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Superalgebras for the 3D Harmonic Oscillator and Morse Quantum Potentials

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Abstract

In addition to obtaining supersymmetric structure related to the partner Hamiltonians, we get another supersymmetric structure via factorization method for both the 3D harmonic oscillator and Morse quantum potentials. These two supersymmetries induce also an additional supersymmetric structure involving simultaneous laddering relations with respect to two different parameters for both models. These lead to the realizing of Heisenberg Lie superalgebras with two, four and six supercharges.

1 Introduction

Since the 3D isotropic (central) harmonic oscillator potential only depends on the radial distance, Hamiltonian has rotational symmetry which can be solved by separating variables. Radial part of the 3D isotropic harmonic oscillator Hamiltonian is a well-known differential equation which can be factorized as the operator products [6]. There are application aspects for the 3D isotropic harmonic oscillator. For example, it has been shown that for the screened 3D isotropic harmonic oscillator there exist an infinite number of closed orbits for suitable angular momentum values so that the dynamical symmetry $SU(2)$ is only preserved at the aphelion (perihelion) points of classical orbits in the eigencoordinate system [26]. The q-deformed 3D harmonic oscillator model is also applied to the description of atomic metal clusters [21], and the successful prediction of magic numbers in alkali metal clusters, of up to 1500 atoms per cluster [4], are in agreement with experiment.

On the other hand, the 1D Morse potential, which was first introduced by Morse in 1929 [19], is one of the most successful models for the states of the diatomic molecules [7]. On a real axis like θ from $-\infty$ to $+\infty$, the Morse partner potentials are defined as $(Ae^{\theta} - B_n)^2 - 2Ae^{\theta}$ and $(Ae^{\theta} - B_{n+1})^2 + 2Ae^{\theta}$. In fact, although the Morse potential does not have the $\theta^{-\gamma}$ dependence with $\gamma > 0$, it gives a good overall approximation for small systems. The Morse potential has been widely used in many areas such as molecular systems, quantum chemistry and, in particular, chemical bonds [20]. In the context of supersymmetric quantum mechanics [2, 6, 15, 23, 25], the energy spectrum and the eigenfunction of the Morse potential have been calculated by variational method for various diatomic molecules in Ref. [9]. In algebraic models of the molecular structure, the Morse potential is related to the Lie algebra $su(2)$, and the bound states are labelled by the representations of the algebra [13]. In the approaches of potential group, Lie algebra $su(1,1)$ and other dynamical algebras many studies have been done for obtaining the energy spectrum and the eigenfunctions of the Morse potential [3]. Considerations pointed out above reveal importance of the following studies.

In recent years, shape invariance [6, 11, 14, 16, 17] with respect to one parameter has been a rich facet of quantum mechanical solvable models [6, 18, 19]. This is because shape invariance with respect to one parameter represents not only supersymmetry algebra [6, 15, 17, 25] but also the parasupersymmetry algebra of arbitrary order [6, 16, 22]. Using the idea of shape invariance one can also construct solvable 2D Schrödinger or Dirac equations [5, 16]. In Ref. [11], using the idea of master function theory we have shown that most of shape invariant potentials are classified in two different classes. In the first class, the superpotentials are labelled in terms of a pair of quantum numbers n and m, whereas in the second class, the superpotentials are labelled in terms of a single quantum number m. This formalism for classifying 1D quantum solvable models leads us to the realization of supersymmetry algebras with more generators [1]. For this, in this paper it will be important to obtain the simultaneous shape invariance with respect to the parameters n and m for both the 3D harmonic oscillator and the Morse solvable models.

In this paper, using factorization [11, 16, 17, 24] and shape invariance of the associated Laguerre differential equation with respect to two parameters m and n [16, 17], we obtain the factorized Schrödinger equations for the 3D radial harmonic oscillator and the Morse Hamiltonians. In this process, quantum states of both models are labelled by both parameters n and m. The partner supersymmetric structure corresponding to the 3D harmonic oscillator and Morse potentials describe the laddering relations for the parameters m and n, respectively. Furthermore, we derive their laddering operators such that their indices are displaced with the inverse order n and m, respectively. These are interpreted as the Heisenberg Lie superalgebra $H(0|2)\oplus H(0|2)$ for the 3D harmonic oscillator and Morse models. Also, we derive a nice symmetry involving simultaneous displacement of both parameters n and m for both 3D harmonic oscillator and Morse quantum states. Finally, we can conclude that the Heisenberg Lie superalgebra $H(0|2)\oplus H(0|2)\oplus H(0|2)$ is realized by laddering operators of both models. Since, in fact, the existence of every pair of raising and lowering operators in quantum mechanics via introducing two supercharges operators and one bosonic operator enable us to construct a representation of the supersymmetry algebra.

2 Mathematical foundation

The differential equation of the so called Laguerre for given non-negative integers $n \geq 0$ and $0 \leq m \leq n$ has been known to be [8, 16, 17]

$$xy'' + (1+\alpha-\beta x)\,y' + \left[\left(n-\frac{m}{2}\right)\beta - \frac{m}{2}\left(\alpha+\frac{m}{2}\right)\frac{1}{x}\right] y = 0\,, \tag{2.1}$$

where its solutions $y = L_{n,m}^{(\alpha,\beta)}(x)$ in the Rodrigues representation are

$$L_{n,m}^{(\alpha,\beta)}(x) = \frac{a_{n,m}(\alpha,\beta)}{x^{\alpha+\frac{m}{2}}e^{-\beta x}}\left(\frac{d}{dx}\right)^{n-m}\left(x^{n+\alpha}e^{-\beta x}\right), \tag{2.2}$$

in which $a_{n,m}(\alpha,\beta)$ is the normalization coefficient. The above differential equation for the special case $m = 0$ transforms into a differential equation corresponding to the Laguerre polynomials. In fact, the associated Laguerre differential equation (2.1) has been obtained by taking m-th derivative of the Laguerre polynomials differential equation, and using the change of function by multiplying it by $x^{m/2}$ as well.

Shape invariance of the associated Laguerre differential equation (2.1) with respect to m, for a given n, is realized as [16]

$$\begin{aligned} A_+(m;x)A_-(m;x)L_{n,m}^{(\alpha,\beta)}(x) &= (n-m+1)\beta L_{n,m}^{(\alpha,\beta)}(x)\,,\\ A_-(m;x)A_+(m;x)L_{n,m-1}^{(\alpha,\beta)}(x) &= (n-m+1)\beta L_{n,m-1}^{(\alpha,\beta)}(x)\,, \end{aligned} \tag{2.3}$$

where the differential explicit forms of the operators $A_+(m;x)$ and $A_-(m;x)$ are, respectively:

$$\begin{aligned} A_+(m;x) &= \sqrt{x}\frac{d}{dx} - \frac{m-1}{2\sqrt{x}}\,,\\ A_-(m;x) &= -\sqrt{x}\frac{d}{dx} - \frac{2\alpha+m-2\beta x}{2\sqrt{x}}\,. \end{aligned} \tag{2.4}$$

One may write down the shape invariance equations (2.3) as the raising and lowering relations:

$$\begin{aligned} A_+(m;x)L_{n,m-1}^{(\alpha,\beta)}(x) &= \sqrt{(n-m+1)\beta}L_{n,m}^{(\alpha,\beta)}(x)\,,\\ A_-(m;x)L_{n,m}^{(\alpha,\beta)}(x) &= \sqrt{(n-m+1)\beta}L_{n,m-1}^{(\alpha,\beta)}(x)\,. \end{aligned} \tag{2.5}$$

On the other hand, the associated Laguerre differential equation (2.1) can be factorized with respect to the parameter n, for a given m, as [17]

$$\begin{aligned} A_+(n,m;x)A_-(n,m;x)L_{n,m}^{(\alpha,\beta)}(x) &= (n-m)(n+\alpha)L_{n,m}^{(\alpha,\beta)}(x)\,,\\ A_-(n,m;x)A_+(n,m;x)L_{n-1,m}^{(\alpha,\beta)}(x) &= (n-m)(n+\alpha)L_{n-1,m}^{(\alpha,\beta)}(x)\,, \end{aligned} \tag{2.6}$$

where the differential operators as functions of the parameters n and m are calculated as follows, respectively:

$$\begin{aligned} A_+(n,m;x) &= x\frac{d}{dx} - \beta x + \frac{1}{2}\left(2n+2\alpha-m\right)\,,\\ A_-(n,m;x) &= -x\frac{d}{dx} + \frac{1}{2}\left(2n-m\right)\,. \end{aligned} \tag{2.7}$$

Note that the shape invariance equations (2.6) can also be written as the raising and lowering relations:

$$\begin{aligned} A_+(n,m;x)L_{n-1,m}^{(\alpha,\beta)}(x) &= \sqrt{(n-m)(n+\alpha)}\,L_{n,m}^{(\alpha,\beta)}(x)\,,\\ A_-(n,m;x)L_{n,m}^{(\alpha,\beta)}(x) &= \sqrt{(n-m)(n+\alpha)}\,L_{n-1,m}^{(\alpha,\beta)}(x)\,. \end{aligned} \tag{2.8}$$

It is clear that realization of the shape invariance equations (2.3) and (2.6) does not impose any condition on the normalization coefficients $a_{n,m}(\alpha,\beta)$. However, realization of the laddering equations (2.5) and (2.8) imposes two recursion relations with respect to m and n, respectively on the coefficients. In other words, in order to realize the equations (2.5) and (2.8), the highest power coefficients of x must be the same on both sides of them. The mentioned method, after some calculations, leads to the following result

$$a_{n,m}(\alpha,\beta)=(-1)^m\sqrt{\frac{\beta^{\alpha+m+1}}{\Gamma(n-m+1)\Gamma(n+\alpha+1)}}\,. \tag{2.9}$$

The normalization coefficient (2.9) has also been so chosen that the associated Laguerre functions $L_{n,m}^{(\alpha,\beta)}(x)$, with the same m but with different n's, with respect to the inner product with the weight function $x^\alpha e^{-\beta x}$ $[\beta>0,\alpha>-1]$ form an orthonormal set in the interval $0\leq x<+\infty$:

$$\int_0^\infty L_{n,m}^{(\alpha,\beta)}(x)\,L_{n',m}^{(\alpha,\beta)}(x)\,x^\alpha e^{-\beta x}dx=\delta_{nn'}. \tag{2.10}$$

3 The 3D harmonic oscillator potential

Using the function $x^{\frac{2\alpha+1}{4}}e^{-\frac{\beta x}{2}}$ in order to do the similarity transformation on the operators $A_\pm(m;x)$ appearing in equations (2.3), together with application of the new variable r, $x=\frac{r^2}{4}$ $(0\leq r<+\infty)$, we can easily get the radial Schrödinger equations. According to this procedure, the explicit forms of the raising and lowering operators corresponding to the parameter m are calculated as

$$A_\pm(m;r)=\pm\frac{d}{dr}+W_m(\beta;r)\,, \tag{3.1}$$

where $W_m(\beta;r)$ is the well-known 3D harmonic oscillator superpotential

$$W_m(\beta;r)=\frac{1}{4}\beta r-\frac{2\alpha+2m-1}{2r}. \tag{3.2}$$

Also, with the help of the associated Laguerre functions $L_{n,m}^{(\alpha,\beta)}(x)$ we obtain the 3D harmonic oscillator quantum states corresponding to the supersymmetric partner [15, 25] Hamiltonians ($\hbar=2M=1$),

$$A_+(m;r)A_-(m;r)\psi_{n,m}(r)=(n-m+1)\beta\psi_{n,m}(r)\,, \tag{3.3}$$

$$A_-(m;r)A_+(m;r)\psi_{n,m-1}(r)=(n-m+1)\beta\psi_{n,m-1}(r)\,, \tag{3.4}$$

as

$$\psi_{n,m}(r)=\left(\frac{r}{2}\right)^{\frac{2\alpha+1}{2}}e^{-\frac{\beta}{8}r^2}L_{n,m}^{(\alpha,\beta)}(\frac{r^2}{4})\,. \tag{3.5}$$

From equations (2.5), we can obtain the raising and lowering relations for the quantum states of 3D harmonic oscillator as follows

$$A_+(m;r)\psi_{n,m-1}(r)=\sqrt{(n-m+1)\beta}\,\psi_{n,m}(r)\,, \tag{3.6}$$

$$A_-(m;r)\psi_{n,m}(r)=\sqrt{(n-m+1)\beta}\,\psi_{n,m-1}(r)\,. \tag{3.7}$$

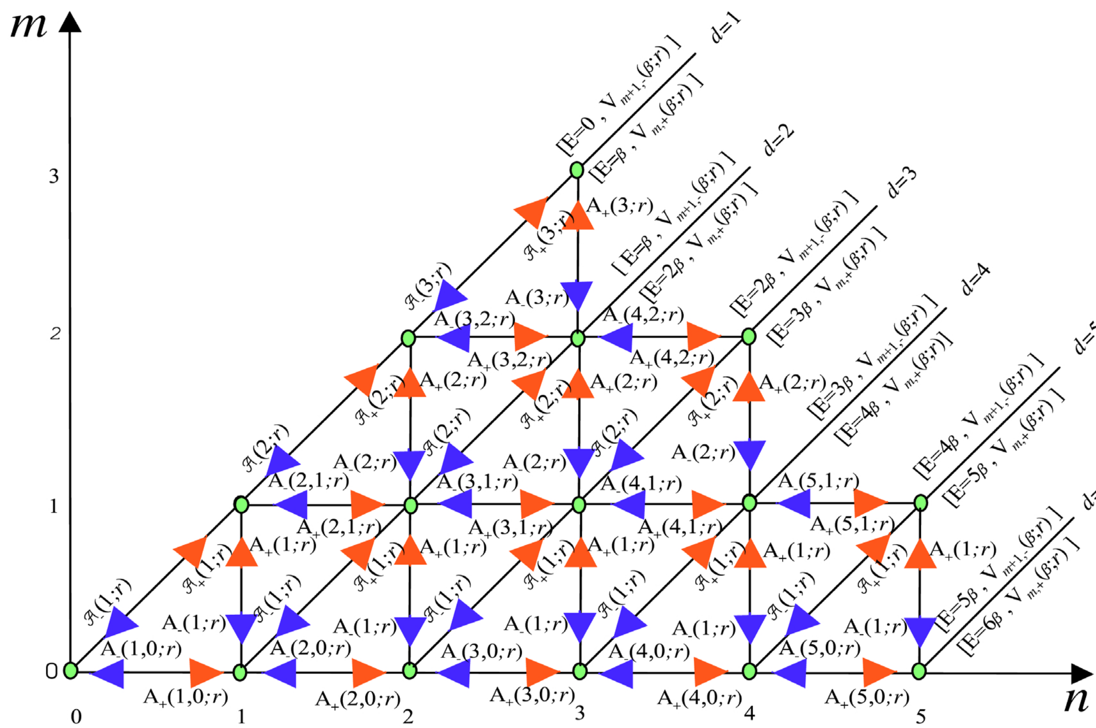

Figure 1. The lattice of points as permissible quantum states for the partner potentials of 3D harmonic oscillator. The Figure shows that the allowed energies corresponding to all quantum states lain on dth oblique line are $(d-1)\beta$ and $d\beta$.

Now, using the equations (2.10) and (3.5), it is easily shown that the set of quantum states $\psi_{n,m}(r)$ for a given value of m form an orthonormal set

$$\int_0^\infty \psi_{n,m}(r)\psi_{n',m}(r)\,dr = \delta_{nn'}. \tag{3.8}$$

The operators $A_+(m;r)$ and $A_-(m;r)$ are Hermitian conjugate of each other with respect to the inner product (3.8). Shape invariance equations (3.3) and (3.4) describe radial part of the 3D harmonic oscillator as supersymmetric partner Hamiltonians with the following partner potentials, respectively:

$$\begin{aligned} V_{m,\pm}(\beta;r) &= W_m^2(\beta;r) \pm \frac{d}{dr}W_m(\beta;r) \\ &= \frac{\beta^2}{16}r^2 + \frac{(2\alpha+2m-1)\left(\alpha+m-\frac{1}{2}\pm 1\right)}{2r^2} - \frac{\beta}{4}\left(2\alpha+2m-1\mp 1\right), \end{aligned} \tag{3.9}$$

which satisfy the following shape invariance condition on the parameter m:

$$V_{m,+}(\beta;r) = V_{m+1,-}(\beta;r) + \beta\,. \tag{3.10}$$

The parameter m distinguishes different potentials of the 3D harmonic oscillator, while the parameter n describes radial quantization for the 3D harmonic oscillator potential.

We have schematically shown all 3D harmonic oscillator quantum states $\psi_{n,m}(r)$ as the points (n,m) with $0 \le m \le n$ limitation in the flat plane with the horizontal n-axis and perpendicular m-axis in Figure 1. According to (3.3) and (3.4), for a given value of n, the quantum state $\psi_{n,n}(r)$ corresponding to the supersymmetric partner potentials $V_{n+1,-}(\beta;r)$ and $V_{n,+}(\beta;r)$ has the least possible energies of 0 and β, respectively. The first-order differential equation (3.6) for $m = n+1$ gives the ground state $\psi_{n,n}(r)$ as

$$\psi_{n,n}(r) = (-1)^n \sqrt{\frac{\beta^{n+\alpha+1}}{\Gamma(n+\alpha+1)}} \left(\frac{r}{2}\right)^{\frac{2n+2\alpha+1}{2}} e^{-\frac{\beta}{8}r^2}, \tag{3.11}$$

where it is in agreement with the analytic solution given for the ground state in (3.5). Now, by making use of the equation (3.7) for given n, one can algebraically calculate all other quantum states on a normal line by the ground state $\psi_{n,n}(r)$ placed on the bisector line

$$\psi_{n,m}(r) = \frac{A_-(m+1;r)A_-(m+2;r)\cdots A_-(n;r)\psi_{n,n}(r)}{\sqrt{\beta^{n-m}\Gamma(n-m+1)}} \qquad m = 0,1,2,\cdots,n-1\,. \tag{3.12}$$

Therefore, according to (3.6) and (3.7), the operators $A_+(m;r)$ and $A_-(m;r)$ displace the lattice points on a vertical line towards up and down, respectively. In other words, they are the shift operators of the second index, m, which describe the different partner potentials of the 3D harmonic oscillator quantum states.

To obtain the raising and lowering operators of the first index n which describes radial quantization for the 3D harmonic oscillator quantum states, we do the similarity transformation by the function $x^{\frac{2\alpha+1}{4}} e^{-\frac{\beta x}{2}}$ on the operators $A_\pm(n,m;x)$ of equations (2.8):

$$A_\pm(n,m;r) = \pm\frac{r}{2}\frac{d}{dr} - \frac{\beta}{8}r^2 + \frac{1}{4}\left(4n + 2\alpha - 2m \mp 1\right), \tag{3.13}$$

in which, the radial variable r has been also used. Hence, the equations (2.8) yield the following relations for the raising and lowering of the 3D harmonic oscillator quantum states with respect to the first index n:

$$A_+(n,m;r)\psi_{n-1,m}(r) = \sqrt{(n-m)(n+\alpha)}\,\psi_{n,m}(r), \tag{3.14}$$

$$A_-(n,m;r)\psi_{n,m}(r) = \sqrt{(n-m)(n+\alpha)}\,\psi_{n-1,m}(r)\,. \tag{3.15}$$

It is obvious that the operators $A_+(n,m;r)$ and $A_-(n,m;r)$ are not Hermitian conjugate of each other with respect to the inner product (3.8). Again, according to equations (3.3) and (3.4), for a given value of m, the energy of $\psi_{m,m}(r)$ corresponding to the supersymmetric partner potentials $V_{m+1,-}(\beta;r)$ and $V_{m,+}(\beta;r)$ has values 0 and β, respectively. For a given m, one can derive the lowest state $\psi_{m,m}(r)$ as equation (3.11), with m instead of n, from the first-order differential equation (3.15) with $n = m$. Now, for given partner potentials $V_{m,+}(\beta;r)$ and $V_{m+1,-}(\beta;r)$, it is rather trivial to show that

$$\begin{aligned}\psi_{n,m}(r) = &\sqrt{\frac{\Gamma(\alpha+m+1)}{\Gamma(n-m+1)\Gamma(n+\alpha+1)}} \times \\ &\times A_+(n,m;r)A_+(n-1,m;r)\cdots A_+(m+1,m;r)\psi_{m,m}(r) \\ &n = m+1, m+2, \cdots\end{aligned} \tag{3.16}$$

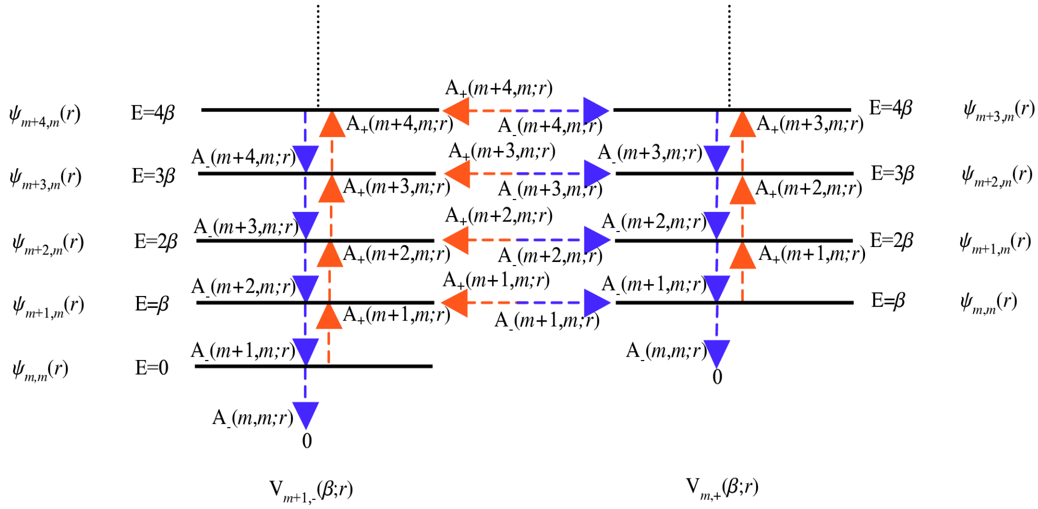

Figure 2. Sketch of the energy levels for the partner potentials of the 3D harmonic oscillator on the basis of the shape invariance with respect to n, for two given partner potentials. In the representation of the related supersymmetry, the energy is shifted by changing quantum number n.

are algebraic solutions corresponding to the other quantum states with more energies. From equations (3.14) and (3.15) it is clear that for a given m the operators $A_+(n,m;r)$ and $A_-(n,m;r)$ shift the lattice points on a horizontal line to the right and left respectively. Also, in Figures 2 and 3, we have schematically shown sketches of the energy levels of the 3D harmonic oscillator on the basis of the shape invariance with respect to the parameters n and m, respectively. According to above discussion, the number of ground states -lain on the bisector line- is infinite, and one may move either to the right on a horizontal line (cf. Figure 2) or down on a vertical line (cf. Figure 3). It is easily seen that number of energy levels for either one of the supersymmetric partner potentials in Figures 2 and 3 are infinite and $n+1$, respectively.

We now give an interesting feature of a simultaneous shape invariance with respect to both parameters n and m of the 3D harmonic oscillator quantum states $\psi_{n,m}(r)$. Defining the differential operators of first-order as the following form which are Hermitian conjugates of each other with respect to the inner product (3.8)

$$\begin{aligned}
\mathcal{A}_+(m;r) &:= A_+(m;r)A_+(n,m-1;r) - A_+(n,m;r)A_+(m;r) = \frac{d}{dr} + W_m(-\beta;r)\,,\\
\mathcal{A}_-(m;r) &:= A_-(n,m-1;r)A_-(m;r) - A_-(m;r)A_-(n,m;r) = -\frac{d}{dr} + W_m(-\beta;r)\,,
\end{aligned} \tag{3.17}$$

and with the help of equations (3.6), (3.7), (3.14) and (3.15) one can obtain the following relations for the raising and lowering of the quantum states $\psi_{n,m}(r)$ with respect to both

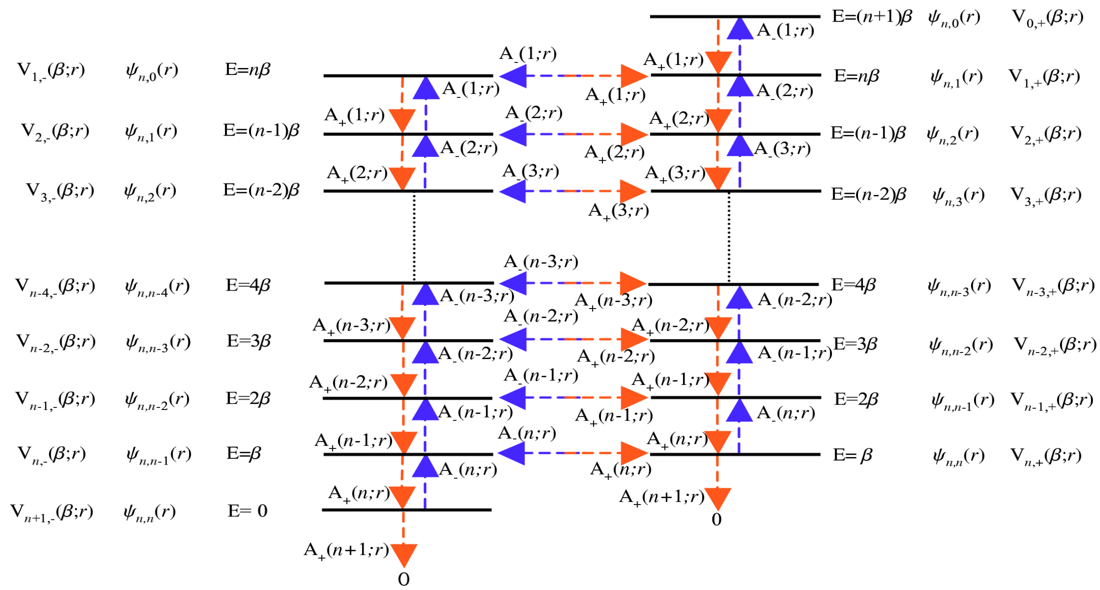

Figure 3. Sketch of the energy levels for the partner potentials of the 3D harmonic oscillator on the basis of the shape invariance with respect to m, for a given n. In the representation of the related supersymmetry, the energy is shifted by changing both partner potentials.

parameters n and m

$$\begin{aligned}\mathcal{A}_+(m;r)\psi_{n-1,m-1}(r) &= \sqrt{(n+\alpha)\beta}\,\psi_{n,m}(r)\\ \mathcal{A}_-(m;r)\psi_{n,m}(r) &= \sqrt{(n+\alpha)\beta}\,\psi_{n-1,m-1}(r).\end{aligned} \tag{3.18}$$

In Figure 1, if we call the bisector line and all their adjacent parallel lines respectively by $d = 1, 2, 3, \cdots$, then one can write the equation of dth line as $n = m + d - 1$. The shape invariance relations (3.3) and (3.4) express that all quantum states on d-th line for the partner potentials $V_{m+1,-}(\beta;r)$ and $V_{m,+}(\beta;r)$ have the same spectra $(d-1)\beta$ and $d\beta$, respectively. Therefore, the laddering equations (3.18) describe displacement of quantum states on an arbitrary line d, in particular the ground states on the bisector line $d = 1$, by the generators $\mathcal{A}_\pm(m;r)$. It must be emphasized that the energy values of all quantum states $\psi_{n,m}(r)$ on d-th line, which satisfy the condition $n = m + d - 1$, are constant. In Figure 4, we have schematically shown sketch of two different energy levels of the 3D harmonic oscillator on the basis of the simultaneous shape invariance with respect to the parameters n and m. As another interesting result, using the relations (3.18), one can obtain the factorized Schrödinger equations with respect to both parameters n and m

$$\begin{aligned}\mathcal{A}_+(m;r)\mathcal{A}_-(m;r)\psi_{n,m}(r) &= (n+\alpha)\beta\,\psi_{n,m}(r)\,,\\ \mathcal{A}_-(m;r)\mathcal{A}_+(m;r)\psi_{n-1,m-1}(r) &= (n+\alpha)\beta\,\psi_{n-1,m-1}(r)\,,\end{aligned} \tag{3.19}$$

which include the 3D harmonic oscillator supersymmetric partner potentials $V_{m,+}(-\beta;r)$ and $V_{m,-}(-\beta;r)$, respectively. In the supersymmetric partner Schrödinger equations (3.19) which are again the radial part of the Hamiltonian corresponding to the 3D isotropic

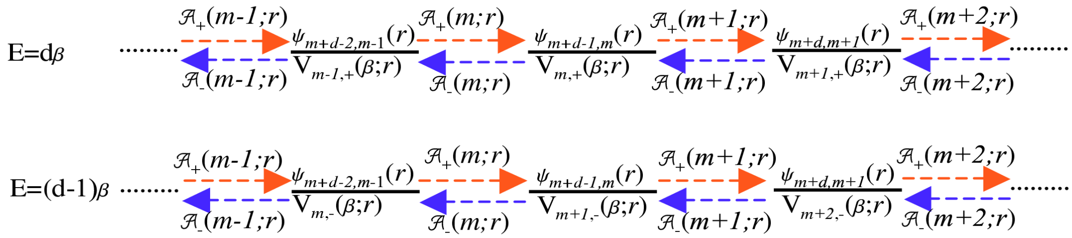

Figure 4. Sketch of the energy levels for the partner potentials of the 3D harmonic oscillator on the basis of the simultaneous shape invariance with respect to n and m, for a given d. In the representation of the related supersymmetry, the energy is shifted by changing a potential to its partner.

harmonic oscillator, the energy levels are independent of m unlike equations (3.3) and (3.4), and they are described only in terms of the radial quantum number n.

Since the following relations are identically satisfied

$$\begin{aligned} &A_+(m;r)A_-(n,m-1;r) - A_-(n,m;r)A_+(m;r) = 0\,, \\ &A_+(n,m-1;r)A_-(m;r) - A_-(m;r)A_+(n,m;r) = 0\,, \end{aligned} \tag{3.20}$$

it becomes obvious that we can not obtain the differential operators of first-order such that they increase one of the indices n and m and decrease the other index by one unit.

Therefore, we have obtained three different types of the laddering relations (3.6), (3.7), (3.14), (3.15) and (3.18) for the 3D harmonic oscillator quantum states. The existence of each of the relations leads us to a representation of the Heisenberg Lie superalgebra $H(0|2)$. Meanwhile, using any type three (or different type two) of the mentioned laddering operators we can construct the Heisenberg Lie superalgebra $H(0|2)\oplus H(0|2)\oplus H(0|2)$ (or $H(0|2)\oplus H(0|2)$). For given n and m, defining the supercharges $Q_i^{\pm}$ and the bosonic operators H_i for $i=1,2,3$, as 6×6 matrices with the following matrix elements:

$$\begin{aligned} &\left(Q_1^+\right)_{ij} = \delta_{i\,1}\delta_{j\,6}\,A_+(m;r)\,, && \left(Q_1^-\right)_{ij} = \delta_{i\,6}\delta_{j\,1}\,A_-(m;r)\,, \\ &\left(Q_2^+\right)_{ij} = \delta_{i\,2}\delta_{j\,5}\,A_+(n,m;r)\,, && \left(Q_2^-\right)_{ij} = \delta_{i\,5}\delta_{j\,2}\,A_-(n,m;r)\,, \\ &\left(Q_3^+\right)_{ij} = \delta_{i\,3}\delta_{j\,4}\,\mathcal{A}_+(m;r)\,, && \left(Q_3^-\right)_{ij} = \delta_{i\,4}\delta_{j\,3}\,\mathcal{A}_-(m;r)\,, \end{aligned}$$

$$\begin{aligned} (H_1)_{ij} &= \delta_{i\,1}\delta_{j\,1}\,A_+(m;r)A_-(m;r) + \delta_{i\,6}\delta_{j\,6}\,A_-(m;r)A_+(m;r)\,, \\ (H_2)_{ij} &= \delta_{i\,2}\delta_{j\,2}\,A_+(n,m;r)A_-(n,m;r) + \delta_{i\,5}\delta_{j\,5}\,A_-(n,m;r)A_+(n,m;r)\,, \\ (H_3)_{ij} &= \delta_{i\,3}\delta_{j\,3}\,\mathcal{A}_+(m;r)\mathcal{A}_-(m;r) + \delta_{i\,4}\delta_{j\,4}\,\mathcal{A}_-(m;r)\mathcal{A}_+(m;r)\,, \end{aligned} \tag{3.21}$$

one can conclude the (anti)commutation relations of the Heisenberg Lie superalgebra $H(0|2)\oplus H(0|2)\oplus H(0|2)$ as follows $(i,j=1,2,3)$

$$\begin{aligned} &\left\{Q_i^+, Q_j^-\right\} = \delta_{ij} H_i\,, \\ &\left\{Q_i^+, Q_j^+\right\} = \left\{Q_i^-, Q_j^-\right\} = 0\,, \\ &\left[H_i, Q_j^{\pm}\right] = [H_i, H_j] = 0\,. \end{aligned} \tag{3.22}$$

Note that the Heisenberg Lie superalgebra $H(0|2) \oplus H(0|2)$ can be extracted in a similar manner. As example, we also remind that the supercharges $Q^{\pm}$ and the bosonic operator H defined as

$$Q^{\pm} = \Sigma_{i=1}^{3} Q_i^{\pm}\,, \qquad\qquad H = \Sigma_{i=1}^{3} H_i\,, \tag{3.23}$$

satisfy the (anti)commutation relations of the Heisenberg Lie superalgebra $H(0|2)$ as follows

$$\begin{aligned}
&\{Q^+, Q^-\} = H\,,\\
&\{Q^+, Q^+\} = \{Q^-, Q^-\} = 0\,,\\
&[H, Q^{\pm}] = 0\,.
\end{aligned} \tag{3.24}$$

It must be pointed out that the 3D harmonic oscillator problem can be transformed to the motion of a charged particle on the flat surface in the presence of a constant magnetic field along z-axis (i.e. Landau problem) with dynamical symmetry group H_4. Based on the results of this section, in Ref. [10], the Hilbert space corresponding to all quantum states of Landau levels has been split into an infinite direct sum of infinite-dimensional Hilbert sub-spaces. For any one of these Hilbert sub-spaces, we have calculated the generalized type of Klauder-Perelomov and Gazeau-Klauder coherent states.

4 The Morse potential

Imposing the similarity transformation by the function $x^{\frac{\alpha}{2}} e^{-\frac{\beta x}{2}}$ on the operators $A_{\pm}(n, m; x)$ together with the change of variable $x = e^{\theta}$ $(-\infty < \theta < +\infty)$ one can obtain the factorized Schrödinger equations with respect to the parameter n for the Morse supersymmetric partner potentials

$$A_+(n, m; \theta) A_-(n, m; \theta) \psi_{n,m}(\theta) = (n - m)(n + \alpha) \psi_{n,m}(\theta)\,, \tag{4.1}$$

$$A_-(n, m; \theta) A_+(n, m; \theta) \psi_{n-1,m}(\theta) = (n - m)(n + \alpha) \psi_{n-1,m}(\theta), \tag{4.2}$$

or the raising and lowering relations of the Morse quantum states with respect to n

$$A_+(n, m; \theta) \psi_{n-1,m}(\theta) = \sqrt{(n - m)(n + \alpha)}\, \psi_{n,m}(\theta)\,, \tag{4.3}$$

$$A_-(n, m; \theta) \psi_{n,m}(\theta) = \sqrt{(n - m)(n + \alpha)}\, \psi_{n-1,m}(\theta)\,. \tag{4.4}$$

The raising and lowering operators of the first index n in terms of the Morse superpotential

$$W_{n,m}(\beta; \theta) = \frac{1}{2}\left(-\beta e^{\theta} + 2n + \alpha - m\right)\,, \tag{4.5}$$

are calculated as

$$A_{\pm}(n, m; \theta) = \pm\frac{d}{d\theta} + W_{n,m}(\beta; \theta)\,. \tag{4.6}$$

In addition to the parameter m, the Morse superpotential also depends on n, contrary to the 3D harmonic oscillator superpotential. Therefore, neither n nor m alone describes quantization on θ-axis. For a given constant value of $k = 2n - m$ the functions

$$\psi_{n,m}(\theta) = e^{\frac{\alpha}{2}\theta - \frac{\beta}{2} e^{\theta}} L_{n,m}^{(\alpha,\beta)}(e^{\theta}) \tag{4.7}$$

are quantum states corresponding to the Morse superpotential $\frac{1}{2}\left(-\beta e^{\theta}+\alpha+k\right)$. Therefore, one of the parameters n or m via the relation $k=2n-m$ describes quantization on θ-axis in a complicated way, and the other parameter is related to the information of the Morse superpotential. Using the relation (2.10) it is easy to show that the set of quantum states $\psi_{n,m}(\theta)$ corresponding to the Morse superpotential for a given m form an orthonormal set as

$$\int_{-\infty}^{+\infty} \psi_{n,m}(\theta)\psi_{n',m}(\theta)\, e^{\theta}\, d\theta = \delta_{nn'}\,. \tag{4.8}$$

The Morse supersymmetric partner potentials corresponding to equations (4.1) and (4.2) are, respectively

$$\begin{aligned} V_{n,m,\pm}(\beta;\theta) &= W^2_{n,m}(\beta;\theta) \pm \frac{d}{d\theta} W_{n,m}(\beta;\theta) \\ &= \frac{1}{4}\left(\beta^2 e^{2\theta} - 2\beta\,(2n+\alpha-m\pm 1)\, e^{\theta} + (2n+\alpha-m)^2\right), \end{aligned} \tag{4.9}$$

which satisfy shape invariance condition on the parameter n as

$$V_{n,m,+}(\beta;\theta) = V_{n+1,m,-}(\beta;\theta) - 2n - \alpha + m - 1\,. \tag{4.10}$$

As an application of this shape invariance, in Ref. [12], we have shown that the quantum states of Morse potential represent the infinite-dimensional Morse Lie algebra. A representation of the Lie algebra $su(1,1)$ has also been generated by means of using the ladder operators (4.3) and (4.4). Furthermore, there we have obtained the Barut-Girardello coherent states as a linear combination of the quantum states corresponding to the Morse potential. The equations (4.1) and (4.2) express that $\psi_{n,m}(\theta)$ is the quantum state of the Morse supersymmetric partner potentials $V_{n,m,+}(\beta;\theta)$ and $V_{n+1,m,-}(\beta;\theta)$ with the spectra $(n-m)(n+\alpha)$ and $(n-m+1)(n+\alpha+1)$, respectively. Therefore, for a given m, the Morse supersymmetric partner potentials $V_{m,m,+}(\beta;\theta)$ and $V_{m+1,m,-}(\beta;\theta)$ with the same quantum state $\psi_{m,m}(\theta)$ have energy values 0 and $m+\alpha+1$, respectively, which are the least possible values. This means that $\psi_{m,m}(\theta)$ is the lowest (or ground) state. With the help of equation (4.4) with $n=m$ one can derive it as

$$\psi_{m,m}(\theta) = (-1)^m \sqrt{\frac{\beta^{\alpha+m+1}}{\Gamma(\alpha+m+1)}}\, e^{\frac{\alpha+m}{2}\theta - \frac{\beta}{2}e^{\theta}}\,. \tag{4.11}$$

From the equation (4.3) it becomes known that any other quantum state can be algebraically calculated by means of operator products

$$\begin{aligned} \psi_{n,m}(\theta) = \sqrt{\frac{\Gamma(\alpha+m+1)}{\Gamma(n-m+1)\Gamma(n+\alpha+1)}} \times& \\ \times A_+(n,m;\theta)A_+(n-1,m;\theta)\cdots A_+(m+1,m;\theta)\psi_{m,m}(\theta)& \\ n = m+1, m+2, \cdots\,.& \end{aligned} \tag{4.12}$$

If we impose similarity transformation by the function $x^{\frac{\alpha}{2}} e^{-\frac{\beta x}{2}}$ together with the change of variable $x=e^{\theta}$ on the laddering equations (2.5), then we get the raising and lowering relations with respect to the second index of the Morse quantum states

$$A_+(m;\theta)\psi_{n,m-1}(\theta) = \sqrt{(n-m+1)\beta}\, \psi_{n,m}(\theta)\,, \tag{4.13}$$

$$A_-(m;\theta)\psi_{n,m}(\theta) = \sqrt{(n-m+1)\beta}\, \psi_{n,m-1}(\theta)\,, \tag{4.14}$$

with

$$A_{\pm}(m;\theta) = \pm e^{-\frac{\theta}{2}}\frac{d}{d\theta} + \frac{\beta}{2}e^{\frac{\theta}{2}} - \frac{1}{4}\left(2\alpha + 2m - 1 \mp 1\right)e^{-\frac{\theta}{2}}\,. \tag{4.15}$$

The operators $A_+(n,m;\theta)$ and $A_-(n,m;\theta)$ are Hermitian conjugates of each other with respect to the inner product (4.8). For a given n, the ground state $\psi_{n,n}(\theta)$ by means of (4.13) is again calculated as (4.11) with n instead of m. The equation (4.14) gives algebraic solutions for the other quantum states as

$$\psi_{n,m}(\theta) = \frac{A_-(m+1;\theta)A_-(m+2;\theta)\cdots A_-(n;\theta)\psi_{n,n}(\theta)}{\sqrt{\beta^{n-m}\Gamma(n-m+1)}} \qquad m = 0,1,2,\cdots,n-1\,. \tag{4.16}$$

Therefore, in the Morse quantum states lattice the operators $A_{\pm}(n,m;\theta)$ and $A_{\pm}(m;\theta)$ displace the points along horizontal and vertical lines, respectively.

Defining the first-order differential operators as

$$\begin{aligned}
\mathcal{A}_+(m;\theta) &:= A_+(m;\theta)A_+(n,m-1;\theta) - A_+(n,m;\theta)A_+(m;\theta)\\
&= e^{-\frac{\theta}{2}}\frac{d}{d\theta} - \frac{\beta}{2}e^{\frac{\theta}{2}} - \frac{1}{2}\left(\alpha + m - 1\right)e^{-\frac{\theta}{2}}\\
\mathcal{A}_-(m;\theta) &:= A_-(n,m-1;\theta)A_-(m;\theta) - A_-(m;\theta)A_-(n,m;\theta)\\
&= -e^{-\frac{\theta}{2}}\frac{d}{d\theta} - \frac{\beta}{2}e^{\frac{\theta}{2}} - \frac{1}{2}\left(\alpha + m\right)e^{-\frac{\theta}{2}}\,,
\end{aligned} \tag{4.17}$$

which are Hermitian conjugates of each other with respect to the inner product (4.8), and by using the equations (4.3), (4.4), (4.13) and (4.14) we can obtain the simultaneous laddering relations with respect to both parameters n and m

$$\begin{aligned}
&\mathcal{A}_+(m;\theta)\psi_{n-1,m-1}(\theta) = \sqrt{(n+\alpha)\beta}\,\psi_{n,m}(\theta)\,,\\
&\mathcal{A}_-(m;\theta)\psi_{n,m}(\theta) = \sqrt{(n+\alpha)\beta}\,\psi_{n-1,m-1}(\theta)\,,
\end{aligned} \tag{4.18}$$

$$\begin{aligned}
&\mathcal{A}_+(m;\theta)\mathcal{A}_-(m;\theta)\psi_{n,m}(\theta) = (n+\alpha)\beta\,\psi_{n,m}(\theta)\,,\\
&\mathcal{A}_-(m;\theta)\mathcal{A}_+(m;\theta)\psi_{n-1,m-1}(\theta) = (n+\alpha)\beta\,\psi_{n-1,m-1}(\theta)\,.
\end{aligned} \tag{4.19}$$

The equations (4.1) and (4.2) show that the displacement on bisector line or the lines parallel to it by means of the operators $\mathcal{A}_{\pm}(m;\theta)$ do not preserve a constant value for the energy, contrary to the 3D harmonic oscillator.

The existence of identities similar to (3.20) for the operators $A_{\pm}(m;\theta)$ and $A_{\pm}(n,m;\theta)$ does not permit us to obtain new operators which increase one of the parameters n and m, and decrease the other remaining parameter. We finally note that the scenario of simultaneous shape invariance with respect to both parameters n and m may also be extended to other quantum solvable models introduced in Refs. [6, 11, 16, 17]. Actually, simultaneous 1D shape invariance with respect to two parameters is a new type of shape invariance leading to new solutions for 1D Dirac equation in the presence of a scalar field. Moreover, they are valuable from the point of view of their transformations to 2D or 3D Schrödinger and Dirac problems. Another interesting problem is to find a representation for the parasupersymmetry algebra using this new type of shape invariance.

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