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Dirac Reduction Revisited

Krzysztof MARCINIAK † *and Maciej BLASZAK* ‡

† *Department of Science and Technology, Campus Norrköping, Linköping University, 601-74 Norrköping, Sweden*
E-mail: krzma@itn.liu.se

‡ *Institute of Physics, A. Mickiewicz University, Umultowska 85, 61-614 Poznań, Poland*
E-mail: blaszakm@amu.edu.pl

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Abstract

The procedure of Dirac reduction of Poisson operators on submanifolds is discussed within a particularly useful special realization of the general Marsden-Ratiu reduction procedure. The Dirac classification of constraints on 'first-class' constraints and 'second-class' constraints is reexamined.

1 Introduction

Dirac bracket as well as Dirac's classification of constraints is nowadays a well recognized and very useful tool in the construction of Poisson dynamics on admissible submanifolds from a given Poisson dynamics on a given manifold. In this paper we consider the Dirac reduction procedure in a more general setting than is usually met in literature. In Section 2 we implement the Dirac reduction procedure into a particularly useful special realization of the general Marsden–Ratiu reduction scheme, based on the concept of transversal distributions. In Section 3 we reconsider the Dirac concept of first class constraints as it seems to be too restrictive.

Firstly we recall few basic notions from Poisson geometry. Given a manifold $\mathcal{M}$, a *Poisson operator* π on $\mathcal{M}$ is a mapping $\pi : T^*\mathcal{M} \to T\mathcal{M}$ that is fibre-preserving (i.e. $\pi|_{T_x^*\mathcal{M}} : T_x^*\mathcal{M} \to T_x\mathcal{M}$ for any $x \in \mathcal{M}$) and such that the induced bracket on the space $C^\infty(\mathcal{M})$ of all smooth real-valued functions on $\mathcal{M}$

$$\{\cdot,\cdot\}_\pi : \ C^\infty(\mathcal{M}) \times C^\infty(\mathcal{M}) \to C^\infty(\mathcal{M}), \qquad \{F,G\}_\pi \stackrel{\text{def}}{=} \langle dF, \pi\, dG \rangle\,, \tag{1.1}$$

where $\langle\cdot,\cdot\rangle$ is the dual map between $T\mathcal{M}$ and $T^*\mathcal{M}$, is skew-symmetric and satisfies Jacobi identity (the bracket (1.1) always satisfies the Leibniz rule $\{F,GH\}_\pi = G\{F,H\}_\pi + H\{F,G\}_\pi$). The symbol d denotes the operator of exterior differentiation. The operator

π can always be interpreted as a bivector, $\pi \in \Lambda^2(\mathcal{M})$ and in a given coordinate system $(x^1, \ldots, x^m)$ on $\mathcal{M}$ we have

$$\pi = \sum_{i<j}^{m} \pi^{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}.$$

A function $C : \mathcal{M} \to \mathbb{R}$ is called a *Casimir function* of the Poisson operator π if for an arbitrary function $F : \mathcal{M} \to \mathbb{R}$ we have $\{F, C\}_\pi = 0$ or, equivalently, if $\pi dC = 0$.

2 Marsden–Ratiu reduction for transversal distributions

The Marsden–Ratiu reduction theorem [1] describes the procedure of reducing a Poisson operator π on arbitrary submanifold $\mathcal{S}$ of our manifold $\mathcal{M}$. This general procedure exists only if some conditions are satisfied. These conditions involve a distribution E (in the original notation of Marsden and Ratiu) that is a subbundle of $T\mathcal{M}$. By a simple assumption, namely that this distribution is transversal, one can, however, satisfy all these conditions automatically. Below we reformulate the Marsden–Ratiu theorem in this more limited but useful setting.

Consider an m-dimensional manifold $\mathcal{M}$ equipped with a Poisson operator π and an s-dimensional submanifold $\mathcal{S}$ of $\mathcal{M}$. Fix a distribution $\mathcal{Z}$ of constant dimension $k = m - s$, that is a smooth collection of m-dimensional subspaces $\mathcal{Z}_x \subset T_x\mathcal{M}$ at every point x in $\mathcal{M}$, which is transversal to S in the sense that no vector field $Z \in \mathcal{Z}$ is at any point tangent to the submanifold $\mathcal{S}$. Hence we have

$$T_x\mathcal{M} = T_x\mathcal{S} \oplus \mathcal{Z}_x$$

for every $x \in \mathcal{S}$ and, similarly,

$$T_x^*\mathcal{M} = T_x^*\mathcal{S} \oplus \mathcal{Z}_x^*,$$

where $T_x^*\mathcal{S}$ is the annihilator of $\mathcal{Z}_x$ and $\mathcal{Z}_x^*$ is the annihilator of $T_x\mathcal{S}$. That means that if α is a one form in $T_x^*\mathcal{S}$ then $\alpha(Z) = 0$ for all vectors $Z \in \mathcal{Z}_x$ and if β is a one-form in $\mathcal{Z}_x^*$ then β vanishes on all vectors in $T\mathcal{S}_x$.

Definition 1. A function $F : \mathcal{M} \to \mathbb{R}$ is invariant with respect to $\mathcal{Z}$ if $L_Z F = Z(F) = 0$ for any $Z \in \mathcal{Z}$.

We observe that for any function $f : \mathcal{S} \to \mathbb{R}$ there exists a unique $\mathcal{Z}$-invariant prolongation $F : \mathcal{M} \to \mathbb{R}$, (so that $F|_\mathcal{S} = f$). Here and in what follows the symbol L_Z means the Lie derivative along the vector field Z.

Definition 2. The operator π is called invariant with respect to the distribution $\mathcal{Z}$ if the functions that are invariant along $\mathcal{Z}$ form a Poisson subalgebra, that is, if F, $G : \mathcal{M} \to \mathbb{R}$ are two functions invariant with respect to $\mathcal{Z}$, then $\{F, G\}_\pi$ is again invariant with respect to $\mathcal{Z}$.

We denote this Poisson subalgebra by $\mathcal{A}$.

Theorem 1 (Marsden and Ratiu [1]). *Let $\mathcal{S}$ be a submanifold of $\mathcal{M}$ equipped with a Poisson operator π and let $\mathcal{Z}$ be a distribution in $\mathcal{M}$ that is transversal to $\mathcal{S}$. If the operator π is invariant with respect to the distribution $\mathcal{Z}$, then the Poisson operator π is reducible on S in the sense that on S there exists a (uniquely defined) Poisson operator π_R such that for any $f,g : S \to \mathbb{R}$ we have*

$$\{f,g\}_{\pi_R} = \{F,G\}_{\pi}|_S \tag{2.1}$$

for the $\mathcal{Z}$-invariant prolongations F and G of f and g respectively.

The proof of this theorem is obvious. Since π is invariant with respect to $\mathcal{Z}$, $\{F,G\}_\pi$ is also invariant along $\mathcal{Z}$ and can thus be considered as a $\mathcal{Z}$-invariant prolongation of a function on $\mathcal{S}$. Moreover, since π satisfies Jacobi identity, so does π_R (because $\pi_R = \pi|_{\mathcal{A}}$).

The above construction, however, is difficult to perform in practice since it is often impossible to find explicit expressions for the prolongations F and G. We now show how this difficulty can be omitted.

Firstly, suppose that our submanifold $\mathcal{S}$ is given by k functionally independent equations $\varphi_i(x) = 0$, $i = 1,\ldots,k$ (constraints) and that our transversal distribution $\mathcal{Z}$ is spanned by k vector fields Z_i chosen such that the following orthogonality relation holds

$$\langle d\varphi_i, Z_j\rangle = Z_j(\varphi_i) = \delta_{ij}, \tag{2.2}$$

(this is no restriction since for any distribution $\mathcal{Z}$ transversal to $\mathcal{S}$ we can choose its basis so that (2.2) is satisfied). We observe that in this case we have $[Z_i, Z_j]\,\varphi_k = 0$ for all k, where $[X,Y] = L_X Y = X(Y) - Y(X)$ is the Lie bracket (commutator) of the vector fields X, Y, so that $[Z_i, Z_j]$ is always tangent to $\mathcal{S}$. Then, in case that the distribution $\mathcal{Z}$ is involutive (integrable), this means that $[Z_i, Z_j] = 0$ for all i,j. Moreover, we define the vector fields X_i as

$$X_i = \pi d\varphi_i, \qquad i = 1,\ldots,k. \tag{2.3}$$

There exists an important class of $\mathcal{Z}$-invariant Poisson operators.

Lemma 1 ([2]). *If*

$$L_{Z_i}\pi = \sum_{j=1}^{k} W_j^{(i)} \wedge Z_j, \qquad i = 1,\ldots,k \tag{2.4}$$

for some vector fields $W_j^{(i)}$, then the Poisson operator π is invariant with respect to $\mathcal{Z}$.

We sketch the proof here for the clarity of the text.

Proof. Assume, that $L_{Z_i}F = L_{Z_i}G = 0$ for all i. We have to show that $L_{Z_i}\{F,G\}_\pi = 0$ for all i, but, due to (2.4)

$$L_{Z_i}\{F,G\}_\pi = L_{Z_i}\langle dF, \pi dG\rangle = \sum_{j=1}^{k}\langle dF, (W_j^{(i)} \wedge Z_j) dG\rangle$$

since $L_{Z_i}(dF) = d(L_{Z_i}F) = 0$ (and similarly for G). On the other hand

$$\langle dF, (W_j^{(i)} \wedge Z_j) dG\rangle = Z_j(G) W_j^{(i)}(F) - Z_j(F) W_j^{(i)}(G) = 0$$

since $Z_j(F) = L_{Z_j}F = 0$ (and similarly for G). ■

The condition (2.4) is sufficient but not necessary. For example, if

$$L_{Z_i}\pi = \sum_{j=1}^{k} W_j \wedge [Z_i, Z_j]\,, \qquad i = 1, \ldots, k$$

for some vector fields W_i, then the operator π is also $\mathcal{Z}$-invariant (one shows it by computations similar to those in the above proof).

In the case π satisfies (2.4) we apply the Lie derivative L_{Z_j} to both sides of the equation (2.3). Due to (2.4) we obtain

$$\begin{aligned}[][Z_j, X_i] = L_{Z_j}X_i = (L_{Z_j}\pi)d\varphi_i &= \left(\sum_l W_l^{(j)} \wedge Z_l\right) d\varphi_i \\ &= \sum_l \left(Z_l(\varphi_i)W_l^{(j)} - W_l^{(j)}(\varphi_i)Z_l\right) = W_i^{(j)} - \sum_l W_l^{(j)}(\varphi_i)Z_l. \end{aligned} \tag{2.5}$$

We observe that, if F and G are two $\mathcal{Z}$-invariant functions on $\mathcal{M}$ and V_j are arbitrary vector fields, then $\langle dF, \sum_j V_j \wedge Z_j\, dG\rangle = 0$ since $\langle dF, V_j \wedge Z_j\, dG\rangle = Z_j(G)V_j(F) - Z_j(F)V_j(G) = 0$. Thus the Poisson operator π and its *deformation* of the form

$$\pi_D = \pi - \sum_j V_j \wedge Z_j \tag{2.6}$$

both act in the same way on the Poisson subalgebra $\mathcal{A}$ so that both can be used to define our restricted operator π_R on $\mathcal{S}$ through (2.1). Of course, the deformed operator π_D does not have to be Poisson, but nevertheless its restriction to $\mathcal{S}$ through (2.1) must be Poisson since it naturally coincides with similar restriction of π to $\mathcal{S}$. It turns out that we can choose our (undetermined so far) vector fields V_j in (2.6) so that

$$\pi_D(\alpha_x) \in T_x\mathcal{S} \quad \text{for any} \quad \alpha_x \in T_x^*\mathcal{M} \tag{2.7}$$

which has a far reaching consequence.

Lemma 2. *The deformation π_D given by (2.6) that also satisfies (2.7) is Poisson.*

Proof. The condition that $\pi_D(\alpha_x)$ is tangent to $\mathcal{S}$ for any $\alpha_x \in T_x^*\mathcal{M}$ is equivalent to the requirement that $\langle d\varphi_i, \pi_D(\alpha_x)\rangle = 0$ for all i. Due to the antisymmetry of π_D this requirement can be rewritten as $\langle \alpha_x, \pi_D(\varphi_i)\rangle = 0$ for all i. Since α_x is arbitrary, the condition attains the form $\pi_D(d\varphi_i) = 0$ for $i = 1, \ldots, k$. We now complete the set of functions φ_i with some functions x_j to a coordinate system (x, φ) on $\mathcal{M}$. Then the matrix of the operator π_D has the last k rows and last k columns equal to zero while the $m - k$ dimensional upper left block coincides with π_R which is Poisson by the Marsden–Ratiu construction. ■

Lemma 3. *The condition (2.7) can be written as*

$$V_i - \sum_{j=1}^{k} V_j(\varphi_i)Z_j = X_i. \tag{2.8}$$

Proof. We know that the condition (2.7) can be written as $\pi_D(d\varphi_i) = 0$ for $i = 1, \ldots, k$. An easy calculation yields now that

$$0 = \pi_D(d\varphi_i) = \pi(d\varphi_i) - \sum_{j=1}^{k} (Z_j(\varphi_i)V_j - V_j(\varphi_i)Z_j) = X_i - V_i + \sum_{j=1}^{k} V_j(\varphi_i)Z_j$$

due to the normalization condition (2.2). ■

We now restrict ourselves to only two limit cases, when all X_i are tangent to $\mathcal{S}$ and when X_i span $\mathcal{Z}$.

2.1 The case when X_i are tangent to $\mathcal{S}$

We firstly assume that all the vectors X_i are tangent to $\mathcal{S}$ and that π satisfies (2.4) (to guarantee the invariance of π with respect to $\mathcal{Z}$). We have then naturally $X_i(\varphi_j) = 0$. This in turn means that $\{\varphi_i, \varphi_j\}_\pi = \langle d\varphi_i, \pi d\varphi_j\rangle = \langle d\varphi_i, X_j\rangle = 0$ so that all the vector fields X_i commute. In this case the simplest solution of (2.8) has the form $V_i = X_i$ and the corresponding deformation (2.6) attains the form

$$\pi_D = \pi - \sum_{i=1}^{k} X_i \wedge Z_i. \tag{2.9}$$

This deformation has been recently widely used for projecting Poisson pencils on symplectic leaves of one of their operators [3, 4, 5].

Lemma 4 ([3]). *The vector fields $W_j^{(k)}$ in (2.4) can, in the case that all X_i are tangent to $\mathcal{S}$, be chosen as tangent to $\mathcal{S}$.*

Proof. Consider the projections $\widetilde{W}_j^{(i)}$ of the vector fields $W_j^{(i)}$ onto $\mathcal{S}$:

$$\widetilde{W}_j^{(i)} = W_j^{(i)} - \sum_{r=1}^{k} W_j^{(i)}(\varphi_r)Z_r.$$

If $W_j^{(i)}$ are in $\mathcal{Z}$, then $\widetilde{W}_j^{(i)} = 0$. The vector field $\widetilde{W}_j^{(i)}$ is indeed tangent to $\mathcal{S}$ since

$$\widetilde{W}_j^{(i)}(\varphi_l) = W_j^{(i)}(\varphi_l) - \sum_{r=1}^{k} W_j^{(i)}(\varphi_r)\delta_{lr} = 0.$$

Now

$$\sum_{j=1}^{k} \widetilde{W}_j^{(i)} \wedge Z_j = \sum_{j=1}^{k} W_j^{(i)} \wedge Z_j - \sum_{j,r=1}^{k} W_j^{(i)}(\varphi_r)Z_r \wedge Z_j$$

the last term being equal to zero since $L_{Z_k}\{\varphi_i, \varphi_j\}_\pi = 0$ implies $W_j^{(i)}(\varphi_r) = W_r^{(i)}(\varphi_j)$. Thus $\sum_{j=1}^{k} W_j^{(i)} \wedge Z_j = \sum_{j=1}^{k} \widetilde{W}_j^{(i)} \wedge Z_j$. ■

Due to this gauge freedom, if we choose $W_j^{(i)}$ as tangent to $\mathcal{S}$ (which means that $W_j^{(i)}(\varphi_r) = 0$) then the formula (2.5) yields that $W_j^{(i)} = [Z_i, X_j]$. Thus, due to the fact that we assumed (2.4),

$$L_{Z_i}\pi = \sum_{j=1}^{k} [Z_i, X_j] \wedge Z_j. \tag{2.10}$$

Remark 1. In the case that the functions φ_i are Casimir functions of π we have $X_i = \pi d\varphi_i = 0$ so that the formula (2.10) yields $L_{Z_i}\pi = 0$ for all i, i.e. the vector fields Z_i are symmetries of π. In this case the Marsden–Ratiu reduction procedure (2.1) coincides with the standard restriction to a level set of Casimir functions (symplectic leaf in case there are no other Casimirs apart from φ_i) [6].

From what we have said above it becomes clear that the Marsden–Ratiu reduction scheme can be interpreted as a two-step procedure: firstly we deform the original Poisson tensor π to a Poisson tensor π_D and then we obtain π_R as standard restriction of π_D to the level set $\mathcal{S}$ of its Casimirs φ_i (thus we need not calculate the prolongations F and G in order to define $\{f, g\}_{\pi_R}$).

Now we check what can be said about our vector fields Z_i.

According to Remark 1 $L_{Z_i}\pi_D = 0$. On the other hand, due to (2.9),

$$0 = L_{Z_i}\pi_D = \sum_{j=1}^{k} [Z_i, X_j] \wedge Z_j - \sum_{j=1}^{k} L_{Z_i}X_j \wedge Z_j - \sum_{j=1}^{k} X_j \wedge L_{Z_i}Z_j$$

so that $\sum_{j=1}^{k} X_j \wedge [Z_i, Z_j] = 0$. Of course one of the possible realizations of this condition is the case that the distribution $\mathcal{Z}$ be integrable since then $[Z_i, Z_j] = 0$. There are, however, other possibilities here. For example, if $[Z_i, Z_j] = \sum_{s=1}^{k} c_{ij}^s X_s$ with $c_{ij}^s = c_{sj}^i$, $\sum_{j=1}^{k} X_j \wedge [Z_i, Z_j] = 0$ as well.

2.2 The case when X_i span $\mathcal{Z}$

This time we assume that $X_i = \sum_k \varphi_{ki} Z_k$ for some real valued functions φ_{ij}, which due to (2.2) yields

$$\varphi_{ij} = \sum_k \varphi_{kj} Z_k(\varphi_i) = X_j(\varphi_i) = \{\varphi_i, \varphi_j\}_\pi\,. \tag{2.11}$$

The functions φ_{ij} define a k-dimensional skew-symmetric matrix $\varphi = (\varphi_{ij})$, $i, j = 1, \ldots, k$. The only condition imposed on φ is related to the demand that X_i span $\mathcal{Z}$, i.e. $\det\varphi \neq 0$. We thus do not have to assume (2.4) this time since now the distribution $\mathcal{Z}$ is spanned by the Hamitlonian vector fields X_i and thus π is automatically invariant with respect to $\mathcal{Z}$ as $L_{X_i}\pi = 0$ for all i. It can be easily shown that

$$[X_j, X_i] = X_{\{\varphi_i,\varphi_j\}_\pi} = \pi\, d\{\varphi_i, \varphi_j\}_\pi = \pi\, d\varphi_{ij}.$$

Now we look for solutions of (2.8) in the simple form $V_i = \alpha X_i$. Inserting this into (2.8) and using the fact that $\varphi_{ij} = -\varphi_{ji}$ we obtain

$$0 = \alpha X_i - \alpha \sum_{j=1}^{k} X_j(\varphi_i) Z_j - X_i = \alpha X_i + \alpha \sum_{j=1}^{k} \varphi_{ji} Z_j - X_i = (2\alpha - 1) X_i$$

so that $a = 1/2$ and $V_i = \frac{1}{2} X_i$. In this case the deformation (2.6) attains the form:

$$\pi_D = \pi - \frac{1}{2} \sum_{i=1}^{k} X_i \wedge Z_i \tag{2.12}$$

and is, as mentioned above, Poisson. It is easy to check that our operator π_D defines the following bracket on $\mathcal{M}$

$$\{F, G\}_{\pi_D} = \{F, G\}_\pi - \sum_{i,j=1}^{k} \{F, \varphi_i\}_\pi (\varphi^{-1})_{ij} \{\varphi_j, G\}_\pi, \tag{2.13}$$

where $F, G : \mathcal{M} \to \mathbb{R}$ are now two *arbitrary* functions on $\mathcal{M}$, which is just the well known *Dirac deformation* [7] of the bracket $\{\cdot, \cdot\}_\pi$ associated with π.

Remark 2. If $C : \mathcal{M} \to \mathbb{R}$ is a Casimir function of π, then it is also a Casimir function of π_D since in this case (2.13) yields

$$\{F, C\}_{\pi_D} = \{F, C\}_\pi - \sum_{i,j=1}^{m} \{F, \varphi_i\}_\pi (\varphi^{-1})_{ij} \{\varphi_j, C\}_\pi = 0 - 0 = 0. \tag{2.14}$$

We also know that the constraints φ_i are Casimirs of the deformed operator π_D. Thus we can state that Dirac deformation preserves all the old Casimir functions and introduces new Casimirs φ_i.

It is now possible to restrict our Poisson operator π_D (or our Poisson bracket $\{\cdot, \cdot\}_{\pi_D}$) to a Poisson operator π_R (bracket $\{\cdot, \cdot\}_{\pi_R}$) on the submanifold $\mathcal{S}$, i.e. the level set $\varphi_1 = \cdots = \varphi_m = 0$ of Casimirs of π_D, in a standard way through the Marsden–Ratiu procedure (2.1), where now we can use arbitrary prolongations F and G of f and g. Again, the Dirac reduction, as a special case of the Marsden–Ratiu reduction scheme, has two steps: we firstly deform π to π_D and then restrict π_D to the level set $\mathcal{S}$.

3 Existence of Dirac reduction

We now present some realizations of the above Dirac case and discuss the classical concept of the Dirac classification of constraints. We will show that the classification of constraints as being either of first-class or of second-class, proposed by Dirac, should be reexamined when one looks at the problem from a more general point of view.

We recall that a constraint φ_k is of *first class* if its Poisson bracket with all the remaining constants φ_i vanishes on $\mathcal{S}$, that is if

$$\{\varphi_k, \varphi_i\}_\pi|_\mathcal{S} = 0, \qquad i = 1, \ldots, m. \tag{3.1}$$

Otherwise φ_k is of *second-class*. In the case that at least one of the constraints is of the first class, the matrix φ_{ij} in (2.11) is singular on $\mathcal{S}$ so that the formula (2.13) cannot be used in order to define π_R. However, it may still be possible to define π_R via the above general scheme. This indicates that the concept of first class constraint is too narrow. Below we demonstrate the examples of Dirac reduction in case when constraints *are* of first class.

We start with a simple example. Consider a $2n$-dimensional manifold $\mathcal{M}$ parametrized by coordinates $(q_1, \ldots, q_n, p_1, \ldots, p_n)$ and equipped with a Poisson operator of the form

$$\pi = \begin{bmatrix} 0 & Q_n \\ -Q_n & 0 \end{bmatrix},$$

where Q is a diagonal matrix of the form $Q_n = \operatorname{diag}(q_1, \ldots, q_n)$. Consider a submanifold $\mathcal{S}$ given by a pair of constraints $\varphi_1(q,p) \equiv q_n = 0$ and $\varphi_2(q,p) \equiv p_n = 0$. Then the matrix φ has the form

$$\varphi = \begin{bmatrix} 0 & q_n \\ -q_n & 0 \end{bmatrix}$$

so that it is clearly singular on $\mathcal{S}$ $(\det(S) = 0 \text{ on } \mathcal{S})$and

$$\varphi^{-1} = \frac{1}{q_n} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

so that the Dirac formula (2.13) cannot be applied. However, the vector fields $Z_1 = q_n^{-1} X_2$ and $Z_2 = -q_n^{-1} X_1$ that span our distribution $\mathcal{Z}$ are not singular on $\mathcal{S}$ since $X_1 = -q_n \partial/\partial p_n$ and $X_2 = q_n \partial/\partial q_n$ so that the deformation (2.12) becomes

$$\pi_D = \pi - \frac{1}{q_n} X_1 \wedge X_2 = \pi - q_n \frac{\partial}{\partial q_n} \wedge \frac{\partial}{\partial p_n} = \sum_{i=1}^{n-1} q_i \frac{\partial}{\partial q_i} \wedge \frac{\partial}{\partial p_i}$$

and can easily be restricted to $\mathcal{S}$. The operator π_R obtained on $\mathcal{S}$ parametrized by coordinates $(q_1, \ldots, q_{n-1}, p_1, \ldots, p_{n-1})$ is

$$\pi_R = \begin{bmatrix} 0 & Q_{n-1} \\ -Q_{n-1} & 0 \end{bmatrix}.$$

This simple example clearly illustrates that Dirac's classification is too strong. As a second example we consider a particle moving in a Riemannian manifold $\mathcal{Q}$ of dimension three with a contravariant metric tensor

$$G = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

given in some coordinates $\left(q^1, q^2, q^3\right)$. Suppose that this particle is subordinated to a holonomic constraint on $\mathcal{Q}$ given by

$$\varphi_1(q) \equiv q^1 q^2 + q^3 = 0. \tag{3.2}$$

This defines a submanifold of $\mathcal{Q}$. The velocity $v = \sum_{i=1}^{3} v^i \partial/\partial q^i$ of this particle must then remain tangent to this submanifold so that

$$0 = \langle d\varphi_k, v \rangle = \sum_{i=1}^{3} \frac{\partial \varphi_k}{\partial q^i} v^i.$$

and thus in our coordinates $v^i = \sum_j G^{ij} p_j$ the motion of the particle in the phase space $\mathcal{M} = T^*\mathcal{Q}$ is constrained not only by (3.2) but also by the relation

$$\varphi_2(q,p) \equiv \sum_{i,j=1}^{3} G^{ij} \frac{\partial \varphi_1(q)}{\partial q^i} p_j \equiv p_1 + p_2 q^1 + p_3 q^2 = 0 \tag{3.3}$$

that is nothing else than the lift of (3.2) to $\mathcal{M}$. The constraints (3.2)–(3.3) define a four-dimensional submanifold $\mathcal{S}$ of $\mathcal{M}$. We now introduce the following Poisson structure on $\mathcal{M}$:

$$\pi = \begin{bmatrix} 0 & 0 & 0 & q^1 & -1 & 0 \\ 0 & 0 & 0 & q^2 & 0 & -1 \\ 0 & 0 & 0 & 2q^3 & q^2 & q^1 \\ -q^1 & -q^2 & -2q^3 & 0 & p_2 & p_3 \\ 1 & 0 & -q^2 & -p_2 & 0 & 0 \\ 0 & 1 & -q^1 & -p_3 & 0 & 0 \end{bmatrix}.$$

Again the matrix φ is singular, since $\varphi_{12} = 2\left(q^1 q^2 + q^3\right) = 2\varphi_1$ which obviously vanishes on $\mathcal{S}$. One can, however, perform the deformation (2.12). A quite lengthy but straightforward computation shows that in this case

$$\pi_D = \begin{bmatrix} 0 & 0 & 0 & q^1 & -1 & 0 \\ 0 & 0 & 0 & q^2 & 0 & -1 \\ 0 & 0 & 0 & -2q^1 q^2 & q^2 & q^1 \\ -q^1 & -q^2 & 2q^1 q^2 & 0 & p_2 & p_3 \\ 1 & 0 & -q^2 & -p_2 & 0 & 0 \\ 0 & 1 & -q^1 & -p_3 & 0 & 0 \end{bmatrix}$$

and this operator can be restricted to $\mathcal{S}$. To do this, one can first pass to the Casimir variables

$$\left(q^1, q^2, \varphi_1(q), \varphi_2(q,p), p_2, p_3\right)$$

since, due to the fact that it is easiest to eliminate q^3 and p_1 from the system of equations $\varphi_1 = \varphi_1(q) = 0$, $\varphi_2 = \varphi_2(q,p) = 0$, we parametrize our submanifold by the coordinates $\left(q^1, q^2, p_2, p_3\right)$. In these variables the operator π_R attains the canonical form

$$\pi_R = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

Our two examples show that the condition (3.1) is only a necessary condition for nonexistence of π_R on $\mathcal{S}$, but is not a sufficient one. Hence the definition of first class constraints has to be made weaker. Even in the case when we deal with a real first class constraint we can obtain π_R on $\mathcal{S}$ coming from the Dirac reduction of π. We demonstrate this below.

Firstly we assume that we have a pair of constraints φ_1, φ_2 that define our submanifold $\mathcal{S} = \{\varphi_1 = 0, \varphi_2 = 0\}$ and such that they are of second class, i.e. that $\varphi_{12}|_{\mathcal{S}} = \{\varphi_1, \varphi_2\}|_{\mathcal{S}} \neq 0$. It is clear that our submanifold $\mathcal{S}$ can be parametrized in infinitely many different ways by constraints $\widetilde{\varphi}_1 = 0$, $\widetilde{\varphi}_2 = 0$, where

$$\widetilde{\varphi}_1 = \psi_1\varphi_1 + \psi_2\varphi_2, \qquad \widetilde{\varphi}_2 = \psi_3\varphi_1 + \psi_4\varphi_2 \tag{3.4}$$

and where ψ_i are some functions on $\mathcal{M}$ such that $\psi_i|_{\mathcal{S}} \neq 0$ and such that

$$D \equiv \left| \frac{D(\widetilde{\varphi}_1, \widetilde{\varphi}_2)}{D(\varphi_1, \varphi_2)} \right| = \psi_1\psi_4 - \psi_2\psi_3 \neq 0. \tag{3.5}$$

One can prove the following

Lemma 5. *The deformations (2.12) given by the pair φ_1, φ_2 of constraints and by the pair $\widetilde{\varphi}_1$, $\widetilde{\varphi}_2$ of constraints define the same reduced Poisson operator π_R on $\mathcal{S}$.*

Proof. For the moment we denote the deformation (2.12) defined through φ_1, φ_2 by π_D and the corresponding deformation defined through $\widetilde{\varphi}_1$, $\widetilde{\varphi}_2$ by $\widetilde{\pi}_D$. Applying (2.12) we easily get that for any two functions $A, B : \mathcal{M} \to R$

$$\{A, B\}_{\pi_D} = \{A, B\}_\pi + \frac{\{A, \varphi_2\}_\pi \{B, \varphi_1\}_\pi - \{A, \varphi_1\}_\pi \{B, \varphi_2\}_\pi}{\{\varphi_1, \varphi_2\}_\pi},$$

where we have assumed that $\{\varphi_1, \varphi_2\}_\pi$ does not vanish on $\mathcal{S}$. Similarly

$$\{A, B\}_{\widetilde{\pi}_D} = \{A, B\}_\pi + \frac{\{A, \widetilde{\varphi}_2\}_\pi \{B, \widetilde{\varphi}_1\}_\pi - \{A, \widetilde{\varphi}_1\}_\pi \{B, \widetilde{\varphi}_2\}_\pi}{\{\widetilde{\varphi}_1, \widetilde{\varphi}_2\}_\pi}, \tag{3.6}$$

where $\{\widetilde{\varphi}_1, \widetilde{\varphi}_2\}_\pi$ does not vanish on $\mathcal{S}$ due to (3.5). Using the relations (3.4) between the deformed constraints $\widetilde{\varphi}_i$ and the original constraints φ_i, the Leibniz property of Poisson brackets and the fact that the functions φ_i vanish on $\mathcal{S}$ we obtain

$$\{\widetilde{\varphi}_1, \widetilde{\varphi}_2\}_\pi \big|_{\mathcal{S}} = D \{\varphi_1, \varphi_2\}_\pi \big|_{\mathcal{S}}$$

and

$$\begin{aligned}(\{A, \widetilde{\varphi}_2\}_\pi \{B, \widetilde{\varphi}_1\}_\pi - \{A, \widetilde{\varphi}_1\}_\pi \{B, \widetilde{\varphi}_2\}_\pi)\big|_{\mathcal{S}} \\ = D(\{A, \varphi_2\}_\pi \{B, \varphi_1\}_\pi - \{A, \varphi_1\}_\pi \{B, \varphi_2\}_\pi)\big|_{\mathcal{S}}\end{aligned}$$

so that the nonzero terms D in the numerator and denominator of (3.6) cancel and we obtain $\{A, B\}_{\pi_D}\big|_{\mathcal{S}} = \{A, B\}_{\widetilde{\pi}_D}\big|_{\mathcal{S}}$ which implies that the projections of π_D and $\widetilde{\pi}_D$ onto $\mathcal{S}$ coincide. ∎

In this nonsingular case the distribution $\mathcal{Z}$ along which we project a Poisson tensor π usually changes after reparametrization, but $\mathcal{Z}|_{\mathcal{S}}$ remains the same as can be easily demonstrated. Thus in case of the second class constraints one has a "canonical" way of projecting π onto $\mathcal{S}$.

We now suppose that the constraints φ_i are of first class, that is $\{\varphi_1,\varphi_2\}_\pi|_{\mathcal{S}} = 0$ and that the singularity in π_D is not removable. We may still attempt to define the projection π_R by reparametrizing $\mathcal{S}$ as in (3.4) above. It turns out that among an infinite set of admissible reparametrizations there are some exceptional which, although they fulfil the condition (3.1), nevertheless eliminate the singularity in π_D. In this case, however, by choosing a new parametrization $\widetilde{\varphi}_1$, $\widetilde{\varphi}_2$ of $\mathcal{S}$ we change the distribution $\mathcal{Z}$ even on $\mathcal{S}$ so that we cannot expect that the projection π_R will be independent of the choice of the parametrization. We lose a natural, "canonical" choice of projection, but we still can perform the projection, although in infinitely many nonequivalent ways. We illustrate this below in a sequence of examples.

Consider a six-dimensional manifold $\mathcal{M}$ parametrized with coordinates $(q_1, q_2, q_3, p_1, p_2, p_3)$ with the following Poisson operator:

$$\pi = \begin{bmatrix} 0 & 0 & 0 & 1 & q_1 & 0 \\ 0 & 0 & 0 & q_1 & 2q_2+1 & q_3 \\ 0 & 0 & 0 & 0 & q_3 & 0 \\ -1 & -q_1 & 0 & 0 & -p_1 & 0 \\ -q_1 & -2q_2-1 & -q_3 & p_1 & 0 & p_3 \\ 0 & -q_3 & 0 & 0 & -p_3 & 0 \end{bmatrix}.$$

Consider now a four-dimensional submanifold $\mathcal{S}$ in $\mathcal{M}$ given by the relations

$$\varphi_1(q,p) = q_3 = 0, \qquad \varphi_2(q,p) = p_3 = 0. \tag{3.7}$$

It is clear that $\{\varphi_1,\varphi_2\}_\pi$ vanishes on the whole manifold $\mathcal{M}$ (and thus on $\mathcal{S}$) so that these constraints do not define any Dirac deformation at all. We now deform (3.7) as

$$\widetilde{\varphi}_1 = \varphi_1 + \varphi_2, \qquad \widetilde{\varphi}_2 = (-p_2 - q_1p_1)\varphi_1 + \varphi_2. \tag{3.8}$$

Calculation shows $\{\widetilde{\varphi}_1,\widetilde{\varphi}_2\}_\pi = (p_3 - q_3)q_3$ so that $\{\widetilde{\varphi}_1,\widetilde{\varphi}_2\}_\pi|_{\mathcal{S}} = 0$. One can show that after introducing the Casimir variables $(q_1, q_2, \widetilde{\varphi}_1, p_1, p_2, \widetilde{\varphi}_2)$ the deformed operator π_D attains the form

$$\pi_D = \begin{bmatrix} 0 & 2\dfrac{q_1q_3}{q_3-p_3} & 0 & 1 & -q_1 & 0 \\ -2\dfrac{q_1q_3}{q_3-p_3} & 0 & 0 & q_1 + 2\dfrac{p_1q_3}{q_3-p_3} & -q_1^2+\theta & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -q_1 - 2\dfrac{p_1q_3}{q_3-p_3} & 0 & 0 & -p_1 & 0 \\ q_1 & -q_1^2-\theta & 0 & p_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

where now $q_3 = q_3(q,p,\widetilde{\varphi})$ and $p_3 = p_3(q,p,\widetilde{\varphi})$ and $\theta = (q_3 + p_2q_3 + q_1q_3p_1)/(q_3 - p_3)$, and as such is clearly singular on $\mathcal{S}$ and thus unreducible. This situation seems to be the most common, i.e. a spontaneous choice of parametrization almost always leads to a singularity. However, if we perform a slightly different deformation of (3.7):

$$\widetilde{\varphi}_1 = \varphi_1, \qquad \widetilde{\varphi}_2 = (-p_2 - q_1 p_1)\varphi_1 + \varphi_2 \tag{3.9}$$

so that $\{\widetilde{\varphi}_1, \widetilde{\varphi}_2\}_\pi = -q_3^2$ is again zero on $\mathcal{S}$, then the operator π_D becomes nonsingular and its projection on $\mathcal{S}$ has the following form

$$\pi_R = \begin{bmatrix} 0 & 0 & 1 & -q_1 \\ 0 & 0 & q_1 & 1 - q_1^2 \\ -1 & -q_1 & 0 & p_1 \\ q_1 & q_1^2 - 1 & -p_1 & 0 \end{bmatrix}$$

in the variables (q_1, q_2, p_1, p_2). Yet another deformation (even this time of the form (3.4)):

$$\widetilde{\varphi}_1 = q_2\varphi_1, \qquad \widetilde{\varphi}_2 = (p_2 + \varphi_2)\varphi_1 \tag{3.10}$$

yields a quite complicated expression on $\{\widetilde{\varphi}_1, \widetilde{\varphi}_2\}_\pi$:

$$\{\widetilde{\varphi}_1, \widetilde{\varphi}_2\}_\pi = (3q_2 + 1)q_3^2 + q_3^3,$$

so that it again vanishes on $\mathcal{S}$, but π_D is again nonsingular and in the same variables (q_1, q_2, p_1, p_2) its projection becomes

$$\pi_R = \begin{bmatrix} 0 & 0 & 1 - \dfrac{q_1^2}{3q_2+1} & 0 \\ 0 & 0 & \dfrac{q_1 q_2}{3q_2+1} & 0 \\ \dfrac{q_1^2}{3q_2+1} - 1 & -\dfrac{q_1 q_2}{3q_2+1} & 0 & -\dfrac{q_1 p_2}{3q_2+1} \\ 0 & 0 & \dfrac{q_1 p_2}{3q_2+1} & 0 \end{bmatrix}$$

which concludes our series of examples.

4 Conclusions

In this article we have focused on two issues involving Dirac reductions of Poisson operators on submanifolds. In the first part of the article we have shown how the Dirac reduction procedure fits in a natural way, i.e. as a result of two natural assumptions about the deformation π_D of π, in the general Marsden–Ratiu reduction scheme. In the second part of our considerations we have demonstrated that the Dirac reduction procedure is often possible even in cases when the constraints that define our submanifold are of first class (in Dirac terminology), possibly after some suitably chosen reparametrization of the submanifold $\mathcal{S}$.

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