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On the Spectral Problem Associated with the Camassa-Holm Equation

Christer BENNEWITZ

Department of Mathematics, Lund University, Box 118, SE-221 00 Lund, Sweden
E-mail: christer.bennewitz@math.lu.se

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Abstract

We give a basic uniqueness theorem in the inverse spectral theory for a Sturm-Liouville equation with a weight which is not of one sign. It is shown that the theorem may be applied to the spectral problem associated with the Camassa-Holm integrable system which models shallow water waves.

1 Introduction

The Camassa-Holm (CH) equation

$$\Phi_t - \Phi_{txx} + 3\Phi\Phi_x + 2\varkappa\Phi_x = 2\Phi_x\Phi_{xx} + \Phi\Phi_{xxx},$$

where $\varkappa$ is a parameter, was first derived by Fokas and Fuchssteiner [12] as an example of an equation with a bi-Hamiltonian structure, and then as a model for shallow water waves by Camassa and Holm [5]; see also Johnson [13]. It was also derived as a model for hyper-elastic rods by Dai [11], and shown to describe geodesic flow on the diffeomorphism group of the circle by Misiolek [15]; see also Constantin and Kolev [7, 8].

Associated with CH there is the spectral problem

$$-u'' + \tfrac{1}{4}u = \lambda w(\cdot, t)u, \tag{1.1}$$

where $w = \Phi - \Phi_{xx} + \varkappa$ and t is just considered a parameter. One would like to copy the scattering—inverse scattering approach for the KdV equation to this situation. Since the spectral problem is no longer the Schrödinger equation there are a number of complications. In particular, one of the more interesting features of the CH equation is the presence of wave breaking. This phenomenon is discussed in Constantin and Escher [6]. It is known, however, that wave breaking will not occur if $w(x, 0) \geq 0$; see Constantin [10]. Thus one should not assume that $w \geq 0$ in (1.1). Standard spectral theory, however, considers (1.1) in an L^2-space with weight w, which is then not possible.

One should instead use $H^1(\mathbb{R})$ as the Hilbert space for (1.1), provided with the slightly modified scalar product

$$\langle u, v\rangle = \int_{\mathbb{R}} (u'\overline{v'} + \tfrac{1}{4}u\overline{v}), \qquad \|u\| = \sqrt{\langle u, u\rangle}.$$

There is then a complete spectral theory for (1.1), and we will sketch this in Section 4.

The plan of the paper is as follows: In Section 2 we briefly discuss what is known about scattering for (1.1). In Section 3 is introduced the selfadjoint operator in $H^1(\mathbb{R})$ which corresponds to (1.1); it is shown to have discrete spectrum if $\varkappa = 0$. Section 4 sketches a general spectral theory for equations like (1.1) which need to be realized in a Hilbert space with a norm of Sobolev type. More details may be found in [1] and, especially, [2]. An inverse spectral theory for such equations is developed in Section 5, and in Section 6 are proved some theorems of Paley-Wiener type which are crucial for the inverse spectral theory. Finally, in Section 7 we briefly describe one way to apply this theory to the Camassa-Holm spectral problem.

2 Scattering

We will only consider solutions of CH which decay sufficiently fast at infinity, so that the coefficient w in (1.1) at least satisfies

$$(1+|x|)(w(x)-\varkappa) \in L^1(\mathbb{R}). \tag{2.1}$$

We will consider both the case $\varkappa = 0$ and the case $\varkappa \neq 0$. In the latter case a simple scaling argument shows that we may as well restrict ourselves to the case $\varkappa = 1$. For the moment we will ignore time dependence in w. There are good reasons to allow non-integrable local singularities in w, at least permitting w to be a measure, possibly even $w \in H^{-1}_{\text{loc}}$, but we will not do this here.

If $w \equiv \varkappa$ the solutions of (1.1) are linear combinations of $e^{\pm ikx}$ where $k = \sqrt{\lambda\varkappa - \frac{1}{4}}$, choosing the branch of the root with argument in $[0, \pi)$. For $\varkappa = 0$ this means that $ik = -1/2$ so that the solutions are linear combinations of $e^{\pm x/2}$.

For $\varkappa = 1$ we have instead $ik < 0$ if $\lambda < \frac{1}{4}$ and $k > 0$ if $\lambda > \frac{1}{4}$, so that for $\lambda > \frac{1}{4}$ we have bounded, oscillatory solutions, whereas for $\lambda < \frac{1}{4}$ there are solutions exponentially increasing at $-\infty$ and exponentially decaying at $+\infty$ and *vice versa*.

A standard consequence of (2.1) is that there is a solution f_+ of (1.1) which is asymptotic to e^{ikx} as $x \to \infty$ and a solution f_- which is asymptotic to e^{-ikx} as $x \to -\infty$; they may be constructed by successive approximations. These are the 'small' solutions except when $k > 0$, in which case also $\overline{f_+}$ and $\overline{f_-}$ are solutions, linearly independent of f_+ and f_- respectively. This occurs if $\varkappa = 1$ and $\lambda > \frac{1}{4}$.

Since three solutions are always linearly dependent, we may in the case $k > 0$ write

$$T(k)f_-(x,k) = \overline{f_+(x,k)} + R_+(k)f_+(x,k),$$

where T is called the *transmission coefficient* and R_+ the *reflection coefficient*.

For certain λ for which $ik < 0$ it may happen that f_- and f_+ are linearly dependent, in which case $f_- \in H^1(\mathbb{R})$; then λ is an *eigenvalue* of (1.1) and we define $c = \|f_-\|^{-2}$, called the *normalization constant* for λ.

Assuming (2.1) and $\varkappa = 0$ eigenvalues can only accumulate at $\pm\infty$, so we may number eigenvalues increasingly with $\lambda_n > 0$ if $n > 0$ and $\lambda_n < 0$ if $n < 0$, with corresponding normalization constants c_n. The sequence (λ_n, c_n), is called the *scattering data* for (1.1) in the case $\varkappa = 0$.

Similarly, assuming (2.1) and $\varkappa = 1$ eigenvalues can only accumulate at $-\infty$, so we may number them in decreasing order as $\lambda_1, \lambda_2, \dots$, with corresponding normalization constants $c_1, c_2, \dots$. The finite or infinite sequence (λ_n, c_n), $n = 1, 2, \dots$, together with T and R_+ is called the *scattering data* for (1.1) in the case $\varkappa = 1$.

Now suppose $w = \Phi - \Phi_{xx} + \varkappa$ where Φ is a sufficiently rapidly decaying solution of the CH equation. Thus the scattering data now also depend on time. It is known that

- Eigenvalues are conserved quantities, *i.e.*, independent of t.
- The transmission coefficient T is a conserved quantity.
- Normalization constants evolve according to

$$c_n(t) = c_n(0) \exp(-\frac{\sqrt{\frac{1}{4} - \lambda_n \varkappa}}{2\lambda_n} t).$$

- The reflection coefficient R_+ evolves according to

$$R_+(k, t) = R_+(k, 0) \exp(\frac{ik}{\lambda} t), \quad k > 0.$$

At this point one would need an inverse scattering theory for (1.1), but as yet no such thing is available. If $w \geq 0$ and w is sufficiently smooth one may use the known inverse scattering theory for the Schrödinger equation by transforming (1.1) to a Schrödinger equation. This was carried out by Constantin [10] and Lenells [14], but will not work if w changes sign. There does, however, exist a spectral theory and some inverse spectral theory for (1.1). Some conclusions may be drawn from this, as we shall see.

3 A Hilbert space operator

If $u \in H^1(\mathbb{R})$ we have $u(x) = \langle u, G(\cdot - x) \rangle$ where $G(x) = e^{-|x|/2} \in H^1(\mathbb{R})$ with $\|G\| = 1$. It follows that if $u_j \rightharpoonup 0$ weakly in $H^1(\mathbb{R})$, then $u_j \to 0$ pointwise and, since we have $|u(x)| \leq \|G\| \|u\| = \|u\|$, boundedly. Consider first the case $\varkappa = 0$ so that $w \in L^1(\mathbb{R})$. If we multiply by w and integrate, writing $\|w\|_1 = \int_{\mathbb{R}} |w|$ we obtain

$$\left| \int_{\mathbb{R}} u\overline{v}w \right| \leq \|w\|_1 \|u\| \|v\|,$$

so that the sesqui-linear form $\int_{\mathbb{R}} u\overline{v}w$ is bounded on $H^1(\mathbb{R})$. It follows that the linear form $u \mapsto \int_{\mathbb{R}} u\overline{v}w$ is bounded on $H^1(\mathbb{R})$ for any $v \in H^1(\mathbb{R})$, so by the Riesz representation theorem there exists an operator T on $H^1(\mathbb{R})$, bounded by $\|w\|_1$ and selfadjoint since w is real-valued, such that

$$\int_{\mathbb{R}} u\overline{v}w = \langle u, Tv \rangle.$$

It is clear that T is compact, since $\|Tu\|^2 = \int_{\mathbb{R}} (Tu)\overline{u}w \leq \|Tu\| \int_{\mathbb{R}} |u||w|$ so that

$$\|Tu_j\| \leq \int_{\mathbb{R}} |u_j||w| \to 0$$

by dominated convergence if $u_j \rightharpoonup 0$ weakly. It is easy to see that T has an (unbounded) inverse if $\operatorname{supp} w = \mathbb{R}$. We have $\int (Tv)'\varphi' = \int (wv - Tv/4)\varphi$ if $\varphi \in H^1$ by the definition of T, so in the distributional sense $-(Tv)'' + Tv/4 = wv$. Thus $u = \lambda Tu$ if and only if $u \in H^1(\mathbb{R})$ and satisfies (1.1), so in this case (1.1) generates an operator in $H^1(\mathbb{R})$ with discrete spectrum.

Now consider the case $\varkappa = 1$. Since $\int |u|^2 \leq 4\|u\|^2$ the sesqui-linear form $\int_{\mathbb{R}} u\overline{v}w$ generates a bounded operator also in this case. This operator is no longer compact but has continuous spectrum in $[0, 4]$. The inverse operator, which exists as an unbounded operator if $\operatorname{supp} w = \mathbb{R}$, thus has discrete spectrum below $1/4$ and (absolutely) continuous spectrum equal to $[1/4, \infty)$.

There are only finitely many eigenvalues if and only if $w \geq 0$ a.e. If $w < 0$ on a set with positive measure and w_- is the negative part of w, one may by rather standard methods (Prüfer transform) prove the asymptotic formula

$$\lambda_n = -\Big\{ \frac{n\pi}{\int_{\mathbb{R}} \sqrt{w_-}} \Big\}^2 (1 + o(1))$$

as $n \to \infty$. Here $\sqrt{w_-} \leq \frac{1}{2}|w-1|$, so that the square root is integrable. Since all eigenvalues are conserved quantities we obtain as a byproduct the following theorem, which was also proved in Constantin and McKean [9].

Theorem 3.1. *In the case $\varkappa = 1$ the integral $\int_{\mathbb{R}} \sqrt{w_-}$ is a conserved quantity under the Camassa-Holm flow. Similarly, if $\varkappa = 0$ the integrals $\int_{\mathbb{R}} \sqrt{w_-}$ and $\int_{\mathbb{R}} \sqrt{w_+}$ are conserved quantities.*

4 Spectral theory

Here we will sketch a general spectral theory for equations of the form

$$-(pu')' + qu = \lambda wu \quad \text{ in } [0, b), \tag{4.1}$$

where $0 < b \leq +\infty$, $p \geq 0$, $q \geq 0$ and $1/p$, q and w are all in $L^1_{\mathrm{loc}}[0, b)$. It is also assumed that $q \not\equiv 0$ and that $w \neq 0$ a.e.

We shall study the equation in the space $\mathcal{H}$ which is the completion of $C^1(0, b)$-functions which are *finite, i.e.*, vanish in a left neighborhood of b, with respect to the norm-square $\|u\|^2 = \int_0^b (p|u'|^2 + q|u|^2)$. We will also study the equation in the space $\mathcal{H}_0$, which is the subspace of $\mathcal{H}$ of codimension 1 obtained by completing $C^1_0(0, b)$ in the same norm. In order to avoid some technicalities we also assume that the form $\int_0^b u\overline{v}w$ is a bounded, hermitian form on $\mathcal{H}$, so that, like in Section 3, there exists a bounded operator T such that for $u, v \in \mathcal{H}$ we have $\int_0^b u\overline{v}w = \langle Tu, v\rangle$ and which is the inverse of a selfadjoint operator P, which in turn is a selfadjoint realization in $\mathcal{H}$ of the differential operator corresponding to (4.1). It is easy to see that all functions u in the domain of P satisfy the boundary condition $pu'(0) = 0$, a Neumann type condition. We denote the resolvent of $P = T^{-1}$ by R_λ. Similarly we obtain a differential operator P_0 in $\mathcal{H}_0$, which now corresponds to the Dirichlet boundary condition $u(0) = 0$.

The Cauchy-Schwarz inequality gives

$$|u(x)| \leq |u(y)| + \Big|\int_y^x u'\Big| \leq |u(y)| + \Big|\int_y^x 1/p\Big|^{1/2} \Big(\int_0^b p|u'|^2\Big)^{1/2},$$

and multiplying by $q(y)$, integrating with respect to y over an interval J such that $\int_J q > 0$ and using Cauchy-Schwarz again gives

$$|u(x)| \le C_x \|u\|,$$

with $C_x \le \big((\int_J q)^{-1} + \int_0^y 1/p\big)^{1/2}$ if $y = \max(x, \sup J)$. Thus pointwise evaluations are bounded linear forms on $\mathcal{H}$.

By Riesz' representation theorem it also follows that there exists a kernel $g(x, y, \lambda)$ which for fixed x and $\lambda \in \mathbb{C} \setminus \mathbb{R}$ is in $\mathcal{H}$, and such that

$$R_\lambda u(x) = \langle u, \overline{g(x, \cdot, \lambda)}\rangle - u(x)/\lambda. \tag{4.2}$$

We will call this kernel *Green's function* for the differential operator P, although one should note that, since the Hilbert space is not L^2, the operator in (4.2) is not a standard integral operator. Nevertheless, it is now a fairly standard matter to see that

$$g(x, y, \lambda) = \varphi(\min(x, y), \lambda)\psi(\max(x, y), \lambda),$$

where $\varphi(x, \lambda)$ is the solution of (4.1) satisfying initial data $\varphi(0, \lambda) = -1/\lambda$, $p\varphi'(0, \lambda) = 0$, and $\psi(x, \lambda)$ is another solution, the *Weyl solution*, which is uniquely determined by the facts that $y \mapsto g(x, y, \lambda)$ is in $\mathcal{H}$ and the Wronskian $p\varphi'\psi - p\psi'\varphi = 1/\lambda$. In fact, if $\theta(x, \lambda)$ is the solution with initial data $\theta(0, \lambda) = 0$, $p\theta'(0, \lambda) = 1$, there is a unique function $m(\lambda)$, analytic in $\mathbb{C} \setminus \mathbb{R}$, satisfying $\overline{m(\overline{\lambda})} = m(\lambda)$, and such that $\psi(x, \lambda) = \theta(x, \lambda) + m(\lambda)\varphi(x, \lambda)$. From $R_\lambda - R_\mu = (\lambda - \mu)R_\lambda R_\mu$ one easily obtains $\|\psi(\cdot, \lambda)\|^2 = \operatorname{Im} m(\lambda)/\operatorname{Im} \lambda$, so that $m(\lambda)$ maps the upper and lower half-planes into themselves. Altogether this means that $m(\lambda)$ is a so called Nevanlinna or Herglotz function. Such functions have a unique representation

$$m(\lambda) = \alpha + \beta\lambda + \int_{-\infty}^{\infty} \Big(\frac{1}{t - \lambda} - \frac{t}{t^2 + 1}\Big)\, d\rho(t), \tag{4.3}$$

where $\alpha \in \mathbb{R}$, $\beta \ge 0$ and $d\rho$, the *spectral measure*, is a positive measure with $\int_{-\infty}^{\infty} \frac{d\rho(t)}{t^2+1} < \infty$. Thus $m(\lambda)$ is completely analogous to the so called Titchmarsh-Weyl m-function which is defined when $w \equiv 1$ and one considers the equation (4.1) in the Hilbert space $L^2(0, b)$.

One can carry out exactly the same procedure for the operator P_0 and find a corresponding m-function m_0 with a corresponding spectral measure $d\rho_0$, but for the moment we will concentrate on the Neumann case.

Now consider L^2_ρ, the L^2-space over $\mathbb{R}$ with respect to the measure $d\rho$. We get a generalized Fourier transform for the equation (4.1) in the following way. For a finite function $u \in \mathcal{H}$ we set $\hat{u}(\lambda) = \langle u, \overline{\varphi(\cdot, \lambda)}\rangle$ which is seen to be an entire function of λ by an integration by parts. If also $v \in \mathcal{H}$ is finite it is easily seen that $\langle R_\lambda u, v\rangle - \hat{u}(\lambda)\overline{\hat{v}(\overline{\lambda})}m(\lambda)$ is an entire function. If we integrate this around an axis-parallel rectangle, intersecting $\mathbb{R}$ in an interval Δ, we get from (4.3), taking some care in interchanging the order of integration,

$$\langle E_\Delta u, v\rangle = \int_\Delta \hat{u}\overline{\hat{v}}\, d\rho,$$

where the left hand side comes from the spectral theorem and E_Δ is the spectral projector associated with Δ. This quite easily leads to the following concrete version of the spectral theorem (for more details see also [1] and especially [2]).

Theorem 4.1. *For $u \in \mathcal{H}$ the integral*

$$\mathcal{F}(u)(t) = \hat{u}(t) = \int_0^b (pu'\varphi'(\cdot,t) + qu\varphi(\cdot,t))$$

converges in L^2_ρ and gives a unitary map $\mathcal{F} : \mathcal{H} \ni u \mapsto \hat{u} \in L^2_\rho$ with inverse given by $u(x) = \int_{\mathbb{R}} \hat{u}(t)\varphi(x,t)\,d\rho(t)$, the integral converging in $\mathcal{H}$, and locally uniformly. We have $u \in \mathcal{H}$ in the domain of P precisely if $t\hat{u}(t) \in L^2_\rho$, and then $\mathcal{F}(Pu)(t) = t\hat{u}(t)$. Finally, if $M \subset \mathbb{R}$ is a Borel set and E_M the corresponding spectral projector for P, then $E_M u(x) = \int_M \hat{u}(t)\varphi(x,t)\,d\rho(t)$

It follows that the spectrum of the operator P is $\sigma(P) = \operatorname{supp} d\rho$, eigenvalues corresponding to point-masses in the measure. If the spectrum is discrete with eigenvalues λ_n and we introduce the normalization constants $c_n = \|\varphi(\cdot,\lambda_n)\|^{-2}$, it is clear that $d\rho = \sum c_n\delta_{\lambda_n}$ where δ_a is the Dirac measure at a. Thus, in this case knowing the spectral measure is equivalent to knowing all eigenvalues and normalization constants.

It is easy to see that the Fourier transform of the Weyl solution $\psi(\cdot,\lambda)$ is $\hat{\psi}(t,\lambda) = 1/(t-\lambda)$. It follows that the Weyl solution tends to 0 in $\mathcal{H}$, and thus locally uniformly, as $\lambda \to \infty$ along non-real rays originating at the origin. Since u is in the domain of P precisely if $u = R_\lambda v$ for some $v \in \mathcal{H}$, and the Fourier transform of $R_\lambda v$ is $\hat{v}(t)/(t-\lambda)$ it also follows that $\hat{u}$ is integrable $(d\rho)$ if u is in the domain of P. We obtain the following lemma.

Lemma 4.2.

1. *$\mathcal{F}(\psi(\cdot,\lambda))(t) = 1/(t-\lambda)$.*
2. *As $\lambda \to \infty$ along any non-real ray originating in the origin we have $\psi(\cdot,\lambda) \to 0$, as an element of $\mathcal{H}$ and locally uniformly.*
3. *$\hat{u}$ is integrable $(d\rho)$ for any u in the domain of P.*
4. *If $u \in \mathcal{H}$, then $u(0) = 0$ precisely if $\langle u, \psi(\cdot,0)\rangle = 0$.*

5 Inverse spectral theory

We shall deal with the following question: *To what extent is the operator P,* i.e., *the interval $[0,b)$ and the coefficients p, q and w, determined by the spectral measure $d\rho$?* We will also consider the same question for the operator P_0. To answer these questions we introduce the concept of a *Liouville transform* as a map $v \mapsto u$ given by $u(x) = f(x)v(g(x))$, where f and g are fixed functions defined in $[0,b)$. We suppose that g is locally absolutely continuous with $g' > 0$ a.e., and that f is at least measurable and $\neq 0$ a.e. It is then easy to see that the inverse of a Liouville transform is also a Liouville transform, as is the composition of two Liouville transforms. Now consider another operator $\breve{P}$ of the same type as P, with interval $[0,\breve{b})$ and coefficients $\breve{p}$, $\breve{q}$ and $\breve{w}$. We define the functions $h(x) = \int_0^x \sqrt{|w|/p}$ on $[0,b)$ and $\breve{h}(x) = \int_0^x \sqrt{|\breve{w}|/\breve{p}}$ on $[0,\breve{b})$ respectively. Our main theorem is the following.

Theorem 5.1. *Suppose that P and $\breve{P}$ have the same spectral measure $d\rho$. Then there is a Liouville transform taking $\breve{P}$ into P, where $g(x) = \breve{h}^{-1}\circ h(x)$ and $f(x) = (\breve{p}(g(x))|\breve{w}(g(x))|/p(x)|w(x)|)^{1/4}$.*

The function f is locally absolutely continuous, strictly positive and $f(0) = 1$, and pf' is also locally absolutely continuous with $pf'(0) = 0$. Exactly the same statements are true for two Dirichlet operators P_0 and $\breve{P}_0$ with the same spectral measure $d\rho_0$.

Sufficient additional information will imply that P and $\breve{P}$ are identical. We give two corollaries of this type.

Corollary 5.2. *Suppose P and $\breve{P}$ have the same spectral measure and that $p = \breve{p}$, $|w| = |\breve{w}|$ in $[0, \min(b, \breve{b}))$. Then $P = \breve{P}$,* i.e., *$b = \breve{b}$, $q = \breve{q}$ and $w = \breve{w}$. The same statement is true for Dirichlet operators P_0 and $\breve{P}_0$.*

Proof. The assumptions together with Theorem 5.1 show that $g(x) = x$ and that $f(x) = 1$, so that P and $\breve{P}$, respectively P_0 and $\breve{P}_0$, are identical. ■

Note that only the absolute value of w need be known, so that all information about sign changes in w is encoded in the spectral measure. Also note that if $p = \breve{p}$, $|w| = |\breve{w}|$ only in $[0, a)$ where $0 < a < \min(b, \breve{b})$ we still have $q = \breve{q}$ and $w = \breve{w}$ in $[0, a)$.

The following corollary may have some interest in the study of the Camassa-Holm equation.

Corollary 5.3. *Suppose P and $\breve{P}$ have the same spectral measure and that $b = \breve{b}$, $p = \breve{p}$ and $q = \breve{q} \not\equiv 0$. Then $P = \breve{P}$,* i.e., *then also $w = \breve{w}$. The same statement is true for Dirichlet operators P_0 and $\breve{P}_0$.*

Proof. We assume that the equation $-(pu')' + qu = \lambda wu$ is transformed into $-(pv')' + qv = \lambda\breve{w}v$ via a Liouville transform F given by $u(x) = f(x)v(g(x))$ where $f(0) = 1$, $pf'(0) = 0$, $g(0) = 0$ and $g(b) = b$. Now let f_1 be the solution of $-(pf_1')' + qf_1 = 0$ with initial data $f_1(0) = 1$, $pf_1'(0) = 0$. Since $p \geq 0$ and $q \geq 0$ this solution is > 0 on $[0, b)$, so we may put $g_1(x) = \int_0^x 1/pf_1^2$. This gives us a Liouville transform F_1 mapping $[0, b)$ onto some interval $[0, c)$, and transforming the equations into $-u_1'' = \lambda w_1 u_1$ and $-v_1'' = \lambda\breve{w}_1 v_1$ respectively. Thus $F_1 F F_1^{-1}$ transforms these two equations into each other. Being a composition of Liouville transforms this is itself a Liouville transform, given, say, by $u_1(x) = f_2(x)v_1(g_2(x))$. By construction we obtain $f_2(0) = 1$ and $f_2'(0) = 0$. Since the highest order coefficients of the two equations are 1, it is easily seen that $f_2^2 g_2'$ is constant, and since the potentials are identically 0 for both equations it follows that $f_2'' = 0$. Altogether this means that $f_2 \equiv 1$ and $g_2(x) = Cx$ for a constant C.

Since g_2 maps $[0, c)$ onto itself we must have $C = 1$, unless $c = +\infty$. However, this is not possible since $c = \int_0^b 1/pf_1^2$, and the latter integral is always convergent. To see this, let $H(x) = pf_1'(x)f_1(x)$. Then $H'(x) = p(f_1')^2 + q(f_1)^2$ so that H increases, and since $q \not\equiv 0$ we have $H > 0$ close to b. It follows that $1/pf_1^2 = p(f_1')^2/H^2 \leq H'/H^2$. Thus, if d is so close to b that $H(d) > 0$ we obtain $\int_d^b 1/pf_1^2 \leq 1/H(d)$. Consequently $C = 1$ so that F is the identity on $[0, b)$. But this means that $\breve{w} = w$ and thus finishes the proof. ■

To prove Theorem 5.1 we first show that the statement about Dirichlet operators follows from that for Neumann operators.

Proposition 5.4. *If the operators P_0 and $\breve{P}_0$ have the same spectral measures, then so have the operators P and $\breve{P}$.*

Proof. By the representation formula (4.3) the m-functions for the operators P_0 and $\breve{P}_0$ can only differ by a linear function. However, it is known that Dirichlet m-functions (in our 'left-definite' case) always tend to 0 as $\lambda \to \infty$ for example along the imaginary axis. See [3, Theorem 3.6] for this result. Thus the m-functions for P_0 and $\breve{P}_0$ coincide.

It is also known, see [3, (2.8)], that $m(\lambda) = -1/m_0(\lambda)$, so that in fact P and $\breve{P}$ also have the same m-functions. By the uniqueness of the spectral measure in the representation (4.3) it follows that P and $\breve{P}$ have the same spectral measures. ∎

We therefore only need to prove Theorem 5.1 for Neumann operators, and to do this we need the following theorem of Paley-Wiener type. For its statement it will be convenient to introduce a special class of entire functions.

Definition 5.5. Suppose r is measurable and $|r|^{1/2}$ is integrable in every compact subset of $[0, b)$. Let $\mathcal{A}(r)$ be the set of entire functions $\hat{u}$ of order $\leq 1/2$ which satisfy

$$\limsup_{t\to\infty} t^{-1} \ln |\hat{u}(t^2\lambda)| \leq \int_0^a \operatorname{Re} \sqrt{-\lambda r} \tag{5.1}$$

for some $a \in (0, b)$ and all $\lambda \in \mathbb{C} \setminus \mathbb{R}$.

Theorem 5.6. *Let $\hat{u}$ be the generalized Fourier transform of $u \in \mathcal{H}$. Then $\hat{u}$ has at most one entire continuation in $\mathcal{A}(w/p)$, and if* $\sup(\operatorname{supp} u) = a < b$ *such a continuation is given by*

$$\hat{u}(\lambda) = \int_0^a pu'\varphi'(\cdot, \lambda) + qu\varphi(\cdot, \lambda)$$

in which case (5.1) *holds with equality for $r = w/p$ and all λ.*

Conversely, if $\hat{u}$ has an entire continuation of order $\leq 1/2$ satisfying (5.1) *for λ on at least two different rays from the origin, then* $\operatorname{supp} u \subset [0, a]$.

We will also need another theorem of Paley-Wiener type. To state it we introduce another transform on $\mathcal{H}$, which is analogous to a Laplace transform. For $v \in \mathcal{H}$, set $\tilde{v}(\lambda) = \langle v, \overline{\psi(\cdot, \lambda)} \rangle$, which is analytic at least for $\lambda \notin \mathbb{R}$. By Parseval's formula $\tilde{v}$ is the Stieltjes transform of the measure $\hat{v}\, d\rho$, so by the uniqueness of the Stieltjes transform it follows that $\tilde{v}$ determines $\hat{v}$ and thus v. We will prove the following theorem.

Theorem 5.7. *Suppose $v \in \mathcal{H}$ and* $\inf \operatorname{supp} v = a \geq 0$. *Then*

$$\limsup_{t\to\infty} t^{-1} \ln |\tilde{v}(t^2\lambda)| = -\int_0^a \operatorname{Re} \sqrt{-\lambda w/p} \tag{5.2}$$

Conversely, if $v(0) = 0$ and the left hand side of (5.2) *is bounded by the right hand side for two non-real λ with different argument, then* $\inf \operatorname{supp} v \geq a$.

We will postpone the proofs of Theorems 5.6 and 5.7 to the next section.

Proof of Theorem 5.1. Let $\mathcal{U} = \breve{\mathcal{F}}^{-1} \circ \mathcal{F} : \mathcal{H} \to \breve{\mathcal{H}}$, which is unitary. Since $\mathcal{U}P = \breve{P}\mathcal{U}$ we also have $\mathcal{U}T = \breve{T}\mathcal{U}$, with natural notation. Thus

$$\int u\overline{v}w = \langle u, Tv \rangle = \langle \mathcal{U}u, \mathcal{U}Tv \rangle = \langle \mathcal{U}u, \breve{T}\mathcal{U}v \rangle = \int \mathcal{U}u\overline{\mathcal{U}v}\breve{w}. \tag{5.3}$$

Now, applying Theorem 5.6 for the rays generated by $\pm i$ it is clear that if $\breve{a} \in (0, \breve{b})$ and $u \in \mathcal{H}$, then $\sup(\operatorname{supp} u) = a$ if and only if $\sup(\operatorname{supp} \mathcal{U}u) = \breve{a}$, where $h(a) = \breve{h}(\breve{a})$, provided there is such an $a \in (0, b)$. This will certainly be the case if $\breve{a}$ is sufficiently close to 0. Suppose for some $\breve{a} \in (0, \breve{b})$ we have $h(b) \le \breve{h}(\breve{a})$. Then, since finite functions are dense in $\mathcal{H}$, the range of $\mathcal{U}$ would be orthogonal to all elements of $\breve{\mathcal{H}}$ with supports in $(\breve{a}, \breve{b})$, contradicting the fact that $\mathcal{U}$ is unitary. A similar reasoning applied to $\mathcal{U}^{-1}$ shows that the mapping $g : [0, b) \ni a \mapsto \breve{a} \in [0, \breve{b})$ is strictly increasing and bijective. The function g equals the composition $\breve{h}^{-1} \circ h$, and since an absolutely continuous, increasing map has an absolutely continuous inverse precisely if its derivative is > 0 a.e., it is clear that g and g^{-1} are both locally absolutely continuous.

By Lemma 4.2(4) we have $u(0) = 0$ if and only $\mathcal{U}u(0) = 0$. Thus, applying Theorem 5.7 for $\lambda = \pm i$ we similarly obtain $\inf(\operatorname{supp} u) = a$ if and only if $\inf(\operatorname{supp} \mathcal{U}u) = g(a)$. Thus the convex hull of $\operatorname{supp} u$ is $[a, c]$ precisely if the convex hull of $\operatorname{supp} \mathcal{U}u$ is $[g(a), g(c)]$. It therefore follows easily from (5.3) that if $x \in (0, b)$, then

$$\int_0^x u\overline{v}w = \int_0^{g(x)} \mathcal{U}u\overline{\mathcal{U}v}\breve{w}.$$

Differentiating this we obtain, after rearrangement,

$$\frac{\breve{w}(g(x))g'(x)}{w(x)} = \frac{u(x)\overline{v(x)}}{\mathcal{U}u(g(x))\overline{\mathcal{U}v(g(x))}},$$

where the left hand side does not depend on u or v. Setting $f(x) = u(x)/\mathcal{U}u(g(x))$ and varying v shows that f independent of u, so $u(x) = f(x)\mathcal{U}u(g(x))$. Since it is clear that the Fourier transform maps real-valued functions to realvalued functions we must have f real-valued, locally absolutely continuous and $\neq 0$, thus of one sign. We also get $\breve{w}(g(x)) = w(x)f^2(x)/g'(x)$. Using this we obtain from $\breve{h}(g(x)) = h(x)$ by differentiating and squaring also that $\breve{p}(g(x)) = p(x)f^2(x)g'(x)$.

Furthermore $\mathcal{F}(\psi(\cdot, \lambda))(t) = (t-\lambda)^{-1} = \breve{\mathcal{F}}(\breve{\psi}(\cdot, \lambda))(t)$ from which follows that $\psi(x, \lambda) = f(x)\breve{\psi}(g(x), \lambda)$. Since $m(\lambda)$ is not linear, but the difference $m(\lambda) - \breve{m}(\lambda)$ is, we obtain $f(0) = 1$ so that $f > 0$. Differentiating and using the formulas for $\breve{w}$ and $\breve{p}$ shows that pf' is also locally absolutely continuous and $pf'(0) = 0$. ■

6 The Paley-Wiener theorems

The proofs of Theorems 5.6 and 5.7 rely on the following lemma, which is taken from [3, Theorem 6.1, Corollary 6.2].

Lemma 6.1. *The following asymptotic formulas hold, locally uniformly for $\lambda \in \mathbb{C} \setminus \mathbb{R}$.*

$$\lim_{t\to\infty} t^{-1} \log(p\varphi'(x, t^2\lambda)) = \int_0^x \sqrt{-\lambda w/p},$$

$$\lim_{t\to\infty} t^{-1} \log(p\psi'(x, t^2\lambda)) = -\int_0^x \sqrt{-\lambda w/p}.$$

The next lemma implies the simple direction of the Theorem 5.6.

Lemma 6.2. *Suppose $u \in \mathcal{H}$ and $\operatorname{supp} u \subset [0, a]$. Then $\hat{u}(\lambda)$ is entire of order $\leq 1/2$ and $\hat{u}(\lambda) = o(|p\varphi'(a, \lambda)|)$ as $\lambda \to \infty$ along any non-real ray originating at the origin.*

Proof. Let $\psi_{(0,a]}(x, \lambda) = \varphi(x, \lambda)/p\varphi'(a, \lambda)$. This function has a quasi-derivative $p\varphi'(x, \lambda)/p\varphi'(a, \lambda)$ which vanishes at 0 and equals 1 at a. Therefore $\psi_{(0,a]}$ is a Weyl solution for the interval $(0, a]$, viewing a as the initial point and with a Neumann boundary condition also at 0. Lemma 4.2(2) continues to hold in this situation, so $\psi_{(0,a]} \to 0$ in norm as $\lambda \to \infty$ along non-real rays from the origin. We have $\hat{u}(\lambda) = p\varphi'(a, \lambda)\langle u, \psi_{(0,a]}(\cdot, \lambda)\rangle$ so the lemma follows. ■

The hard direction of Theorem 5.6 follows from the next lemma.

Lemma 6.3. *Suppose $u \in \mathcal{H}$, that $\hat{u}$ has an entire continuation of order $\leq 1/2$ and that $\hat{u}(\lambda) = \mathcal{O}(1/|p\psi'(a, \lambda)|)$ as $\lambda \to \infty$ along two different non-real rays originating at the origin. Then $\operatorname{supp} u \subset [0, a]$ and $\hat{u}(\lambda) = \langle u, \overline{\varphi(\cdot, \lambda)}\rangle$.*

Proof. Consider $F(\lambda) = \langle R_\lambda u, v\rangle - \hat{u}(\lambda)\langle \psi(\cdot, \lambda), v\rangle$. We shall show that if $v = Tf$, where $f \in \mathcal{H}$ has compact support in (a, b), then F has an entire continuation of order $\leq 1/2$ which tends to 0 along the given rays. By Phragmén-Lindelöf's theorem it follows that F is bounded everywhere and is therefore constant by Liouville's theorem, thus actually identically 0. Now $F(\lambda) = \int_0^b (R_\lambda u - \hat{u}(\lambda)\psi(\cdot, \lambda))\overline{f}w$, so f being essentially arbitrary in (a, b), it follows that $R_\lambda u - \hat{u}(\lambda)\psi(\cdot, \lambda)$ has support in $[0, a]$. Applying the differential operator it follows that also u has support in $[0, a]$. For $x > a$ the formula (4.2) gives $R_\lambda u(x) = \psi(x, \lambda)\langle u, \overline{\varphi(\cdot, \lambda)}\rangle$ from which follows that $\psi(x, \lambda)(\hat{u}(\lambda) - \langle u, \overline{\varphi(\cdot, \lambda)}\rangle) = 0$. The lemma follows from this.

To prove that F is entire, Parseval's formula and Lemma 4.2(1) shows that

$$F(\lambda) = \int_{-\infty}^{\infty} \frac{\hat{u}(t) - \hat{u}(\lambda)}{t - \lambda}\, \overline{\hat{v}(t)}\, d\rho(t).$$

It is obvious that this is an entire function, at least if we can bound the integrand properly. To do this and see that the order is at most $1/2$, note that for $|t - \lambda| \leq 1$ we may estimate the integrand by $\sup_{|z|\leq 1} |\hat{u}'(\lambda + z)||\hat{v}(t)|$. For $|t - \lambda| > 1$ we may estimate the integrand by $|\hat{u}(t)\hat{v}(t)| + |\hat{u}(\lambda)||\hat{v}(t)|$. Hence we have locally uniform convergence of the integral and

$$|F(\lambda)| \leq \|u\|\|v\| + (\sup_{|z|\leq 1} |\hat{u}'(\lambda + z)| + |\hat{u}(\lambda)|) \int |\hat{v}| d\rho,$$

which is the desired estimate, the integral being finite by Lemma 4.2(3).

Finally, to show that F tends to 0 along the rays, we first note that $\langle \psi(\cdot, \lambda), Tf\rangle = \int_0^b \psi(\cdot, \lambda)\overline{f}w$. Now, for $x \geq a$, $\psi_{[a,b)}(x, \lambda) = \psi(x, \lambda)/p\psi'(a, \lambda)$ is the Weyl solution for our equation considered on the interval $[a, b)$. Assuming $\operatorname{supp} f \subset [a, b)$ we have $\int_0^b \psi(\cdot, \lambda)\overline{f}w = p\psi'(a, \lambda)\int_a^b \psi_{[a,b)}(\cdot, \lambda)\overline{f}w$ so that Lemma 4.2(2) implies $\langle \psi(\cdot, \lambda), Tf\rangle = o(|p\psi'(a, \lambda)|)$. Since $R_\lambda \to 0$ strongly as $\operatorname{Im}\lambda \to \infty$, it follows that F tends to 0 along the given rays. This finishes the proof. ■

Theorem 5.6 is now a simple consequence of Lemmas 6.1–6.3 and we will leave out the details. See, however, the proof of Theorem 3.2 of [4]. The proof of Theorem 5.7 is quite similar, and relies on the following two lemmas.

Lemma 6.4. *Suppose $v \in \mathcal{H}$ and $v = 0$ in $[0, a]$. Then $\tilde{v}(\lambda) = o(|p\psi'(a, \lambda)|)$ as $\lambda \to \infty$ along any non-real ray starting at 0.*

Proof. Setting $\psi_{[a,b)}(x, \lambda) = \psi(x, \lambda)/p\psi'(a, \lambda)$ we obtain the Weyl solution for the interval $[a, b)$ with Neumann's boundary condition at a. Thus $\langle v, \overline{\psi_{[a,b)}(\cdot, \lambda)}\rangle = o(1)$ so that $\tilde{v}(\lambda) = o(|p\psi'(a, \lambda)|)$ as $\lambda \to \infty$ along any non-real ray starting at 0. ■

Lemma 6.5. *Suppose $v \in \mathcal{H}$ with $v(0) = 0$ and also that $\tilde{v}(\lambda) = \mathcal{O}(1/|p\varphi'(a, \lambda)|)$ as $\lambda \to \infty$ along two different non-real rays starting at 0. Then* $\inf \operatorname{supp} v \geq a$.

Proof. Let $u \in \mathcal{H}$ and assume $\operatorname{supp} u \subset [0, a)$ and put $F(\lambda) = \langle R_\lambda v, u\rangle - \tilde{v}(\lambda)\overline{\hat{u}(\overline{\lambda})}$. By Theorem 5.6 and the assumption it follows that $F \to 0$ along the given rays. We shall show that F may be continued to an entire function of order $\leq 1/2$. Then the Phragmén-Lindelöf theorem shows that $F = 0$, *i.e.*, $\langle R_\lambda v - \tilde{v}(\lambda)\varphi(\cdot, \lambda), u\rangle = 0$. Now setting $v_1 = R_\lambda v - \tilde{v}(\lambda)\varphi(\cdot, \lambda)$ we have $pv_1'(0) = 0$ and $-(pv_1')' + qv_1 = \lambda w v_1 + wv$. Thus integrating by parts we obtain $0 = \langle v_1, u\rangle = \int_0^a (\lambda v_1 + v)\overline{u}w$. Since u is essentially arbitrary in $[0, a]$ it follows that $v = -\lambda v_1$ in $[0, a]$. Thus $v(0) = pv'(0) = 0$ and v satisfies $-(pv')' + qv = 0$ in $[0, a]$, so that v vanishes in this interval.

To see that $\langle v_1, u\rangle$ is entire of order $\leq 1/2$, note that according to (4.2) and using $v(0) = 0$, partial integration gives

$$v_1(x) = \theta(x, \lambda)\int_0^x v\lambda\varphi(\cdot, \lambda)w - \lambda\varphi(x, \lambda)\int_0^x v\theta(\cdot, \lambda)w,$$

which is entire of order $\leq 1/2$, locally uniformly in x. As before we have $\langle v_1, u\rangle = \int(\lambda v_1 + v)\overline{u}w$, so that we are done. ■

7 An application to the Camassa-Holm equation

Corollary 5.3 gives a uniqueness theorem for the CH spectral problem on a half-line $[0, \infty)$, with Dirichlet or Neumann boundary conditions at 0. If w is integrable the spectrum is discrete, and in this case w is determined by the eigenvalues and their normalization constants. However, it is clear that the whole line case is of more interest. In this section we shall see that in some cases also this case may be handled by Corollary 5.3.

If we consider (4.1) on a open interval (a, b), where $1/p$, q, w are only locally integrable, we may transform this problem by a Liouville transform. Thus we introduce new independent and dependent variables g and $\tilde{u}$ by setting $u(x) = f(x)\tilde{u}(g(x))$, where f is a given, sufficiently smooth function with no zeros. If u has compact support in (a, b) and $A = g(a)$, $B = g(b)$ we obtain

$$\int_a^b (p|u'|^2 + q|u|^2) = \int_A^B (\partial|\tilde{u}'|^2 + \tilde{q}|\tilde{u}|^2) \tag{7.1}$$

where $\partial(g(x)) = p(x)f^2(x)g'(x)$ and $\tilde{q}(g(x)) = f(x)(-(pf')' + qf)/g'(x)$. Thus, if we let f be a non-vanishing solution to $-(pf')' + qf = 0$ near a and choose $g(x) = \int_a^x 1/pf^2$ the transformed norm-square will be $\int_0^B (|\tilde{u}'|^2 + \tilde{q}|u|^2)$ where $\tilde{q} \equiv 0$ near a. We have here assumed that the integral converges at a, but there are in fact always non-vanishing solutions for which this is true. Since we will give an explicit f for the CH case we will not prove this here.

Now assume $q \not\equiv 0$ in both (a, c) and (c, b) and choose f as the solution of $-(pf')' + qf = 0$ with $f(c) = 1$, $pf'(c) = 0$ in $(a, c]$ and $f(x) = 1$ for $x > c$. Then $f \neq 0$ everywhere and $\tilde{q} \equiv 0$ in (a, c) but $\tilde{q} \not\equiv 0$ in (c, b).

The equation (7.1) is transformed to

$$-\tilde{u}'' + \tilde{q}u = \lambda\tilde{w}\tilde{u} \text{ on } (0, B), \tag{7.2}$$

where $\tilde{w}(g(x)) = f^2(x)w(x)/g'(x)$. If $\tilde{w}$ is integrable near 0, then the spectral theory of Section 4 and inverse spectral theory of Section 5 apply. The assumption on $\tilde{w}$ means that $f^2 w$ is integrable near a. Now consider the case of (1.1) with $\varkappa = 0$. We may then choose $f(x) = \cosh(x/2)$ for $x < 0$ and $f(x) = 1$ otherwise, and we have to assume $e^{-x}w(x) \in L^1(-\infty, 0)$ in addition to (2.1). For $\varkappa = 0$ we have discrete spectrum and the solution f_- is asymptotic to $e^{x/2}$ at $-\infty$. It is easy to see that this means that f_- transforms to the solution φ of (7.2) with initial data $\varphi(0, \lambda) = 0$, $\varphi'(0, \lambda) = 1$. It follows that the Dirichlet normalization constants for the transformed equation coincide with those of the original equation. Thus, Corollary 5.3 tells us that the eigenvalues and their normalization constants determine w uniquely. In other words, in the case $\varkappa = 0$ the scattering data determine the weight w. Presumably the assumption that $e^{-x}w(x)$ is integrable near $-\infty$ is actually superfluous.

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