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von Neumann Quantization of Aharonov-Bohm Operator with δ Interaction: Scattering Theory, Spectral and Resonance Properties

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Abstract

Using the theory of self-adjoint extensions, we study the interaction model formally given by the Hamiltonian $H_\alpha + V(r)$, where H_α is the Aharonov-Bohm Hamiltonian and $V(r)$ is the δ-type interaction potential on the cylinder of radius R . We give the mathematical definition of the model, the self-adjointness of the Hamiltonian and provide relevant spectral properties, results for resonance effects and stationary scattering characteristics.

1 Introduction

The Aharonov-Bohm effect has received much attention in recent years [2, 5, 6, 8]. Recently, Dabrowski and Stovicek described a quantum particle interacting with a thin solenoid and a magnetic flux with point interaction [6]. In this article, using the von Neumann theory of self-adjoint (*s.a.*) extensions of linear symmetric operators [4, 3, 7, 9] we investigate such physical properties as the stationary scattering theory, the spectral and resonance properties for the non relativistic Aharonov-Bohm type Hamiltonian formally expressed in polar coordinates as

$$H_\alpha + V(r), \tag{1.1}$$

where

$$H_\alpha = -\frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}(i\frac{\partial}{\partial \phi} - \alpha)^2 \tag{1.2}$$

is the well known Aharonov-Bohm Hamiltonian acting in the Hilbert space $\mathcal{H}$;

$$V(r) = \xi\delta(r-R), \quad \text{with } \xi \in \mathbb{R},\ R > 0. \tag{1.3}$$

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In (1.2), we have fixed $\hbar = 1$, $m = 1/2$. Besides, without loss of generality, we restrict our study to the case $0 < \alpha < 1$.

2 The Model: Definition and Relevant Physical Properties

Consider the radial equation for δ - cylinder interaction deduced from (1.1) using (1.2) and (1.3), and formally given by the expression:

$$\left[-\frac{d^2}{dr^2} + \frac{(\alpha+m)^2 - 1/4}{r^2} + \xi_m \delta(r-R)\right] f_m(k,r) = k^2 f_m(k,r). \tag{2.1}$$

Then, we assume the function $f_m(k,r)$ continuous at $r = R$ as follows:

$$f_m(k, R_+) = f_m(k, R_-) \equiv f_m(k, R). \tag{2.2}$$

Integrating the equation (2.1) between $r = R-\epsilon$ and $r = R+\epsilon$ and taking the limit when $\epsilon \longrightarrow 0$, we have:

$$f'_m(k, R_+) - f'_m(k, R_-) = \xi_m f(k, R). \tag{2.3}$$

Let us consider in $L^2(\mathbb{R}^2)$ the closed and non-negative operator $\dot{H}_\alpha = \overline{H_\alpha|_{\{C_0^\infty(\mathbb{R}^2 \setminus \{\partial\overline{\Gamma(O,R)}\})\}}}$, with the domain

$$D(\dot{H}_\alpha) = \{f \in L^2(\mathbb{R}^2) \cap H^{2,2}_{loc}(\mathbb{R}^2)/f(\partial\overline{\Gamma(O,R)}) = 0,\ H_\alpha f \in L^2(\mathbb{R}^2)\}, \tag{2.4}$$

where $H^{m,n}_{loc}(\Omega)$ is the local Sobolev space of indices (m,n). Let us now decompose the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^2)$, $L^2(\mathbb{R}^2) = L^2(\mathbb{R}^+) \bigotimes L^2(S^1)$, S^1 being the unit circle in $\mathbb{R}^2$. The isomorphism U is then introduced in order to remove the weight factor r from the measure:

$$U : \left\{ \begin{array}{c} L^2((0,\infty); r dr) \longrightarrow L^2((0,\infty); dr) \equiv L^2((0,\infty)) \\ f \longmapsto (Uf)(r) = \sqrt{r} f(r), \end{array} \right. \tag{2.5}$$

so that we get the following decomposition of $L^2(\mathbb{R}^2)$:

$$L^2(\mathbb{R}^2) = \bigoplus_{m=-\infty}^{m=+\infty} U^{-1}(L^2(\mathbb{R}^+)) \bigotimes \left[\frac{e^{im\phi}}{\sqrt{2\pi}}\right],\ m \in \mathbb{Z}. \tag{2.6}$$

Provided this decomposition $\dot{H}_\alpha = \bigoplus_{m=-\infty}^{m=+\infty} U^{-1} \dot{h}_{\alpha,m} U \bigotimes \mathbb{1}$, where the operator $\dot{h}_{\alpha,m}$ in $L^2(]0,\infty[)$ is defined by

$$\dot{h}_{\alpha,m} = -\frac{d^2}{dr^2} + \frac{(\alpha+m)^2 - 1/4}{r^2}, \tag{2.7}$$

with the domain

$$\begin{aligned} \mathcal{D}(\dot{h}_{\alpha,m}) \ = \ & \Big\{ f \in L^2(]0,\infty[, dr) \cap H^{2,2}_{loc}(]0,\infty[)\,; \\ & f(0_+) = 0 \text{ if } (\alpha+m)^2 - 1/4 = 0; f(R_\pm) = 0\,; \\ & -f'' + ((\alpha+m)^2 - \frac{1}{4}) r^{-2} f \in L^2((0,\infty)) \Big\},\ m \in\ Z. \end{aligned} \tag{2.8}$$

The adjoint operator $\dot{h}^*_{\alpha,m}$ of $\dot{h}_{\alpha,m}$ is defined by

$$\dot{h}^*_{\alpha,m} = -\frac{d^2}{dr^2} + \frac{(\alpha+m)^2 - 1/4}{r^2},$$

with the domain

$$\begin{aligned} D(\dot{h}^*_{\alpha,m}) \;=\; & \{f \in L^2(]0,\infty[, dr) \cap H^{2,2}_{loc}(]0,\infty[-\{R\}); f(0_+) = 0 \text{ if} \\ & (\alpha+m)^2 - 1/4 = 0; f(R_+) = f(R_-) \equiv f(R); \\ & \left(-\frac{d^2}{dr^2} + \frac{(\alpha+m)^2 - 1/4}{r^2}\right) f \in L^2(]0,\infty[)\},\quad m \in \mathbb{Z}. \end{aligned} \tag{2.9}$$

Consequently, we obtain $\dot{H}^*_\alpha = \bigoplus_{m=-\infty}^{m=+\infty} U^{-1}\dot{h}^*_{\alpha,m}U \bigotimes \mathbb{1}$. The indicial equation reads $h^*_{\alpha,m} f_m(k,r) = k^2 f_m(k,r)$, or equivalently

$$\left[-\frac{d^2}{dr^2} + \frac{(\alpha+m)^2 - 1/4}{r^2}\right] f_m(k,r) = k^2 f_m(k,r). \tag{2.10}$$

Next, selecting, in the two-dimensional space of solutions, the solution which vanishes at the point $r = 0$ and satisfies the boundary conditions (2.2) at $r = R$, we arrive at the function

$$f_{|\alpha+m|}(k,r) = \begin{cases} G^{(0)}_{|\alpha+m|-\frac{1}{2}}(k,R) \times F^{(0)}_{|\alpha+m|-\frac{1}{2}}(k,r)\ ; & r \le R\ , \\ F^{(0)}_{|\alpha+m|-\frac{1}{2}}(k,R) \times G^{(0)}_{|\alpha+m|-\frac{1}{2}}(k,r)\ ; & r \ge R\ , \end{cases} \tag{2.11}$$

where

$$\begin{aligned} F^{(0)}_\nu(k,r) &= \left(\tfrac{k}{2}\right)^{-\nu-\frac{1}{2}} \Gamma\left(\nu+\tfrac{3}{2}\right) r^{\frac{1}{2}} J_{\nu+\frac{1}{2}}(kr)\ , \\ G^{(0)}_\nu(k,r) &= \tfrac{-i\pi}{2}\tfrac{1}{\Gamma(\nu+\frac{3}{2})}\left(\tfrac{k}{2}\right)^{\nu+\frac{1}{2}} r^{\frac{1}{2}} H^{(2)}_{\nu+\frac{1}{2}}(kr)\ . \end{aligned} \tag{2.12}$$

$J_l(z)$ and $H^{(2)}_l(z)$ are the Bessel and Hankel functions of order l, respectively [1]. Putting (2.12) into (2.11), we get

$$f_{|\alpha+m|}(k,r) = \begin{cases} \frac{i\pi}{2} R^{1/2} H^{(2)}_{|\alpha+m|}(kR) r^{1/2} J_{|\alpha+m|}(kr)\ ; & r \le R\ , \\ \frac{i\pi}{2} R^{1/2} J_{|\alpha+m|}(kR) r^{1/2} H^{(2)}_{|\alpha+m|}(kr)\ ; & r \ge R\ . \end{cases} \tag{2.13}$$

Since the indicial equation admits one solution, $\dot{h}_{\alpha,m}$ has deficiency indices $(1,1)$ and, consequently, all self-adjoint (s.a) extensions of $\dot{h}_{\alpha,m}$ are given by a 1-parameter family of (s.a.) operators [4] which is defined by

$$h_{\alpha,m,\xi_m} = -\frac{d^2}{dr^2} + \frac{(\alpha+m)^2 - 1/4}{r^2},$$

with the domain

$$\begin{aligned} D(h_{\alpha,m,\xi_m}) \;=\; & \{f \in L^2(]0,\infty[, dr) \cap H^{2,2}_{loc}(]0,\infty[\backslash\{R\}); f(0_+) = 0 \text{ if} \\ & (\alpha+m)^2 - 1/4 = 0; f(R_+) = f(R_-) \equiv f(R);\ f'(R_+) - f'(R_-) = \xi_m f(R); \\ & \left(-\frac{d^2}{dr^2} + \frac{(\alpha+m)^2 - 1/4}{r^2}\right) f \in L^2(]0,\infty[)\}, \end{aligned} \tag{2.14}$$

$m \in \mathbb{Z}$, $-\infty < \xi_m \leq +\infty$. The case $\xi_m = 0$ coincides with the free kinetic energy Hamiltonian $\dot{h}_{\alpha,m,0}$ for fixed quantum number m. Let $\xi = \{\xi_m\}_{m\in\mathbb{Z}}$ and introduce in $L^2(\mathbb{R}^2)$ the operator

$$H_{\alpha,\xi} = \bigoplus_{m=-\infty}^{m=+\infty} U^{-1} h_{\alpha,m,\xi_m} U \bigotimes \mathbb{1}. \tag{2.15}$$

By definition, $H_{\alpha,\xi}$ is the rigorous mathematical formulation of the formal expression (1.1). Actually, it provides a slight generalization of (1.1), since ξ may depend on $m \in \mathbb{Z}$.

2.1 The resolvent equation

We get the following:

Theorem 1. *(i) The resolvent of h_{α,m,ξ_m} is given by*

$$(h_{\alpha,m,\xi_m} - k^2)^{-1} = (h_{\alpha,m,0} - k^2)^{-1} + \mu_m(k) \left(f_{|\alpha+m|}(-\overline{k}), .\right) f_{|\alpha+m|}(k), \tag{2.16}$$

$k^2 \in \rho(\dot{h}_{\alpha,m,\xi_m})$, $\mathcal{I}m(k) > 0$; $m \in \mathbb{Z}$, where $\mu_m(k) = -\xi_m[1 + \xi_m g_{m,k}(R,R)]^{-1}$ and $(h_{\alpha,m,0} - k^2)^{-1}$, is the free resolvent with integral kernel

$$g_{m,k}(r,r') = \begin{cases} G^{(0)}_{|\alpha+m|-\frac{1}{2}}(k,r) \times F^{(0)}_{|\alpha+m|-\frac{1}{2}}(k,r') \, ; & r' \leq r \, , \\ F^{(0)}_{|\alpha+m|-\frac{1}{2}}(k,r) \times G^{(0)}_{|\alpha+m|-\frac{1}{2}}(k,r') \, ; & r' \geq r \, . \end{cases} \tag{2.17}$$

We note that $g_{m,k}(R,r) = f_{|\alpha+m|}(k,r)$, $\mathcal{I}m(k) > 0$.
(ii) The resolvent of $H_{\alpha,\xi}$ is given by

$$\left(H_{\alpha,\xi} - k^2\right)^{-1} = \left(H_{\alpha,0} - k^2\right)^{-1} + \bigoplus_{m=-\infty}^{m=+\infty} \mu_m(k) \left(|.|^{-1} f_{|\alpha+m|}(-\overline{k}) \frac{e^{im\phi}}{\sqrt{2\pi}}, . \right) |.|^{-1} f_{|\alpha+m|}(k) \frac{e^{im\phi}}{\sqrt{2\pi}}, \tag{2.18}$$

$k^2 \in \rho(H_{\alpha,\xi})$, $\mathcal{I}m(k) > 0$.

Theorem 2. *The domain $D(h_{\alpha,m,\xi_m})$ consists of functions of the type $\psi_m(k,r) = F_{\alpha,m}(k,r) + \mu_m(k) F_{\alpha,m}(k,R) g_{m,k}(R,r)$, $F_{\alpha,m} \in D(h_{\alpha,m,0})$ and $k^2 \in \rho(h_{\alpha,m,\xi_m})$, $\mathcal{I}m(k) > 0$. This decomposition is unique and with $\psi_m \in D(h_{\alpha,m,\xi_m})$ of this form, we obtain $(h_{\alpha,m,\xi_m} - k^2)\psi_m = (h_{\alpha,m,0} - k^2) F_{\alpha,m}$.*

Proof. One may follow step by step [3], where a similar result was obtained for point interaction. ■

2.2 Spectral properties

Spectral properties of h_{α,m,ξ_m} are provided by the following theorem where $\sigma(.)$, $\sigma_{ess}(.)$, $\sigma_{ac}(.)$, $\sigma_{sc}(.)$ and $\sigma_p(.)$ denote the spectrum, essential spectrum, absolutely continuous spectrum, singularly continuous spectrum and point spectrum, respectively.

Theorem 3. *For all $\xi_m \in (-\infty, \infty)$, $\sigma_{ess}(h_{\alpha,m,\xi_m}) = \sigma_{ac}(h_{\alpha,m,\xi_m}) = [0, \infty)$, $\sigma_{sc}(h_{\alpha,m,\xi_m}) = \emptyset$, $\sigma_p(h_{\alpha,m,\xi_m}) \cap [0, \infty) = \emptyset$. The negative eigenvalues of h_{α,m,ξ_m} are obtained from the equation $1 + \xi_m g_{m,i\sqrt{-E}}(R,R) = 0$, $E < 0$, which has at most one solution $E_0 < 0$.*

2.3 Resonances of h_{α,m,ξ_m}

Using the boundary conditions, the resolvent equation is given by

$$(h_{\alpha,m,\xi_m} - k^2)^{-1} = (h_{\alpha,m,0} - k^2)^{-1} - \xi_m[1 + \xi_m g_{m,k}(R,R)]^{-1} \left(f_{\alpha,m}(-\overline{k}), .\right) f_{\alpha,m}(k)$$

$k^2 \in \rho(h_{\alpha,m,\xi_m})$, $\mathcal{I}m(k) > 0$; $m \in \mathbb{Z}$. The resonance equation is then $1+\xi_m g_{m,k}(R,R) = 0$, or equivalently $1 - \xi_m i\frac{\pi}{2}RH^{(2)}_{|\alpha+m|}(kR)J_{|\alpha+m|}(kR) = 0$. This equation generates an infinite set of resonances off the imaginary axis for h_{α,m,ξ_m}, whatever the partial wave characterized by the quantum number m and ξ_m [8]. The only difficulties one encounters to prove this statement arise from the complexity of the mathematical expressions that appear more and more less treatable analytically as the quantum number m raises. For instance, taking just $\alpha = 1/2$ and $m = 0$ or -1 leads to an intricate nonlinear system. Numerical computations allow to go round these difficulties.

2.4 Stationary Scattering Theory for the pair $(h_{\alpha,m,\xi_m};h_{\alpha,m,o})$

The phase shifts of h_{α,m,ξ_m} may be obtained through the asymptotic behavior of $\mathcal{F}_{m,\alpha,\xi_m}(k,r)$ as $r \longrightarrow \infty$. So doing, we get

$$\begin{aligned}
\mathcal{F}_{m,\alpha,\xi_m}(k,r) \quad \underset{r\to\infty}{\overset{k>0}{\longrightarrow}} \quad & A_m(k)\sin\left(kr - \frac{\pi(|\alpha+m|-1/2)}{2}\right) + \\
& +\mu_m(k)F^{(0)}_{|\alpha+m|-1/2}(k,R)F^{(0)}_{|\alpha+m|-1/2}(k,r)B_m(k) \\
& \times \exp\left[-i\left(kr - \frac{\pi(|\alpha+m|-1/2)}{2}\right)\right] \\
= \quad & \left[C^2_{1,m}(k) + C^2_{2,m}(k)\right]^{1/2}\sin\left(kr - \frac{\pi(|\alpha+m|-1/2)}{2} + \delta_{m,\xi_m}(k)\right) \\
& +o(1) \qquad (2.19)
\end{aligned}$$

and the phase shifts express as

$$\delta_{m,\xi_m}(k) = -\arctan\frac{B_m(k)\mu_m(k)(F^{(0)}_{|\alpha+m|-1/2}(k,R))^2}{A_m(k) - iB_m(k)\mu_m(k)(F^{(0)}_{|\alpha+m|-1/2}(k,R))^2} \qquad (2.20)$$

where $A_m(k) = 2^{-(|\alpha+m|-1/2)}(k^{-|\alpha+m|-1/2})\Gamma(2|\alpha+m|+1)\Gamma(|\alpha+m|+1/2)^{-1}$ and $B_m(k) = 1/(kA_m(k))$. The corresponding on-shell scattering matrix is defined by

$$S_{m,\xi_m}(k) = 1 - 2ikB^2_m(k)\mu_m(k)(F^{(0)}_{|\alpha+m|-1/2}(k,R))^2, \qquad (2.21)$$

while the on-shell scattering amplitude $f_\xi(k,\omega,\omega')$ corresponding to $H_{\alpha,\xi}$ is given by

$$f_\xi(k,\omega,\omega') = 4\pi\sum_{m=-\infty}^{m=+\infty} f_{m,\xi_m}(k)[\frac{e^{-im\omega'}}{\sqrt{2\pi}}][\frac{e^{im\omega}}{\sqrt{2\pi}}], \qquad (2.22)$$

$k \geq 0 \quad ; \quad \omega\,,\,\omega' \in S^1$. The partial wave scattering amplitude $f_{m,\xi_m}(k)$ reads

$$f_{m,\xi_m}(k) = -B_m^2(k)\mu_m(k)(F^{(0)}_{|\alpha+m|-1/2}(k,R))^2. \tag{2.23}$$

The on-shell scattering operator $S_\xi(k)$ in $L^2(S^1)$ corresponding to $H_{\alpha,\xi}$ is defined by

$$(S_\xi(k)\phi)(\omega) = \phi(\omega) - \frac{k}{2\pi i}\int_{S^1} d\omega' f_\xi(k,\omega,\omega')\phi(\omega'), \tag{2.24}$$

$k \geq 0 \quad ; \quad \omega\,,\,\omega' \in S^1$, $S_\xi(k) = 1 + 2ik\sum_{m=-\infty}^{m=+\infty} f_{m,\xi_m}(k)\left(\frac{e^{im(.)}}{\sqrt{2\pi}},.\right)\frac{e^{im(w)}}{\sqrt{2\pi}}$.

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