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Self-Invariant Contact Symmetries

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Abstract

Every smooth second-order scalar ordinary differential equation (ODE) that is solved for the highest derivative has an infinite-dimensional Lie group of contact symmetries. However, symmetries other than point symmetries are generally difficult to find and use. This paper deals with a class of one-parameter Lie groups of contact symmetries that can be found and used. These symmetry groups have a characteristic function that is invariant under the group action; for this reason, they are called 'self-invariant.' Once such symmetries have been found, they may be used for reduction of order; a straightforward method to accomplish this is described. For some ODEs with a one-parameter group of point symmetries, it is necessary to use self-invariant contact symmetries before the point symmetries (in order to take advantage of the solvability of the Lie algebra). The techniques presented here are suitable for use in computer algebra packages. They are also applicable to higher-order ODEs

1 Introduction

A second-order scalar ordinary differential equation (ODE) of the form

$$y'' = \omega(x, y, y') \tag{1.1}$$

can be solved (that is, reduced to quadrature) if its Lie algebra $\mathcal{L}_p$ of point symmetry generators has dimension two or more. (The ways in which this can be done are described in many texts [2, 4, 9, 12, 13, 17, 18, 20].) If the dimension of $\mathcal{L}_p$ is less than two, there is no guarantee that a reduction to quadrature can be found; however, the ODE may be solved in any of the following circumstances.

1. A contact transformation is known that maps (1.1) to a solvable ODE. (At present, there is no systematic way of finding such transformations.)

2. The ODE (1.1) has nonlocal potential symmetries relating it to a solvable ODE [2].

3. The ODE has a one-parameter Lie group of variational point symmetries.

4. The Lie algebra is one-dimensional, enabling (1.1) to be reduced to a first-order ODE, and the reduced ODE can be solved by using new symmetries (which are called Type II hidden symmetries [1]).

Even if none of the above apply, the problem may still be tractable, because every scalar ODE (1.1) has infinitely many independent one-parameter Lie groups of contact symmetries [16]. If it is possible to find enough of these groups, they can be used to construct the solution. Two such groups are sufficient, provided that differential invariants and canonical coordinates can be constructed; this proviso is the chief obstacle to the use of contact symmetries that are not point symmetries.

Contact symmetries of higher-order ODEs and scalar partial differential equations (PDEs) can be found systematically, but it is not usually possible to find even one contact symmetry of (1.1) by inspection. The problem is similar to that of finding point symmetries of first-order ODEs (as discussed in [5, 11]). The classification of all real Lie groups of contact symmetries that act on the (x, y) plane has been completed recently [8]. This classification holds up to equivalence under contact transformations; however, it does not enable one to construct a contact transformation that reduces (1.1) to a simpler form.

The current paper outlines a method for testing a given ODE (1.1) for the presence of Lie contact symmetries whose characteristic function is invariant under the group action. These *self-invariant* contact symmetries and their differential invariants and canonical coordinates may be constructed systematically.

Note that this paper deals only with scalar ODEs; contact symmetries of systems of ODEs are discussed in [7] and the references therein.

2 Contact symmetries of second-order ODEs

Sophus Lie devoted much effort to investigating contact transformations [14], but much of his work is not yet well-known.[1] Therefore we begin by stating some useful facts about Lie contact symmetries of (1.1). A one-parameter (local) Lie group of contact transformations of some domain in (x, y, y') space is a set of smooth transformations

$$\begin{aligned}\tilde{x} &= x + \epsilon\xi(x, y, y') + O(\epsilon^2),\\ \tilde{y} &= y + \epsilon\eta(x, y, y') + O(\epsilon^2),\\ \tilde{y}' &= y' + \epsilon\eta^{(1)}(x, y, y') + O(\epsilon^2),\end{aligned}$$

defined for all ϵ in some symmetric neighbourhood of zero, where the contact condition requires that

$$\eta^{(1)} = \frac{\mathrm{d}\eta}{\mathrm{d}x} - y'\frac{\mathrm{d}\xi}{\mathrm{d}x}\,. \tag{2.1}$$

Note that $\eta^{(1)}$ is independent of y'', and so (2.1) yields

$$\eta_{y'} - y'\xi_{y'} = 0.$$

(Throughout this paper, variable subscripts denote partial derivatives with respect to that variable.) It is helpful to use the characteristic function

$$Q(x, y, y') \equiv \eta - y'\xi$$

[1]Some of Lie's results are summarized in English in [3].

in terms of which

$$\xi = -Q_{y'}, \qquad \eta = Q - y'Q_{y'}, \qquad \eta^{(1)} = Q_x + y'Q_y.$$

The action of the Lie group extends to higher derivatives:

$$\tilde{y}^{(k)} = y^{(k)} + \epsilon\eta^{(k)}(x, y, \ldots, y^{(k-1)}, y^{(k)}) + O(\epsilon^2),$$

where

$$\eta^{(k)} = \frac{\mathrm{d}\eta^{(k-1)}}{\mathrm{d}x} - y^{(k)}\frac{\mathrm{d}\xi}{\mathrm{d}x} = \frac{\mathrm{d}^kQ}{\mathrm{d}x^k} - y^{(k+1)}Q_{y'}\,.$$

The infinitesimal generator of the Lie group is

$$X = \xi\partial_x + \eta\partial_y,$$

and the action of the group on derivatives of order k or less is generated by the k^{th} prolongation of the generator, namely

$$X^{(k)} = \xi\partial_x + \eta\partial_y + \eta^{(1)}\partial_{y'} + \cdots + \eta^{(k)}\partial_{y^{(k)}}.$$

Transformations in the Lie group are contact symmetries of (1.1) if and only if

$$X^{(2)}\left(y'' - \omega(x, y, y')\right) = 0 \quad \text{when} \quad y'' = \omega(x, y, y'),$$

which amounts to the following constraint on Q:

$$\begin{aligned} &Q_{xx} + 2y'Q_{xy} + y'^2Q_{yy} + (2Q_{xy'} + 2y'Q_{yy'} + Q_y)\omega + Q_{y'y'}\omega^2 \\ &\qquad = -Q_{y'}\omega_x + (Q - y'Q_{y'})\omega_y + (Q_x + y'Q_y)\omega_{y'}. \end{aligned} \tag{2.2}$$

For any given smooth $\omega(x, y, y')$, the symmetry condition (2.2) has infinitely many nontrivial solutions, any one of which may be used (in principle) to reduce the ODE to one of first order. There are two main difficulties, the first of which is to find a solution of (2.2). Usually, attention is restricted to point symmetries, for which ξ and η are independent of y'. Then the ansatz

$$Q(x, y, y') = \eta(x, y) - y'\xi(x, y)$$

is substituted into (2.2), and powers of y' are equated to obtain an overdetermined system of PDEs for ξ and η; it is usually straightforward to solve this system. If the Lie point symmetries of (1.1) are insufficient to solve the ODE, one can try an ansatz with Q nonlinear in y'. To have a good chance of success the ansatz should not be too restrictive, but (generally speaking) the more restrictive the ansatz, the easier it is to calculate Q from (2.2). The ansatz for point symmetries achieves a good balance between generality and calculability.

The second difficulty lies in finding two independent first order differential invariants $r(x, y, y')$ and $v(x, y, y')$ with which to reduce the ODE. These differential invariants satisfy

$$X^{(1)}r = 0, \qquad X^{(1)}v = 0, \tag{2.3}$$

where

$$X^{(1)} \equiv -Q_{y'}\partial_x + (Q - y'Q_{y'})\partial_y + (Q_x + y'Q_y)\partial_{y'}. \tag{2.4}$$

Although (2.3) can be solved in principle by the method of characteristics, the characteristic equations

$$\frac{\mathrm{d}x}{-Q_{y'}} = \frac{\mathrm{d}y}{Q - y'Q_{y'}} = \frac{\mathrm{d}y'}{Q_x + y'Q_y} \tag{2.5}$$

are typically hard to solve. An advantage of using point symmetries is that the first equation in (2.5) can be solved before the second one; this yields a differential invariant r (that is independent of y') which can assist in the calculation of v.

Writing the ODE (1.1) in terms of the differential invariants, we obtain a first-order ODE of the form

$$\frac{\mathrm{d}v}{\mathrm{d}r} = \Omega(r, v). \tag{2.6}$$

The general solution of (2.6) is a first-order ODE

$$F\left(r(x, y, y'), v(x, y, y'); c_1\right) = 0, \qquad c_1 \text{ constant}, \tag{2.7}$$

that is invariant under the group generated by X. If this group consists of point symmetries, (2.7) is solved by introducing a canonical coordinate $s(x, y)$ such that

$$Xs = 1.$$

Then $v(x, y, y')$ is a function of $r(x, y)$ and $\frac{\mathrm{d}s}{\mathrm{d}r}$ only, and so (2.7) can be written (locally) in the form

$$\frac{\mathrm{d}s}{\mathrm{d}r} = f(r; c_1),$$

and solved by quadrature. A similar approach works for (non-point) contact symmetries. Here the canonical coordinate is of the form $s(x, y, y')$, and satisfies

$$X^{(1)}s = 1. \tag{2.8}$$

Direct computation shows that

$$X^{(2)}\frac{\mathrm{d}s}{\mathrm{d}r} = 0,$$

and therefore there is some function Φ such that

$$\frac{\mathrm{d}s}{\mathrm{d}r} = \Phi\left(r, v, \frac{\mathrm{d}v}{\mathrm{d}r}\right). \tag{2.9}$$

If (2.7) is solvable for either r or v, then (2.6), (2.7) and (2.9) can be combined to determine s by quadrature. Generally speaking, the solution to (1.1) will be given in parametric form; in some cases, the parameters can be eliminated to obtain the general solution in the form

$$G(x, y; c_1, c_2) = 0,$$

where c_1, c_2 are arbitrary constants (see [6] for details of parameter elimination algorithms).

3 Self-Invariance

In view of the difficulty of finding differential invariants of non-point Lie contact symmetries, we restrict attention to symmetries whose characteristic function $Q(x, y, y')$ is invariant under the group action. The idea is that having found one differential invariant, $r(x, y, y') = Q$, it is easier to find $v(x, y, y')$ and $s(x, y, y')$. The characteristic function is invariant if and only if

$$X^{(1)}Q = QQ_y = 0.$$

To be of any use, the symmetry group must be non-trivial, so we ignore the trivial solution $Q = 0$ and describe Lie contact symmetries as *self-invariant* if $Q_y = 0$. For self-invariant contact symmetries, the condition (2.2) reduces to

$$Q_{xx} + 2Q_{xy'}\omega + Q_{y'y'}\omega^2 = -Q_{y'}\omega_x + (Q - y'Q_{y'})\omega_y + Q_x\omega_{y'}. \tag{3.1}$$

This symmetry condition can be decomposed into an overdetermined system by equating powers of y, provided that $\omega_y \neq 0$. Subject to this proviso, the self-invariant contact symmetries can be calculated explicitly. If $\omega_y = 0$, the ODE is immediately reducible to a first-order ODE for $z = y'$; henceforth, we assume that $\omega_y \neq 0$.

Once Q has been found, v and s are determined from the following results. If $Q_{y'} \neq 0$ then solve $Q = Q(x, y')$ for y' and calculate

$$\Psi(x, Q) \equiv \int y'(x, Q)\,\mathrm{d}x, \tag{3.2}$$

treating Q as a parameter. Even if a closed-form solution cannot be found, Ψ can be determined as a power series if Q is locally analytic in y'.

Theorem 1. *Given a self-invariant contact symmetry group with $Q_{y'} \neq 0$, define Ψ according to (3.2), and let*

$$r = Q, \qquad v = \Psi - Q\Psi_Q - y, \qquad s = -\Psi_Q, \tag{3.3}$$

writing v and s in terms of (x, y, y'). Then

$$X^{(1)}r = 0, \qquad X^{(1)}v = 0, \qquad X^{(1)}s = 1.$$

Proof. Apply $X^{(1)}$ to each of r, v and s in turn, and use the chain rule. ∎

In some instances, it is easier to solve Q for x instead of y'. Then it is possible to construct v and s by defining

$$\Theta(y', Q) \equiv \int x(y', Q)\,\mathrm{d}y'. \tag{3.4}$$

Theorem 2. *Given a self-invariant contact symmetry group with $Q_x \neq 0$, define Θ according to (3.4), and let*

$$r = Q, \qquad v = Q\Theta_Q + y'\Theta_{y'} - \Theta - y, \qquad s = \Theta_Q, \tag{3.5}$$

writing v and s in terms of (x, y, y'). Then

$$X^{(1)}r = 0, \qquad X^{(1)}v = 0, \qquad X^{(1)}s = 1.$$

Proof. The proof is the same as in the previous theorem. ■

Note that in each theorem r and v are functionally independent differential invariants. With either of the above choices of (r, v, s), a straightforward calculation shows that (2.9) amounts to

$$\frac{\mathrm{d}s}{\mathrm{d}r} = \frac{1}{r}\frac{\mathrm{d}v}{\mathrm{d}r}, \tag{3.6}$$

and so (2.7) can be reduced to quadrature if it can be solved for r or v. For completeness, note that the only remaining case is $Q = 1$, which yields point symmetries with

$$r = x, \qquad s = y, \qquad v = \frac{\mathrm{d}s}{\mathrm{d}r} = y'.$$

Having found generators of self-invariant contact symmetries (and possibly of a one-parameter group of point symmetries as well), it is straightforward to determine the structure of the minimal Lie algebra, $\mathcal{L}$, that contains the known generators. So far, we have focused on characteristic functions, rather than on the generators which are determined by (2.4). The structure of $\mathcal{L}$ may be found directly from the characteristic functions, using the following result:

$$[X_1, X_2] = X_3 \;\Leftrightarrow\; Q_3 = (Q_{1x} + y'Q_{1y})Q_{2y'} + Q_1Q_{2y} - (Q_{2x} + y'Q_{2y})Q_{1y'} - Q_2Q_{1y}\,.$$

(Here characteristic functions and their corresponding generators are labelled by the same suffix.) With a slight abuse of notation, define the commutator of Q_1 and Q_2 to be

$$[Q_1, Q_2] \equiv (Q_{1x} + y'Q_{1y})Q_{2y'} + Q_1Q_{2y} - (Q_{2x} + y'Q_{2y})Q_{1y'} - Q_2Q_{1y}\,. \tag{3.7}$$

The set of characteristic functions of all infinitesimal generators in a given Lie algebra inherit the structure of that Lie algebra, and so it is possible to examine solvability directly, without having to construct the infinitesimal generators.

Suppose that Q_1 and Q_2 both generate self-invariant contact symmetries (in (x, y, y') coordinates). Then (3.7) simplifies to

$$[Q_1, Q_2] = Q_{1x}Q_{2y'} - Q_{2x}Q_{1y'}. \tag{3.8}$$

The commutator of Q_1 and Q_2 is independent of y, and therefore the set of all Lie contact symmetries that are self-invariant in a particular coordinate system forms a Lie algebra. In particular, self-invariant contact symmetries commute if and only if their characteristic functions are functionally dependent.

4 Example

To illustrate the method, consider the ODE

$$y'' = y' + \frac{(\ln(y') - x)y'^2}{y - y'}. \tag{4.1}$$

The only Lie point symmetries of (4.1) are those whose characteristic function is

$$Q_1 = y - y'.$$

The differential invariants of the group whose characteristic function is Q_1 are $r_1 = e^{-x}y$ and $v_1 = y'/y$. Writing (4.1) in terms of (r_1, v_1) yields the first order ODE

$$\frac{\mathrm{d}v_1}{\mathrm{d}r_1} = \frac{v_1^2\{1 - \ln(r_1 v_1)\}}{r_1(1-v_1)^2} - \frac{v_1}{r_1}\,, \tag{4.2}$$

which seems to be intractable.

Having failed to solve (4.1) using its point symmetries, we now seek self-invariant contact symmetries. Substituting the right-hand side of (4.1) for ω in the symmetry condition (3.1), then equating powers of $(y - y')$, yields the following overdetermined system of PDEs:

$$Q_{xx} + 2y'Q_{xy'} + y'^2 Q_{y'y'} = Q_x\,, \tag{4.3}$$

$$y'^2\{\ln(y') - x\}\left(Q_{xy'} + y'Q_{y'y'}\right) = y'\left\{\ln(y') - x - \tfrac{1}{2}\right\}Q_x + \tfrac{1}{2}y'^2\,Q_{y'}, \tag{4.4}$$

$$y'^2\{\ln(y') - x\}Q_{y'y'} = Q_x + y'Q_{y'} - Q. \tag{4.5}$$

Generally speaking, it is easiest to solve this type of system with the aid of computer algebra. Reliable packages are available that will reduce overdetermined systems of PDEs to a differential Gröbner basis that is far easier to solve than the original system [15, 19]. However, our example is easily solved by hand, because (4.3) can be factorized into

$$\left(\frac{\partial}{\partial y'} + \frac{1}{y'}\frac{\partial}{\partial x}\right)^2 Q = 0.$$

Therefore the general solution of (4.3) is

$$Q = y'F\left(\ln(y') - x\right) + G\left(\ln(y') - x\right), \tag{4.6}$$

where F and G are arbitrary functions. Substituting (4.6) into (4.4) and (4.5) yields the results

$$F(z) = 0, \tag{4.7}$$

$$G''(z) - G'(z) + \frac{1}{z}G(z) = 0. \tag{4.8}$$

Thus the Lie algebra of self-invariant characteristic functions is two-dimensional and is spanned by

$$Q_2 = \ln(y') - x, \qquad Q_3 = \{\ln(y') - x\}\int^{\ln(y')-x} z^{-2}e^z\,\mathrm{d}z.$$

Note that Q_3 is a function of Q_2, so $\mathrm{Span}(Q_2, Q_3)$ is abelian. Furthermore

$$[Q_1, Q_2] = -Q_2, \qquad [Q_1, Q_3] = -Q_3,$$

so $\mathcal{L} = \mathrm{Span}(Q_1, Q_2, Q_3)$ is a solvable non-abelian Lie algebra. The reduction of order using the point symmetries generated by Q_1 is non-normal, and it is common for non-normal reductions of order to produce ODEs as apparently intractable as (4.2). To avoid this problem, either Q_2 or Q_3 should be used first; we use Q_2, as it is in closed form.

Differential invariants corresponding to self-invariant contact symmetries are found by the method outlined earlier. Here $\Psi(x, Q) = \exp(x + Q)$, and therefore

$$r = \ln(y') - x, \qquad v = \{1 + x - \ln(y')\}y' - y, \qquad s = -y'. \tag{4.9}$$

Then (4.1) reduces to the first order ODE

$$\frac{\mathrm{d}v}{\mathrm{d}r} = v, \tag{4.10}$$

which inherits the symmetries generated by Q_1 as scalings in v. The general solution of (4.10) is

$$v = c_1 e^r,$$

where c_1 is an arbitrary constant. Therefore, from (3.6), we obtain

$$s = c_1 \int^r z^{-1} e^z \,\mathrm{d}z + c_2,$$

where c_2 is a second arbitrary constant. Substituting these results into (4.9) yields the general solution of (4.1) in parametric form:

$$x = \ln\left(-c_1 \int^r z^{-1} e^z \,\mathrm{d}z - c_2\right) - r, \tag{4.11}$$

$$y = (r - 1)\left(c_1 \int^r z^{-1} e^z \,\mathrm{d}z + c_2\right) - c_1 e^r. \tag{4.12}$$

5 Conclusions

The method of finding and using self-invariant contact symmetries is easy to use, and can yield solutions to ODEs that are not solvable by other means. It is capable of implementation as a computer algebra program; the development of such a program would enhance existing symmetry-finding software (see [10] for a review).

The point of view taken in this paper is that the standard method of finding Lie point symmetries amounts to substituting a good ansatz into the symmetry condition (2.2). Here 'good' means that an appropriate balance is struck between generality and calculability. The set of all possible Lie point symmetries is characterized by two functions of two variables, $\xi(x, y)$ and $\eta(x, y)$, whereas the set of all possible self-invariant contact symmetries is characterized by a single function of two variables, namely $Q(x, y')$. Thus the self-invariant contact symmetries are less general than point symmetries. However, in view of the difficulty of performing symmetry reduction with non-point Lie groups, it is perhaps surprising that such a rich class of symmetries should be both calculable and usable. We have focused on second-order ODEs, but the results of Theorems 1 and 2 can also be used to solve higher-order ODEs that have self-invariant contact symmetries.

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