\partial}\alpha\right) + \bar{\partial}\left(\tan^2(\Phi/2)\partial\alpha\right) = 0. \tag{3.12}$$

Returning to the notation of the Painlevé test, we complete the definition of u by a similar one for w

$$w = \sin(\Phi/2)\exp(-i\alpha). \tag{3.13}$$

According to [21] we impose constraints on the function defining the singularity manifold $F(z, t) = 0$. This may be done without actually changing the manifold [17]. The constraints have the form of Riccati equations

$$F_z = 1 - AF - BF^2, \qquad F_t = -C + (AC + C_z)F + (BC - D)F^2, \tag{3.14}$$

where A, B, C, D are functions of both z and t, satisfying the following cross-derivative compatibility conditions

$$A_t = -(AC)_z - C_{zz} + 2D, \qquad B_t = D_z - 2BC_z - B_zC - AD. \tag{3.15}$$

When $F \to 0$, $u = u_0/F + O(1)$, whence $\Phi = \pm 2i\ln F + O(1)$. Therefore the truncated expansion of Φ should read (with the + sign)

$$\Phi = 2i\ln F + \vartheta. \tag{3.16}$$

The other dependent variable α should be regular and nonzero when $F \to 0$ (up to a set of lower dimensionality) since

$$\exp(2i\alpha) = \frac{u}{w} = \frac{u_0 + O(F)}{w_0 + O(F)}. \tag{3.17}$$

Hence α is also regular at the singularity manifold. It may be expanded in non-negative powers of F.

Substitution of (3.16) into equation (3.11) together with the constraints (3.14) and the compatibility conditions (3.15) yields a polynomial in F containing powers from F^0 to F^{16}. The expansion of α will not spoil the truncation if we assume its truncation at F^4, that is,

$$\alpha = \alpha_0 + \alpha_1 F + \alpha_2 F^2 + \alpha_3 F^3 + \alpha_4 F^4. \tag{3.18}$$

Now we substitute (3.16) and (3.18) into the CSG system in the form (3.11) and (3.12), make use of the constraints and compatibility conditions, put both equations in a polynomial form and compare coefficients at subsequent powers of F. This way we obtain a system of 28 differential equations. Detailed analysis shows that those equations are compatible if and only if

$$\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0, \tag{3.19}$$

and either

$$\alpha_0 = \text{const}, \tag{3.20a}$$

or

$$\vartheta = \text{const}, \tag{3.20b}$$

where by const we understand a quantity independent both of z and t. In the first case

$$\vartheta_{xt} = (i/16)\left[B\exp(i\vartheta) - B^{-1}\exp(-i\vartheta)\right], \tag{3.21}$$

where the function B, introduced by the constraints (3.14), must also be a constant to ensure compatibility of the system. It may play the role of the spectral parameter in the integration scheme. However, the restrictions (3.19) and (3.20a) imply $\alpha = \text{const}$, which reduces the CSG system (3.11) and (3.12) to the usual sine-Gordon equation. This way we have regained the Bäcklund transformation for that equation in the version [21].

The second case, (3.19) and (3.20b), allows for limited variation of the phase α, namely α may be the linear function of x and t

$$\alpha = ax + bt + c, \tag{3.22}$$

where a, b and c are arbitrary constants. However, the constraint (3.20b) makes this case trivial.

The above analysis indicates that the only Bäcklund autotransformation obtainable by a removal of the solution's singularity is that of the usual sine-Gordon equation.

4 On equivalence of two forms of the complex sine-Gordon equation

Throughout this paper we investigate the CSG equation in the form of equations (2.6). Another form of the CSG equation was given in [18, 19] as follows

$$\left(\frac{q_\xi}{\sqrt{1+qp}}\right)_\eta = 4q, \qquad \left(\frac{p_\xi}{\sqrt{1+qp}}\right)_\eta = 4p. \tag{4.1}$$

We formulate the following statement for systems (2.6) and (4.1).

Proposition 3. *Equation (4.1) is transformed into CSG equation (2.6) through the following relations*

$$\xi = \frac{1}{4}\bar{z}, \qquad \eta = -\frac{1}{4}z, \qquad q = -\sin\Phi\exp(-i\beta), \qquad p = \sin\Phi\exp(i\beta), \tag{4.2}$$

where the phase is given by

$$\beta = \int_{z_0}^{z} \left\{ \left[1 + \tan^2(\Phi(z', \bar{z})/2)\right] \partial' \alpha(z', \bar{z}) dz' + \left[1 - \tan^2(\Phi(z, \bar{z}')/2)\right] \bar{\partial}' \alpha(z, \bar{z}') d\bar{z}' \right\}. \tag{4.3}$$

The lower limit of integration, z_0, is fixed and depends on the initial conditions.

Proof. Substitution of (4.2) and (4.3) into (4.1) yields the system (3.11) and (3.12). ■

Note that both equations, (2.6) and (4.1), depend on their phases α and β, respectively, through their derivatives only, except for linear dependence on factors $\exp(i\alpha)$ and $\exp(i\beta)$. Therefore any change of β, which leaves its derivatives unchanged, for example one that arises from deformation of the integration contour in (4.3), does not affect equivalence of those equations. Moreover for those u, which satisfy the CSG equation (2.6), the integrand is an exact differential and the path of integration does not even affect the value of the phase.

Note also that the form of the equation determining the evolution of the phase (3.12) suggests integration in terms of an arbitrary potential $\psi(z, \bar{z})$

$$\tan^2(\Phi/2)\partial\alpha = \partial\psi, \qquad \tan^2(\Phi/2)\bar{\partial}\alpha = -\bar{\partial}\psi. \tag{4.4}$$

However, the potential is not arbitrary since the compatibility condition $\partial\bar{\partial}\alpha = \bar{\partial}\partial\alpha$ imposes a constraint of the form similar to the original phase equation (3.12)

$$\partial\left(\cot^2(\Phi/2)\bar{\partial}\psi\right) + \bar{\partial}\left(\cot^2(\Phi/2)\partial\psi\right) = 0. \tag{4.5}$$

Obviously, if α is a solution of (3.12) for a given Φ, then ψ solves the same equation for Φ shifted by an odd multiple of π or subtracted from such a multiple. However, this is not a symmetry of the CSG equation (2.6) as the signs in the amplitude equation (3.11) are changed by such a transformation. Finally repetition of the transformation brings us back to the original equation. A similar property holds for the CSG equation in the form (4.1) for which the phase equation may be written as

$$\partial\left(\cos^{-1}\Phi\bar{\partial}\beta\right) - \bar{\partial}(\cos\Phi\partial\beta) = 0, \tag{4.6}$$

where the change of independent variables has already been performed.

This symmetry has an additional consequence. The transformations (4.2) and (4.3) from equations (2.6) to (4.1) have some features of a Bäcklund transformation, namely the definition of β by means of the contour integral (4.3) is a solution of a coupled pair of differential equations

$$\partial\beta = \left[1 + \tan^2(\Phi/2)\right]\partial\alpha, \qquad \bar{\partial}\beta = \left[1 - \tan^2(\Phi/2)\right]\bar{\partial}\alpha. \tag{4.7}$$

System (4.7) is overdetermined as the right hand sides of the above equations must satisfy the compatibility condition $\partial\bar{\partial}\beta = \bar{\partial}\partial\beta$. This condition is indeed satisfied as it proves to be equivalent to equation (3.12). Thus we obtain (3.12) in two ways: either through direct substitution of (4.7) to (4.6) or from the above compatibility condition.

The transformation defined by (4.2) and (4.3) may be extended to the inverse scattering method. The inverse scattering technique for (4.1) was given in [18]. Using (4.2) and (4.3), we obtain the Lax pair for (2.6)

$$\partial X = -\frac{1}{4\lambda}[Y_1, X], \qquad \bar{\partial} X = \frac{1}{4}[Y_2 + \lambda Y, X], \tag{4.8}$$

where

$$Y = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}, \quad Y_1 = \begin{pmatrix} i\cos\Phi & \sin\Phi e^{i\beta} \\ -\sin\Phi e^{-i\beta} & -i\cos\Phi \end{pmatrix},$$
$$Y_2 = 2i \begin{pmatrix} 0 & \dfrac{\bar{\partial}(\sin\Phi e^{i\beta})}{\cos\Phi} \\ \dfrac{\bar{\partial}(\sin\Phi e^{-i\beta})}{\cos\Phi} & 0 \end{pmatrix}. \tag{4.9}$$

Further extension to the Weierstrass system (2.1) is also possible by means of the transformation (2.4). The matrices Y_1 and Y_2 expressed in terms of u become

$$Y_1 = \begin{pmatrix} i(1-2u^2e^{-2i\alpha}) & \sqrt{1-(1-2u^2e^{-2i\alpha})^2}e^{i\beta} \\ -\sqrt{1-(1-2u^2e^{-2i\alpha})^2}e^{-i\beta} & -i(1-2u^2e^{-2i\alpha}) \end{pmatrix},$$
$$Y_2 = \begin{pmatrix} 0 & \dfrac{\bar{\partial}\left(\sqrt{1-(1-2u^2e^{-2i\alpha})^2}e^{i\beta}\right)}{1-2u^2e^{-2i\alpha}} \\ \dfrac{\bar{\partial}\left(\sqrt{1-(1-2u^2e^{-2i\alpha})^2}e^{-i\beta}\right)}{1-2u^2e^{-2i\alpha}} & 0 \end{pmatrix}. \tag{4.10}$$

From (2.4) the complex functions u and v are given in terms of ψ_1, $\bar{\psi}_2$ and φ_1, $\bar{\varphi}_2$, respectively. Taking into account that the functions u and v satisfy the same equation (2.6), we can describe the resulting Lax pair (4.8) for Weierstrass system (2.1) by a system of five two by two matrices Y_1, Y_2 in terms of $(\psi_1, \bar{\psi}_2)$ and $(\varphi_1, \bar{\varphi}_2)$ respectively, and the constant matrix Y.

5 Multivortex solutions

At this point we derive, through the link between first order system (2.1) and equations (2.6), a procedure for constructing multivortex solutions in explicit form. We concentrate on a certain class of multivortex solutions of equations (2.6) in polar coordinates (r,θ) on the plane determined by

$$u = A_n(r)e^{in\theta}, \qquad n \in \mathbb{Z}. \tag{5.1}$$

Equation (2.6), under the assumption (5.1) is reducible to a second order ODE of the form

$$\frac{d^2A_n}{dr^2} + \frac{1}{r}\frac{dA_n}{dr} \mp \frac{A_n}{1 \pm A_n^2}\left[\left(\frac{dA_n}{dr}\right)^2 \pm \frac{n^2}{r^2}\right] + \left(1 \pm A_n^2\right)A_n = 0. \tag{5.2}$$

By a homographic transformation of the dependent variable

$$A_n = c\frac{1+w(z)}{1-w(z)}, \qquad z = r, \tag{5.3}$$

where $c = -i$ for the CShG system and $c = 1$ for the CSG system, equation (5.2) has the structure of the fifth Painlevé (P5) equation

$$w'' = \frac{3w-1}{2w(w-1)}w'^2 - \frac{w'}{z} + \frac{(w-1)^2}{z}\left(\alpha w + \frac{\beta}{w}\right) + \frac{\gamma}{z}w + \delta\frac{w(w+1)}{w-1}, \tag{5.4}$$

with the coefficients α and β parametrized by a number $n \in \mathbb{Z}$ and γ, δ fixed as follows

$$\alpha = -\beta = \frac{n^2}{8}, \qquad \gamma = 0, \qquad \delta = -2. \tag{5.5}$$

Such a reduction to P5 has recently been performed [7]. In general equation P5 is not integrable in terms of known classical transcendental functions. However, for specific values of the parameters, solutions of equation (5.4) can be reduced to two types of non-transcendental functions, that is, to solutions of a Riccati equation with one arbitrary parameter or to three types of rational solutions of equation P5 [17, 22]. According to [23] equation (5.4) with the coefficients (5.5) can be written in an equivalent form as a first order system of ODEs,

$$\begin{aligned} z\frac{dp}{dz} &= -\frac{\epsilon n}{2} - \epsilon np - pq - p^2 q, \qquad \epsilon = \pm 1, \\ z\frac{dq}{dz} &= -2z^2 + \epsilon nq - 4z^2 p + \frac{q^2}{2} + pq^2, \end{aligned} \tag{5.6}$$

where $p = w/(1-w)$. The function $q(z)$ satisfies a Painlevé-type equation of the form

$$q'' = \frac{q}{q^2-4z^2}q'^2 - \frac{q^2+4z^2}{q^2-4z^2}\frac{q'}{z} + \frac{q}{4z^2(q^2-4z^2)}\left[16nz^2(2\epsilon - n) - \left(q^2-4z^2\right)^2\right].$$

The function $q^2 - 4z^2$ has two roots at $q = 2z$. Using the transformation

$$y(z) = \frac{q+2z}{q-2z}, \qquad q \neq 2z,$$

we get that $y(z)$ is also a solution of equation P5 with parameters

$$\tilde{\alpha} = -\tilde{\beta} = \frac{(1-\epsilon n)^2}{8}, \qquad \tilde{\gamma} = 0, \qquad \tilde{\delta} = -2. \tag{5.7}$$

Propositions 4 to 6 are special cases studied by V. Gromak [22] (Chapter 12, section 14) concerning the fifth Painlevé equations with specific parameters. This analysis is used to construct solutions to Weierstrass system (2.1).

Proposition 4. *Let $w = w(z)$ be a solution of the fifth Painlevé equation P5 (5.4) with parameters given by (5.5) such that the function,*

$$\Phi_1(w) \equiv zw' - \frac{\epsilon n}{2}w^2 + 2zw + \frac{\epsilon n}{2} \neq 0, \tag{5.8}$$

does not vanish for any $n \in \mathbb{Z}$. Then the function,

$$w_1 = 1 - \frac{4z}{\Phi_1(w)} \tag{5.9}$$

is a solution of the fifth Painlevé equation (5.4) with parameters given by (5.7).

Proposition 4 establishes the Auto-Bäcklund transformation (Auto-BT) for equation P5 when $\gamma = 0$, $\delta = -2$ and $\alpha = -\beta$ are parametrized by $n \in \mathbb{Z}$.

We discuss the link between equations P5 with different values of the parameter δ, namely, $\delta \neq 0$ and $\delta = 0$. Note that, for $\delta \neq 0$, the solutions of equation (5.4) are expressible in terms of Bessel functions whereas, for $\delta = 0$, they can be expressed in terms of Umemura polynomials. This is presented below.

Proposition 5. *Let $u(z) \neq 0$ be a solution of equation P5 with parameters given by (5.5). Then the function*

$$\tilde{u}(z) = \frac{f^2(\sqrt{z})}{f^2(\sqrt{z}) - 1}, \tag{5.10}$$

where $f(z)$ is defined by

$$f(z) = \frac{d}{dz} \ln u(z) - \frac{n}{4z}\left(u(z) - \frac{1}{u(z)}\right), \qquad n \in \mathbb{Z},$$

is a solution of equation P5 with parameters

$$\tilde{\alpha} = \frac{(1+n^2)^2}{2}, \qquad \tilde{\beta} = \tilde{\delta} = 0, \qquad \tilde{\gamma} = -\frac{1}{2}. \tag{5.11}$$

Based on reference [22] and using the result of Proposition 6 in that reference, we can find in our case the relation between equations P3 with $\gamma\delta \neq 0$

$$w'' = \frac{w'^2}{w} - \frac{w'}{w} + \frac{1}{z}(\alpha\gamma w^2 + \beta) + \gamma^2 w^3 + \frac{\delta}{w}, \tag{5.12}$$

and P5 with coefficients given by (5.11). Indeed the third Painlevé equation (5.12) can be written as a first order system of ODEs

$$\begin{aligned} zw' &= (\epsilon\alpha - 1)w + \epsilon\gamma z w^2 + zv, \qquad \epsilon = \pm 1, \\ zwv' &= \beta w + \delta z + (\epsilon\alpha - 2)wv + zv^2. \end{aligned} \tag{5.13}$$

From system (5.13) the elimination of w gives

$$\begin{aligned} v'' - \frac{v}{v^2+\delta} v'^2 + \frac{v'}{z} + \frac{\beta^2 - (2-\epsilon\alpha)^2\delta}{z^2(v^2+\delta)} v \\ + \epsilon\gamma(v^2+\delta) - \frac{2\delta\beta}{z^2}\frac{(\epsilon\alpha-2)}{v^2+\delta} + \frac{\beta}{z^2}(\epsilon\alpha - 2) = 0. \end{aligned} \tag{5.14}$$

By a homographic transformation of the dependent variable and a change of the independent variable

$$v = -i\sqrt{\delta}\frac{y+1}{y-1}, \qquad z = \sqrt{2\tau}, \tag{5.15}$$

we obtain from (5.14) equation P5

$$y'' + \frac{3y-1}{2y(y-1)}y'^2 + \frac{y'}{\tau} + \frac{1}{32\delta\tau^2}\left[(y^2-1)\left(Ay + \frac{B}{y}\right)\right] + \frac{\epsilon}{\tau}\gamma(-\delta)^{1/2}y = 0, \qquad (5.16)$$

where A and B are defined as

$$A = \beta^2 + 4(-\delta)^{1/2}\beta - \delta\alpha^2 - 4\delta - 2(-\delta)^{1/2}\epsilon\alpha\beta + 4\epsilon\delta\alpha,$$
$$B = \delta\alpha^2 - 2(-\delta)^{1/2}\epsilon\alpha\beta + 4(-\delta)^{1/2}\beta + 4\delta - \beta^2 - 4\epsilon\delta\alpha. \qquad (5.17)$$

Proposition 6. *Let $y = y(z)$ be a solution of the fifth Painlevé equation (5.16) with parameters given by (5.17) such that the function*

$$r(z) = w' - (\epsilon\alpha - 1)\frac{w}{z} - \epsilon\gamma w^2 - 1 \neq 0,$$

does not vanish. Then the function,

$$S(\tau) = 1 - 2r^{-1}\left(\sqrt{2\tau}\right) \qquad (5.18)$$

is a solution of the third Painlevé equation (5.12) with parameters $\gamma \neq 0$ and $\delta = -2$.

The τ-functions for the rational class of solutions of the Painlevé equation P3 can be constructed [23, 24] in terms of the Umemura polynomials, $T_n = T_n(z,l)$, which are determined by a sequence of polynomials in z and defined through the recurrence relation

$$T_{n+1}T_{n-1} = \left(\frac{z}{8} - l + \frac{3}{4}n\right)T_n^2 + \frac{\partial T_n}{\partial z}T_n + z\left[\frac{\partial^2 T_n}{\partial z^2}T_n - \left(\frac{\partial T_n}{\partial z}\right)^2\right], \qquad (5.19)$$

with initial conditions $T_0 = T_1 = 1$. Based on reference [23] we have the following result:

Proposition 7. *For the existence of rational solutions of equation P3 of the form*

$$w(z) = \frac{T_{n+1}(z,l-1)T_n(z,l)}{T_{n+1}(z,l)T_n(z,l-1)}, \qquad (5.20)$$

where the Umemura polynomials, $T_n = T_n(z,l)$, satisfy the recurrence relation (5.19), it is necessary and sufficient that the parameters of equation P5 satisfy

$$\alpha = 4(n+l), \qquad \beta = 4(n-l), \qquad \gamma = -\delta = 4.$$

Note that from system (5.13), and the transformation (5.15), there is a connection between solutions w of the third Painlevé equation (5.12) and the solutions y of the fifth Painlevé equation (5.16),

$$y = \frac{zw' - [(4\epsilon(n+l)-1) + 4\epsilon zw]w + 2z}{zw' - [(4\epsilon(n+l)-1) + 4\epsilon zw]w - 2z}. \qquad (5.21)$$

Substituting the rational solutions (5.20) into formula (5.21) and next replacing the w which appears in (5.3) by the function u so obtained, we get multivortex solutions of equations (2.6). Consequently, by applying Proposition 3 to the multivortex solution of (2.6) obtained, we can find certain classes of solutions of system (2.1).

Another class of vortex solutions to CSG equations (2.5) can be provided if we define functions u and v in the polar form as

$$u = A_n(r)e^{in\theta}, \qquad v = A_{n-1}(r)e^{i(n-1)\theta}, \qquad n \in \mathbb{Z}, \tag{5.22}$$

which transforms the CSG system (2.5) into

$$\begin{aligned} &(i) \quad \frac{dA_n}{dr} + \frac{n}{r}A_n = \left(1 - A_n^2\right) A_{n-1}, \\ &(ii) \quad -\frac{dA_{n-1}}{dr} + \frac{(n-1)}{r}A_{n-1} = \left(1 - A_{n-1}^2\right) A_n. \end{aligned} \tag{5.23}$$

When $n = 1$, the second equation (5.23) is solved by taking $A_0 = \pm 1$ and then the first equation (5.23) becomes a Riccati equation which can be linearized by a Cole–Hopf transformation and solved in terms of Bessel functions. The vortex solution (5.22) takes the form

$$u = \frac{I_1(r)}{I_0(r)}e^{i\theta}, \qquad v = \epsilon = \pm 1, \tag{5.24}$$

where I_1 is the Bessel function of the first order, that is $I_1 = I_0'(r)$, and the prime denotes differentiation with respect to r. Such a reduction of (5.23) has been recently obtained [7]. Consequently, from transformation (2.4), we get

$$\psi_1 = \frac{I_1(r)}{I_0(r)}e^{i\theta}\bar{\psi}_2, \qquad \varphi_1 = \epsilon\bar{\varphi}_2. \tag{5.25}$$

Substituting (5.25) into Weierstrass system (2.1) and solving the resulting equations, we obtain

$$\varphi_2 = F\left(re^{-i\theta}\right), \tag{5.26}$$

where F is an arbitrary function of one variable $re^{-i\theta}$ and the function ψ_2 satisfies the PDE

$$\frac{\partial\psi_2}{\partial r} + \frac{i}{r}\frac{\partial\psi_2}{\partial\theta} = 2e^{-i\theta}R(r)|\psi_2|^2\psi_2, \qquad R(r) = 1 - \frac{I_0'^2(r)}{I_0^2(r)}. \tag{5.27}$$

Equation (5.27) has a solution of the form

$$\psi_2 = g\left(v(r)e^{-i\theta}\right), \tag{5.28}$$

where the functions g and v satisfy the following ODEs,

$$v' + \frac{v}{r} - R(r)\lambda = 0, \qquad \dot{g}\lambda + |g|^2 g = 0, \qquad \lambda \in \mathbb{C},$$

and $\dot{g}$ denotes the derivative of g with respect to $s = v(r)e^{-i\theta}$. These two equations can be integrated to give the following expressions

$$v(r) = \frac{\lambda}{r}\int_0^r \tau R(\tau)d\tau, \qquad \frac{g^\lambda}{\bar{g}^{\bar{\lambda}}} = c. \tag{5.29}$$

From equations (5.25), (5.26) and (5.28) we can summarize the results as follows,

$$\psi_1 = \frac{I_0'(r)}{I_0(r)} e^{i\theta} \bar{g} \left(\frac{\bar{\lambda} e^{i\theta}}{r} \int_0^r \tau R(\tau) d\tau \right), \qquad \psi_2 = g \left(\frac{\lambda e^{-i\theta}}{r} \int_0^r \tau R(\tau) d\tau \right),$$
$$\varphi_1 = \epsilon F \left(r e^{-i\theta} \right), \qquad \varphi_2 = F \left(r e^{-i\theta} \right), \tag{5.30}$$

where g is a function of one variable, which is restricted by relation (5.29). Note that the solution for the function u in (5.24) has the form of a scalar field which has appeared in the study of the vortex solutions of superconductivity with asymptotic behavior of the radial part of the solution going to zero as r goes to zero and to constant as r goes to infinity. Consequently solutions (5.30) of the Weierstrass system (2.1) possess similar asymptotic behavior when the functions F and g are bounded.

6 Examples of solving the Weierstrass system via the complex sine-Gordon equation

We now investigate the possibility of generating new multisoliton solutions by taking products of known solutions of the CSG system (2.6). Thus we can formulate the following:

Proposition 8. *Suppose u is a solution of equation (2.6) with constant modulus $|u|^2 = |c|^2 \neq 1$. Suppose also that a complex function w exists which satisfies $|w|^2 = 1$ and the differential constraint equation*

$$u \left(\partial \bar{\partial} w \mp \frac{\bar{w} |c|^2}{1 \pm |c|^2} (\partial w)(\bar{\partial} w) \right) + \frac{1}{1 \pm |c|^2} (\bar{\partial} u \partial w + \partial u \bar{\partial} w) = 0. \tag{6.1}$$

Then the product function $U = uw$ is a solution of system (2.6).

Proof. Differentiating the function $U = uw$, we get the following expressions

$$\bar{\partial}(uw) = (\bar{\partial} u) w + u (\bar{\partial} w), \qquad \partial(uw) = (\partial u) w + (\partial w) u,$$
$$\partial \bar{\partial}(uw) = (\partial \bar{\partial} u) w + (\bar{\partial} u)(\partial w) + (\partial u)(\bar{\partial} w) + u (\partial \bar{\partial} w).$$

Substituting U into equation (2.6), using $|u|^2 = |c|^2$ and $|w|^2 = 1$, we obtain that

$$\begin{aligned} &(\partial \bar{\partial} u) w + (\bar{\partial} u)(\partial w) + (\partial u)(\bar{\partial} w) + u (\partial \bar{\partial} w) \\ &\quad \mp \frac{\bar{u} \bar{w}}{1 \pm |c|^2} ((\bar{\partial} u) w + u (\bar{\partial} w))((\partial u) w + u (\partial w)) + \frac{uw}{4} \left(1 \pm |c|^2 \right) \\ &\quad = (\partial \bar{\partial} u) w + (\bar{\partial} u)(\partial w) + (\partial u)(\bar{\partial} w) + u (\partial \bar{\partial} w) \mp \frac{\bar{u} w}{1 \pm |c|^2} (\partial u)(\bar{\partial} u) \\ &\quad \mp \frac{|c|^2}{1 \pm |c|^2} (\partial u)(\bar{\partial} w) \mp \frac{|c|^2}{1 \pm |c|^2} (\bar{\partial} u)(\partial w) \mp \frac{|c|^2 u \bar{w}}{1 \pm |c|^2} (\partial w)(\bar{\partial} w) + \frac{uw}{4} \left(1 \pm |c|^2 \right). \end{aligned} \tag{6.2}$$

Substituting the second derivative $\partial \bar{\partial} u$ from equation (2.6) into equation (6.2) and next collecting terms with respect to first derivatives of u and w and simplifying, we obtain the differential constraint (6.1). ■

Note that in the case of the CShG equation (2.5) the constant c need not necessarily have modulus different from one. So there is no singularity in the $\left(1+|c|^2\right)^{-1}$ term in equation (6.2).

At this point we illustrate Proposition 8 for constructing a solution to system (2.6) with an elementary example. The simplest solution of analytic type, a vacuum solution, is given by

$$u = ce^{(\bar{A}z - A\bar{z})}, \tag{6.3}$$

where c and A are complex constants. By substituting the function u in (6.3) into (2.6), we easily show that this is a solution provided that the following constraint holds between constants c and A,

$$2\epsilon|A| = 1 \pm |c|^2, \qquad \epsilon = \pm 1.$$

Suppose that f is a complex-valued function of one complex variable z and define the function w as follows,

$$w = \frac{f(z)}{\bar{f}(\bar{z})}, \qquad |w|^2 = 1,$$

such that $f(z)$ satisfies the constraint (6.1), namely,

$$\partial f(z)\bar{\partial}\bar{f}(\bar{z}) + A\partial f(z)\bar{f}(\bar{z}) + \bar{A}f(z)\bar{\partial}\bar{f}(\bar{z}) = 0. \tag{6.4}$$

Then Proposition 8 implies that the function,

$$U = ce^{(\bar{A}z - A\bar{z})}\frac{f(z)}{\bar{f}(\bar{z})} \tag{6.5}$$

is also a solution to system (2.6) and represents a one-soliton solution.

We introduce a new dependent variable

$$y = \frac{\partial f}{f}, \qquad \bar{y} = \frac{\bar{\partial}\bar{f}}{\bar{f}}. \tag{6.6}$$

Then (6.4) takes the form

$$y\bar{y} + Ay + \bar{A}\bar{y} = 0. \tag{6.7}$$

We write y and A in terms of real and imaginary parts, namely $y = a(z,\bar{z}) + ib(z,\bar{z})$ and $A = A_r + iA_i$. Substituting them into (6.7), we obtain

$$a^2 + b^2 + 2A_r a - 2A_i b = 0.$$

Note that the above expression is quadratic in $a(z,\bar{z})$ and $b(z,\bar{z})$. So we can solve this expression for the imaginary part $b(z,\bar{z})$ in terms of $a(z,\bar{z})$ to give

$$b = A_i + \epsilon\left(A_i^2 - a(z,\bar{z})^2 - 2A_r a(z,\bar{z})\right)^{1/2}. \tag{6.8}$$

To ensure that (6.8) gives a real-valued $b(z,\bar{z})$, we require that $a(z,\bar{z})$ satisfy the inequality

$$A_i^2 > a(z,\bar{z})^2 + 2A_r a(z,\bar{z}).$$

Thus we obtain two possible solutions for y

$$y = a + i\left(A_i + \epsilon\left(A_i^2 - a(z,\bar{z})^2 - 2A_r a(z,\bar{z})\right)^{1/2}\right), \qquad \epsilon = \pm 1.$$

Substituting y into (6.6), we can integrate (6.6) to obtain

$$\ln f = \int \left(a + i\left(A_i + \epsilon\left(A_i^2 - a^2 - 2A_r a\right)^{1/2}\right)\right) dz + \bar{q}(\bar{z}). \tag{6.9}$$

Here $\bar{q}(\bar{z})$ is an arbitrary function of $\bar{z}$. Now from (6.5) and (6.9), we obtain

$$u = ce^{\bar{A}z - A\bar{z}} \frac{\exp\left(\bar{q}(\bar{z}) + \int \left(a + i\left(A_i + \epsilon\left(A_i^2 - a^2 - 2A_r a\right)^{1/2}\right)\right) dz\right)}{\exp\left(q(z) + \int \left(a - i\left(A_i + \epsilon\left(A_i^2 - a^2 - 2A_r a\right)^{1/2}\right)\right) d\bar{z}\right)} \tag{6.10}$$

Using (2.5) we obtain the explicit form for the expression for v,

$$\begin{aligned} v = {} & \frac{2c}{1 \pm |c|^2} e^{\bar{A}z - A\bar{z}} \frac{f(z)}{\bar{f}(\bar{z})} \left[\bar{A} + a + i\left(A_i + \epsilon\left(A_i^2 - a^2 - 2A_r a\right)^{1/2}\right)\right. \\ & \left. - \frac{1}{\bar{f}(\bar{z})}\left(\partial q + \left(a - i\left(A_i + \epsilon\left(A_i^2 - a^2 - 2A_r a\right)^{1/2}\right)\right)\right)\right]. \end{aligned} \tag{6.11}$$

Once functions u and v are found, equations (2.1) and (2.4) allow us to determine the functions ψ_i and φ_i. Consider the exponential solution (6.3) for $A = -ia$. Elimination of u from the pair of equations in (2.5) results in an equation which is identical to (2.6) but with u replaced by v. We can assign the solution obtained from the second order equation (2.6) to either the u or in the v variable. We take for example $u = ce^{ia(z+\bar{z})}$. The second function v is obtained from (2.5), that is

$$u = ce^{ia(z+\bar{z})}, \qquad v = \frac{2\partial u}{1 \pm |u|^2} = i\epsilon c e^{ia(z+\bar{z})}. \tag{6.12}$$

Functions u and v satisfy (2.5) provided that $2a\epsilon = 1 \pm |c|^2$. From (2.4) we can write $\psi_1 = u\bar{\psi}_2$ and $\varphi_1 = v\bar{\varphi}_2$, and calculate the quantities Q_1 and Q_2 from (2.1)

$$Q_1 = |\psi_2|^2\left(1 \pm |c|^2\right), \qquad Q_2 = |\varphi_2|^2\left(1 \pm |c|^2\right). \tag{6.13}$$

We now show that we can find an explicit class of solutions which satisfy (2.1). Eliminating u and v from equations (2.1) and (6.13), we get

$$\bar{\partial}\psi_2 = |\psi_2|^2\left(1 \pm |c|^2\right)\psi_2. \tag{6.14}$$

From equation (6.14) the function ψ_2 obeys $\psi_2 = e^{b(z-\bar{z})}$ provided that $b = -2a\epsilon$ be consistent with the condition $2\epsilon a = 1 \pm |c|^2$. Using equations (2.1) and (6.13), we obtain a solution of the form

$$\varphi_2 = e^{2a\epsilon(z-\bar{z})}.$$

In this case equations (2.1) can be integrated and give

$$\begin{aligned}
&\psi_1 = ce^{ia\epsilon(z+\bar{z})}e^{2a\epsilon(z-\bar{z})}, \qquad \psi_2 = e^{-2a\epsilon(z-\bar{z})},\\
&\varphi_1 = ice^{ia\epsilon(z+\bar{z})}e^{-2a\epsilon(z-\bar{z})}, \quad \varphi_2 = e^{2a\epsilon(z-\bar{z})}.
\end{aligned} \tag{6.15}$$

Substituting (6.15) into (2.3) and integrating with respect to z and $\bar{z}$, we obtain the parametric form of a surface

$$\begin{aligned}
&X^1 = -a\epsilon(z-\bar{z})^2 + 2a\epsilon|z|^2 = a\epsilon\left(y^2+2r^2\right),\\
&X^2 = a\epsilon(z-\bar{z})^2 - 2a\epsilon|z|^2 = -a\epsilon\left(y^2+2r^2\right),\\
&X^3 = 0, \qquad X^4 = i|c|^2(z-\bar{z}) = -|c|^2 y,
\end{aligned} \tag{6.16}$$

where we set $z-\bar{z} = iy$ and $|z|^2 = r^2$. Treating y and $r > 0$ as parameters, one can plot X^1, X^2 and X^4 to get the surface, which has the form of a parabolic cylinder. Moreover, from (6.16), we can calculate the components of the induced metric

$$\begin{aligned}
&g_{zz} = \sum_{i=1}^{4}(X_z^i)^2 = 8a^2\left(z^2 - 4|z|^2 + 4\bar{z}^2\right) - |c|^4 = \bar{g}_{\bar{z}\bar{z}},\\
&g_{z\bar{z}} = \sum_{i=1}^{4} X_z^i X_{\bar{z}}^i = 8a^2(2z+\bar{z})(z-2\bar{z}) + |c|^4.
\end{aligned}$$

A procedure for obtaining solutions to (2.1) as a result of using Proposition 8 can be developed from the above example. The new product solution can be called either u or v. One substitutes this in the corresponding equation in (2.5) to obtain the remaining unknown solution v or u. Using these results in (2.4) to eliminate functions $(\bar{\psi}_2, \bar{\varphi}_2)$, one tries to integrate the nonlinear system (2.1) to obtain the required complex functions ψ_1 and φ_1. The solutions of the Weierstrass system (2.1) obtained in this way are used to construct the surface by means of (2.3).

7 Final remarks

Equations (2.1) and (2.5) under investigation here have long been of interest in field theory. In particular there has been the extensive use of applying soliton solutions to construct models of extended particles [6, 25]. The sine-Gordon equation, it seems, is the only Lorentz-invariant, nonlinear equation whose initial value problem has been solved [4]. This equation also describes a completely integrable Hamiltonian system. It would certainly be of great interest to find other Lorentz-invariant integrable systems. Of more recent interest is the study of vortex tubes [26]. The motion of vortex tubes in an inviscid incompressible fluid is described by the Biot–Savart law. The recently proposed localized induction equation is the simplest model to capture the leading order behavior of the three-dimensional self-induced motion of a vortex filament. This type of equation is in fact related to the cubic nonlinear Schrödinger equation for a complex variable, and implies that the localized induction equation is completely integrable. Note that from solutions (5.30), when the functions F and g are real polynomials in a single variable, the vortex structure of the solutions is preserved at the level of the functions ψ_α and φ_β. These functions are applied to generate surfaces. Consequently, by plotting such results in two or three dimensions, these surfaces could model a vortex filament in such a fluid [25, 26].

We have presented a new approach to the study of the Weierstrass system (2.1) in connection with CShG and CSG equations (2.6). It proved to be particularly effective in constructing multivortex solutions of (2.1) in terms of τ-functions based on rational solutions of the third and fifth Painlevé equations. It is worth noting that the approach to the Weierstrass system (2.1) proposed here can be applied, with some necessary modifications, to more general cases of Weierstrass type systems describing more diverse surfaces immersed in multi-dimensional Minkowski and pseudo-Riemannian spaces. The task of obtaining new types of minimal surfaces described by system (2.1) will be undertaken in our future work.

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References

[1] Pohlmeyer K, Integrable Hamiltonian systems and interactions through quadratic constraints, *Commun. Math. Phys.* **46** (1976), 207–221;
Neveu A and Papanicolaou N, Integrability of the Classical $[\bar{\psi}_i\psi_i]^2_2$ and $[\bar{\psi}_i\psi_i]^2_2 - [\bar{\psi}_i\gamma_5\psi_i]^2_2$ Interactions, *Commun. Math. Phys.* **58** (1978), 31–64.

[2] Bobenko A I and Eitner U, Painlevé Equations in the Differential Geometry of Surfaces, Lecture Notes in Mathematics, Vol. 1753, Springer Verlag, New York, 2000.

[3] Makhankov V G and Pashaev O K, Integrable Pseudospin Models in Condensed Matter, *Sov. Sci. Rev. Math. Phys.* **9** (1992), 1–152.

[4] Lund F and Regge T, Unified Approach to String and Vortices with Soliton Solutions, *Phys. Rev.* **D14** (1976), 1524–1535;
Lund F, Example of a Relativistic, Completely Integrable, Hamiltonian System, *Phys. Rev. Lett.* **38** (1977), 1175–1178;
Lund F, Classically Solvable Field Theory Model, *Ann. Phys.* **115** (1978), 251–268.

[5] Getmanov B S, A New Lorentz-Invariant System with Exact Multisoliton Solutions, *Sov. Phys., JETP Lett.* **25** (1977), 132–136.

[6] Zakrzewski W, Low Dimensional Sigma-Models, Hilger, New York, 1989.

[7] Kosevich V L and Pitaevskii L P, On a Unified Approach to Ginsburg-Landau Type Models, *Sov. Phys. JETP* **34** (1958), 858–869;
Lifshitz E M and Pitaevskii L P, Statistical Physics, Part 2, Pergamon Press, Oxford, 1980;
Barashenkov I V and Pelinovsky D E, Exact Vortex Solutions of the Complex Sine-Gordon Theory on the Plane, *Phys. Lett.* **436** (1998), 117–124;
Ginzberg V I and Pitaevskii L P, On the Theory of Superfluidity, *Sov. Phys. JETP* **7** (1958), 585-590.

[8] Nelson D, Piran T and Weinberg S, Statistical Mechanics of Membranes and Surfaces, World Scientific, Singapore, 1992.

[9] Ou-Yang Zhong-Can, Liu Ji-Xing and Xie Yu-Zhang, Geometric Methods in the Elastic Theory of Membranes in Liquid Crystal Phases, World Scientific, Singapore, 1999.

[10] Darboux G, Lecons sur la théorie générale des surfaces, Vol. 3, Lignes géodésiques et courbure géodésique paramètres différentiels-deformation des surfaces, Ed. Gauthier-Villars, Paris, 1894.

[11] Konopelchenko B and Landolfi G, Generalized Weierstrass Representation for Surfaces in Multi-Dimensional Riemann Spaces, *J. Geom. Phys.* **29** (1999), 319–333.

[12] Tafel J, Two-Dimensional Reductions of the Self-Dual Yang–Mills Equations in Self-Dual Spaces, *J. Math. Phys.* **34** (1993), 1892–1907.

[13] Barashenkov I and Getmanov B, The Unified Approach to Integrable Relativistic Equations: Soliton Solutions over Nonvanishing Backgrounds, *J. Math. Phys.* **34** (1993), 3054–3072.

[14] Ablovitz M J, Ramani A and Segur H, A Connection between Nonlinear Evolution Equations and Ordinary Differential Equations of P-Type. I., *J. Math. Phys.* **21** (1980), 715–721.

[15] Weiss J, Tabor M and Carnevale G, The Painlevé Property for Partial Differential Equations, *J. Math. Phys.* **24** (1983), 522–526;
Weiss J, The Sine-Gordon Equations: Complete and Partial Integrability, *J. Math. Phys.* **25** (1984), 2226–2235.

[16] Musette M, Painlevé Analysis for Nonlinear Partial Differential Equations, in The Painlevé Property One Century Later, Editor R Conte, Springer Verlag, New York, 1999, Chapter 8, 517–572.

[17] Conte R, The Painlevé Approach to Nonlinear Ordinary Differential Equations, in The Painlevé Property One Century Later, Editor R Conte, Springer Verlag, New York, 1999, Chapter 3, 77–180.

[18] Weiss M A, Dissertation, Flots isospectraux et moments dans des algèbras de lacets appliqués à des équations différentielles nonlinéaires, CRN Preprint, Université de Montréal, 1993.

[19] Bruschi M and Ragnisco O, Nonlinear Evolution Equations Associated with the Chiral-Field Spectral Problem, *Nuovo Cimento (11)*, **B88** (1985), 119–139;
Sym A, Ragnisco O, Levy D and Bruschi M, Soliton Surfaces. VII. Relativistic String in External Field: General Integral and Particular Solutions, *Lett. Nuovo Cimento* **44** (1985), 529–536.

[20] Pickering A, A New Truncation in Painlevé Analysis, *J. Phys. A: Math. Gen.* **26** (1993), 4395–4405.

[21] Pickering A, The Singular Manifold Method Revisited, *J. Math. Phys.* **37** (1996), 1894–1927.

[22] Gromak V, Bäcklund Transformations of Painlevé Equations and Their Applications, in The Painlevé Property One Century Later, Editor R Conte, Springer Verlag, New York, 1999, Chapter 12, 687–734.

[23] Kajiwara K and Masuda T, On the Umemura Polynomials for the Painlevé III Equation, *Phys. Letts.* **A260** (1999), 462–467.

[24] Okamoto K, The Hamiltonians Associated to the Painlevé Equations, in The Painlevé Property One Century Later, Editor R Conte, Springer Verlag, New York, 1999, Chapter 13, 735–789.

[25] Rajamaran R, Solitons and Instantons, North-Holland, Amsterdam, 1982.

[26] Fukumoto Y and Miyajima M, The Localized Induction Hierarchy and the Lund–Regge Equation, *J. Phys. A: Math. Gen.* **29** (1996), 8025–8034.